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Asset Pricing in a Production Economy with Incomplete Information

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ABSTRACT

This paper analyzes an economy in which investors operate under partial information about technology-relevant state variables. It is shown that for Gaussian information structures under incomplete observations, the consumer's problem can be transformed into an equivalent program with a completely observed state: the conditional expectation of the underlying unobservable state variables. A consequence of this transformation is that classic results in finance remain valid under an appropriate reinterpretation of the state variables.

THE STATIC CAPITAL ASSET Pricing Model (CAPM) of Sharpe [24], Lintner [16], and Mossin [20] states that the expected premium on any risky asset is proportional to the premium on the market as a whole. This theoretical linear relationship, however, is subject to criticisms on two grounds. First, it does not seem to be supported by the empirical evidence.[1] Second, it is based on the premises of a constant opportunity set. Merton [19] subsequently relaxes this assumption, and shows that when the opportunity set fluctuates over time due to changes in the state of the economy, individual securities are also priced with respect to selected portfolios (funds), providing a hedge against unanticipated fluctuations.[2] More recently, Breeden [3] has extended Merton's results by emphasizing the role of aggregate consumption in pricing relationships.

In this paper, we seek to further our understanding of asset pricing by relaxing the assumption that investors observe the state of the economy. In the Merton-Breeden framework, observed state variables affect the means and variances of asset returns, implying deformations of the opportunity set over time. From a practical point of view, it seems that this complete information assumption is violated. Investors in financial markets operate under partial knowledge of the microvariables which influence production and the returns on assets. Their decisions rely mostly on information through noisy channels such as newspapers, technical reviews, or weather reports. In this paper, we attempt to model such

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[1] See, for instance, Black, Jensen, and Scholes [2] or Jensen [15].

[2] Merton focuses on the case of a unique state variable. A treatment considering a vector of state variables is found in Richard [22].

behavior and analyze the consequences for individual demands and asset pricing relationships.

The economy considered is populated by J agents who differ in terms of preferences and initial wealth, but who observe the same set of signals conveying information about an underlying unobservable vector of state variables. The information structure modeled is Gaussian: initial conditional beliefs are normally distributed, and the joint evolution of the state and the observation process has an instantaneous mean which is linear in the vector of state variables. This structure has the property that the distribution of the unobserved state at time t conditional on the information accumulated up to this point is Gaussian. For this class of distributions it is well-known that the mean only depends on the conditioning factor. This property is also true in the dynamic case investigated here: the mean follows a diffusion process depending on the observations collected whereas the variance is deterministic. It follows that the mean is a sufficient statistic for the conditional distribution.

A well-known result of optimal control with partial observations is that the consumer's choice problem can be performed as a two-stage procedure for this class of information structures.[3] In a first step the unobserved state variables are estimated; in a second step the consumption-portfolio choice is made on the basis of the estimated state variables. Thus the structure with incomplete information is equivalent to the Merton-Breeden model where a substitute state vector (the conditional mean of the underlying unobservable state variables) becomes the new state.[4] It follows immediately that classical propositions in finance (portfolio separation, multibeta and consumption capital asset pricing models) hold under this reinterpretation.

These features have been exploited by Feldman [10] and Detemple [9] within the context of a general equilibrium model with a representative agent in order to develop valuation formulas for financial assets. Furthermore, Feldman proposes information imperfections as an explanation for the observed broad range of shapes of the term structure of interest rates. In this context he shows that the yield curve can be characterized by non-monotonic functions of considerable variation when information is incomplete.

In the second section of the paper the structure of the model is detailed and the filtering problem discussed. The third section focuses on the consumer program and its reduction to the Merton-Breeden formulation with the substitute state variables.

[3] From a formal point of view, the equivalence between the original program given the partially observable information structure and the program based on the conditional mean taken as a substitute state variable is only valid under stringent conditions on the optimal controls for the second program (Fleming and Rishel [11, Theorem 11.2]). A detailed presentation of this separation result appears in Gennotte [12].

[4] In a discrete time-infinite horizon model [8] with a similar information structure, I have shown the existence of a value function and an equilibrium price system which depend on wealth, and the conditional mean and variance if the period utility function and the technological processes satisfy a number of restrictions, including a boundedness condition. The existence of an equilibrium in the continuous time model described here is assumed.

I. The Economy

A version of the general equilibrium model of Cox, Ingersoll, and Ross [6] is considered, in which the state variables affecting production are unobserved by consumers. Our approach is similar to that used in Breeden [4]. We shall consider a model with J investors who are homogeneously, but incompletely, informed and have a common lifetime, $[0, T]$.

