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Equilibrium Interest Rates and Multiperiod Bonds in a Partially Observable Economy

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ABSTRACT

This paper analyzes the market for financial assets in a production and exchange economy with several realized outputs and a single unobservable source of nondiversifiable risk. The paper demonstrates that, for a large class of diffusion outputs and preferences, optimizing consumers first estimate the realizations of the unobservable factor and then use these estimates to determine portfolio and consumption rules. Moreover, the explicit consideration of this unobservable productivity factor affects equilibrium demands and prices. The equilibrium spot rate of interest emerges as the "best estimate" of the unobservable factor, and multiperiod default-free bonds arise as the optimal hedge for the unobservable changes of the stochastic investment opportunity set.

THIS PAPER ANALYZES THE market for financial assets in a production and exchange economy with several realized outputs and a single unobservable source of nondiversifiable risk. The paper poses and answers three questions:

1. How do consumers determine equilibrium demands and prices when their investment opportunities are unobservable?
2. In equilibrium, what financial assets provide an optimal hedge for the unobservable changes of the investment opportunity set?
3. What is the equilibrium price structure of these financial assets?

Consumers in a partially observable economy use a separation type result to determine equilibrium demands and prices. If the information structure is conditionally Gaussian, i.e., if the distribution of the unobservable economic state variable conditional on the observations is Gaussian, consumers first estimate the realizations of this variable and then use their estimates and the precision of these estimates in their optimization problem. If the information structure does not yield a Markovian optimization problem, a state vector solution is intractable.

Optimal hedging is achieved by complex trading strategies in realized outputs. Clearly, the direct procedure of writing claims on the unobservable factor is impossible; however, these trading strategies, surprisingly, are equivalent to holding multiperiod default-free bonds.

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Once the first two questions are answered, previously introduced valuation techniques determine prices of default-free bonds. While the above questions are answered for any concave preferences, logarithmic preferences are assumed in order to obtain explicit solutions.

The unobservable factor in this economy represents important but not directly observable influences on the business cycle. Examples of such influences include labor input measured in units of efficiency and technological progress measured in aggregate terms. The following analysis asserts that the explicit consideration of such an unobservable factor affects equilibrium prices and suggests an additional role for financial assets.

Apart from the unobservable state variable, the framework in which these questions are answered is in the spirit of Cox, Ingersoll, and Ross [8, 9]. The environment is described by a single investment and consumption good, a finite number of diffusion realized outputs, and risk-averse expected utility-maximizing consumers. An unobservable state variable or factor affects all the realized outputs, although its effect on different outputs may vary.

Section I describes the economy, and Section II delivers the separation type result and derives the necessary and sufficient conditions for the dynamic general equilibrium. Intuitively, the requirement for separation is that estimates can be updated recursively in a Markovian fashion, and the requirement for this property is a Markovian innovation process, i.e., a process that describes the deviations of realizations from their expected values. When consumers face a posterior distribution of the unobservable state variable, which consists of a sufficient statistic and Markovian updating, they can identify an endogenous Markovian economy where consumption and portfolio rules are optimal if and only if they are optimal in the non-Markovian exogenous economy. A separation result is also analyzed in Detemple [11] and Gennotte [22].

Section III obtains the explicit interest rate for logarithmic preferences and demonstrates that its instantaneous variance is a quadratic function of the dynamic filtering error, and Section IV analyzes the equilibrium in the partially observable economy. It identifies (the systematic and the nonsystematic) sources of covariance used in the inference process and characterizes the different relationships between the volatility of the estimated investment opportunity set and the volatility of the true one. Furthermore, it demonstrates that, by contrast to