A single good is produced in m different technologies. The j[th] investor seeks to allocate his wealth, W_t^j, between consumption, c_t^j, investment in the m technologies $\omega_t^j \equiv [\omega_{1t}^j, \cdots, \omega_{mt}^j]$, and acquisition of q financial claims, $\pi_t^j \equiv [\pi_{1t}^j, \cdots, \pi_{qt}^j]$. Productive technologies are constant stochastic returns to scale, and solve the stochastic differential equations specified below in assumption II.2. The instantaneous mean and variance of these processes depend upon a vector of technological parameters and disturbances (e.g., technological progress, weather, research activities, etc.) which constitute the underlying state of the economy. It is postulated that the state vector evolves according to a diffusion process. In the process of its evolution, it deforms the opportunity set faced by investors. This is a standard feature in the financial literature.[5] In previous papers, however, the state of the economy is observed by investors. Here, we relax this assumption and consider a case in which investors have a diffuse knowledge of the state's value. Investment opportunities are evaluated on the basis of the information available, which is provided either through observable market clearing quantities and prices or through exogenous signals. The information structure is specified in assumption II.3 and later discussed in more detail. The following assumptions are made:

Assumption II.1 (state variables): There is an N-dimensional vector of state variables which affects production and satisfies the stochastic differential equation $d\vartheta_t = [a_0 + a_1\vartheta_t]dt + b\, dz(t)$. The process, $z(t)$, is a vector Wiener process of dimension $N + L + m$, and the parameters, a_0, a_1, and b, are time-dependent, conformable vectors and matrices.

Assumption II.2 (technologies): There are m constant returns to scale technological processes. The vector of production satisfies: $d\eta_t = I_\eta \cdot [(\alpha_0 + \alpha_1\vartheta_t)dt + G\, dz(t)]$, where I_η is a diagonal matrix with the vector of investments η on its diagonal and zeroes elsewhere. The quantity $d\eta_t$ represents the increase in the stock of the good when η_t is invested in the technologies. The parameters α_0, α_1, and G depend on time and have appropriate dimensions.

Assumption II.3 (information): All investors know the structure of the economy, the processes for output and the state vector. They have initial beliefs about the position ϑ_0 of the state which are conditionally normal with mean m_0 and variance-covariance matrix V_0. The investors do not observe the realizations of the state process $\{\tilde{\vartheta}_t\colon 0 \leqq t \leqq T\}$, but draw inferences about its position from endogenous quantities (wealth, prices, etc.), as well as from an exogenous signal ε_t. The informative signal ε_t is observed by all investors and has dimension L. It

[5] Merton [19], Breeden [3], Cox, Ingersoll, and Ross [6], and Richard and Sundaresan [23] use this specification with the implicit assumption that the state vector is perfectly observed by agents.

satisfies the stochastic differential equation: $d\varepsilon_t = [A_0 + A_1\vartheta_t]dt + B\,dz(t)$ where A_0, A_1, and B are conformable time-dependent parameters.

The signal ε_t should be interpreted as public information, for example, a weather forecast or a specialized review. The stochastic differential equation describing the evolution of the signal results from a relationship of the form $\varepsilon_t = \lambda_1(t)\vartheta_t + \lambda z(t)$. The economic interpretation of ε_t follows. This signal is indeed a linear combination of the true position of the state ϑ_t and of a noise component $z(t)$. Investors use their observation in order to infer the position of the state, but can only do so imperfectly due to the noise. Since new information hits the market continuously, investors update their beliefs about the position of the state continuously.

Assumption II.4 (preferences): The j[th] investor chooses a consumption-investment mix so as to maximize his expected lifetime utility: $E\{\int_0^T u^j[c^j(s), s]ds \mid I_0\}$. The operator $E\{\cdot \mid I_0\}$ designates the expectation conditional on the initial information I_0. The utility function $u^j[\cdot, t]$ is strictly increasing, twice continuously differentiable and strictly concave for all j. It is also assumed that $u_c^j[0, t] = +\infty$ $\forall t \in [0, T]$ where $u_c^j[\cdot, \cdot]$ is the derivative of $u^j[\cdot, \cdot]$ with respect to its first argument.

Assumption II.5 (financial markets): (a) Investors can borrow or lend at an instantaneous risk-free rate r_t. (b) Financial claims are traded in a competitive market. There are q different assets available, with prices $f_{1t}, \cdots, f_{qt}$ at time t. The vector of prices is denoted by f_t. Each asset generates a stream of payments $\delta_{1t}, \cdots, \delta_{qt}$ (δ_t in vector notation). These cash flows are specified contractually and are only contingent upon observable quantities.[6]

Investors use their observations in order to update their beliefs about the position of the state. For the structure described, under additional conditions, the conditional distribution of the state given the information collected is Gaussian. Furthermore, the conditional mean $\hat{\vartheta}_t$ and the conditional variance-covariance matrix γ_t are the unique continuous solution to the system of equations:[7]

$$d\hat{\vartheta}_t = [a_0 + a_1\hat{\vartheta}_t]dt + z_t[d\zeta_t - (C_0 + C_1\hat{\vartheta}_t)dt] \tag{1}$$

$$z_t = S_t[DD']^{-1} \tag{2}$$

$$\dot{\gamma}_t = a_1\gamma_t + \gamma_t a_1' + bb' - z_t S_t' \tag{3}$$

$$\hat{\vartheta}_0 = m_0;\ \gamma_0 = V_0 \tag{4}$$

where $\zeta_t' = [\eta_t', \varepsilon_t']$; $C_0' \equiv [\alpha_0', A_0']$; $C_1' \equiv [\alpha_1', A_1']$; $D' \equiv [G', B']$ and $S_t \equiv [bD' + \gamma_t C_1']$ (for notational convenience the time argument of all of these parameters

[6] In this economic structure financial claims are issued by investors and consequently can only be functions of the common information available.

[7] For a proof of these statements and a listing of the conditions imposed on the structure, see Liptser and Shiryayev [17, Volume 2, p. 31].

is omitted). Finally when the matrix γ_0 is positive definite, then the matrices γ_t, $0 \leq t \leq T$ have the same property.

The variance-covariance matrix γ_t is by definition the mean square deviation of the vector $\tilde{\vartheta}_t$ from its conditional mean $\hat{\vartheta}_t$ and therefore constitutes a measure of the quality of the information collected. Its evolution is deterministic and depends on the parameters of the structure. The independence of the conditional variance from observations is a well-known property of Normal distributions. The Bayesian updating of beliefs performed by the investor in addition generates changes in the precision of the estimation of the position of the state. Whether the quality of the information improves or deteriorates over time depends on the initial position and the parameters of the information structure. A better understanding of the adjustments which take place is gained by investigating the unidimensional case with constant parameters (see the Appendix). In this situation, for γ_0 positive, the conditional variance remains positive when T is finite. Furthermore, under conditions on the parameters of the structure, the variance converges to a positive fixed point γ^* independently of the initial condition V_0. At this point, the learning process is stabilized, i.e., the sharpness of the information collected is invariant. Beliefs, then, only respond to observations and are characterized by changes in the conditional mean. When the initial variance V_0 is above (below) the steady state γ^*, the quality improves (deteriorates) over time.

The conditional mean $\hat{\vartheta}_t$ is the solution to the stochastic differential Equation (1). Since the conditional distribution is Normal, the mean is a sufficient statistic: it conveys all the information imbedded in the signals.

The evolution of the economy takes place through the vector of wealth $W_t = [W_t^1, \cdots, W_t^J]$ and through beliefs $(\hat{\vartheta}_t, \gamma_t)$. Equilibrium prices consequently should be of the form $f(W_t, \hat{\vartheta}_t, \gamma_t, t)$. To avoid problems connected with the inclusion of the distribution of wealth in the arguments of the price function we shall suppose that there exists an equilibrium of the form $f(\hat{\vartheta}_t, \gamma_t, t)$. Under smoothness assumptions, the price process then satisfies the stochastic differential equation:

$$df_t = I_f[\mu_f(\cdot)dt + \sigma_f(\cdot)dz] - \delta_t dt \tag{5}$$

where I_f is a diagonal matrix with the vector of prices on its diagonal. The functions $\mu_f(\cdot)$ and $\sigma_f(\cdot)$ depend on the parameters of conditional beliefs and on the vector of state variables since the diffusion of $\hat{\vartheta}_t$ depends on the underlying state ϑ_t through the observation process ε_t. In the next section we analyze the consumer program under the assumption that equilibrium prices satisfy (5).

II. The Consumer's Program

The j[th] investor maximizes his expected lifetime utility by choosing the controls

c_t^j, ω_t^j, and π_t^j. He solves the program:

$$\underset{\{c_t^j,\, \omega_t^j,\, \pi_t^j\}}{\text{Max}} \quad E\{\textstyle\int_0^T u[c^j(s), s]ds \mid I_0\} \tag{6}$$

$$\text{s.t. } dW_t^j = [\omega_t^{j\prime}\ (\mu_\eta - r\cdot\underline{1}) + \pi_t^{j\prime}\ (\mu_f - r\cdot\underline{1}) + r]W_t^j dt \\ - c_t^j dt + \omega_t^{j\prime}\ W_t^j \sigma_\eta dz + \pi_t^{j\prime}\ W_t^j \sigma_f dz \tag{7}$$

where the information structure is described in assumptions II.1–II.3.

Consider now the program (6) subject to the dynamic budget equations (7) and the dynamics of the conditional mean and variance (1)–(4). Let $(\hat{c}_t^j, \hat{\omega}_t^j, \hat{\pi}_t^j)$ be the optimal controls for this problem. The separation principle (Fleming and Rishel [11, Theorem 11.2, p. 194]) states that the original control program with partial observations solved by the consumer is equivalent to this new program if the optimal feedback controls $(\hat{c}_t^j, \hat{\omega}_t^j, \hat{\pi}_t^j)$ exist and satisfy some technical conditions (measurably, boundedness and Lipschitz conditions). To proceed we will suppose that these conditions are satisfied.

From a practical point of view the equivalence between the two programs means that the investor can solve his or her allocation problem via a two-step procedure. First, the true unobservable state variables are estimated by their conditional expectation (optimal estimator for quadratic loss function). Second, the optimal portfolio weights and the consumption rate are computed taking as substitute state variables the estimate $\hat{\vartheta}_t$ and the estimation error γ_t.[8]

Let $J[W_t^j, \hat{\vartheta}_t, \gamma_t, t]$ be the value function associated with the new program. The Bellman equation is:

$$0 = \max_{\{c_t,\, \omega_t,\, \pi_t\}} \quad \{u[c_t, t] + E[dJ[W_t, \hat{\vartheta}_t, \gamma_t, t] \mid I_t]\} \tag{8}$$

This program is similar to the Bellman equation appearing in Merton [18, 19] and Breeden [3, 4] with the substitution of $\hat{\vartheta}_t$ for the stochastic state variables. It follows that their results hold under the appropriate reinterpretation of the state.

III. Conclusions

This paper has reconsidered classic results of financial theory in the case where information about the state is incomplete. It was shown under existence assumptions that the structural form of optimal demands and of equilibrium relation-

[8] This two-stage solution procedure is a standard result in discrete time, incomplete information problems with quadratic loss functions and linear, Gaussian state variables systems (Astrom [1, Chapter 8]). Some intuition can be derived from this well-known result by noticing that, in the continuous time model analyzed in the paper, the marginal utility of the consumer is locally quadratic (only the first two moments of the change in consumption matter) and the stochastic process for the state variables is linear and locally Gaussian. As a consequence we would expect the separation principle to apply in our case.

ships remains unaltered, but that the interpretation differs significantly. Indeed, information emerges as a pivotal concept in the theory developed, whereas its importance went somewhat unnoticed in theoretical formulations based on complete information.

The critical feature of the model which has led to these results is the assumption of a Gaussian information structure. Since there is no economic reason for which state variables should conform to these restrictive processes, a natural question which emerges from the analysis and which may warrant further research is the impact of alternative structures on asset pricing models.

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Appendix: The Evolution of γ_t for a Unidimensional System

Consider the system:

$$d\vartheta_t = (a_0 + a_1\vartheta_t)dt + \textstyle\sum_{i=1}^{2} b_i dz_i(t) \qquad 1 \times 1 \tag{A1}$$

$$d\varepsilon_t = (A_0 + A_1\vartheta_t)dt + \textstyle\sum_{i=1}^{2} B_i dz_i(t) \qquad 1 \times 1 \tag{A2}$$

$$\tilde{\vartheta}_0 \mid \varepsilon_0 \sim N(m_0, \sigma_0^2). \tag{A3}$$

For this system γ_t solves:

$$\gamma_t = 2a_1\gamma_t + (b_1^2 + b_2^2) - (b_1B_1 + b_2B_2 + \gamma_tA_1)^2(B_1^2 + B_2^2)^{-1}. \tag{A4}$$

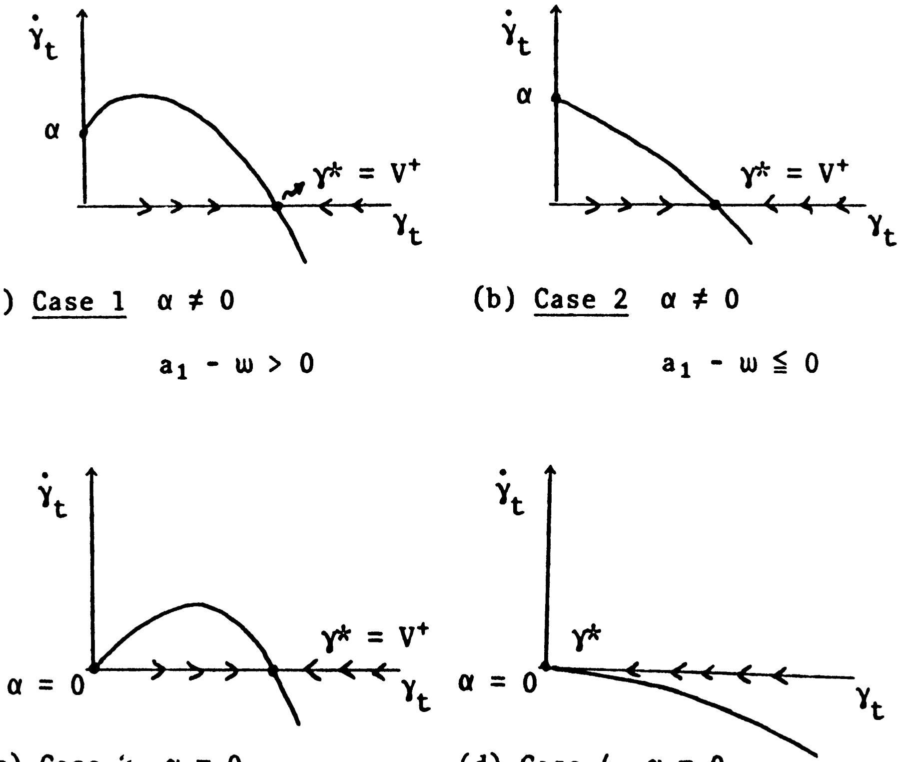
A general solution to the Riccati equation is computed by using the methodology exposited in Reid [21, Chapter 1]:

$$\gamma_t = \begin{cases} \gamma_0[\delta\gamma_0 t + 1]^{-1} & \text{for} \quad \alpha = 0;\ a_1 - \omega = 0 \\ \dfrac{V^+ + V^-\left(\dfrac{V^+ - \gamma_0}{\gamma_0 - V^-}\right)e^{-2\sqrt{(\omega - a_1)^2 + \alpha\delta}\, t}}{1 + \left(\dfrac{V^+ - \gamma_0}{\delta_0 - V^-}\right)e^{-2\sqrt{(\omega - a_1)^2 + \alpha\delta}\, t}} & \text{otherwise.} \end{cases}$$

where

$$\begin{aligned} \omega &= A_1(b_1B_1 + b_2B_2)(B_1^2 + B_2^2)^{-1} \\ \alpha &= (b_1B_2 - b_2B_1)^2(B_1^2 + B_2^2)^{-1} \\ \delta &= A_1^2(B_1^2 + B_2^2)^{-1} \\ V^+ &= \delta^{-1}[a_1 - \omega + \sqrt{(\omega - a_1)^2 + \alpha\delta}] \\ V^- &= \delta^{-1}[a_1 - \omega - \sqrt{(\omega - a_1)^2 + \alpha\delta}]. \end{aligned}$$

The evolution of the conditional variance γ_t depends on the values of the parameters a_1, A_1, b_1, b_2, B_1, and B_2. The following four cases arise:



(a) Case 1 $\alpha \neq 0$, $a_1 - w > 0$

(b) Case 2 $\alpha \neq 0$, $a_1 - w \leqq 0$

(c) Case 3 $\alpha = 0$, $a_1 - w > 0$

(d) Case 4 $\alpha = 0$, $a_1 - w \leqq 0$

Figure A1. Dynamics of the Conditional Variance

In all four cases, $\gamma_t > 0$ when $\gamma_0 > 0$ and $t \leq T < \infty$. In graphs (a), (b), and (c) in addition $\gamma_t \rightarrow \gamma^* > 0$ as $t \rightarrow \infty$, when $\gamma_0 > 0$.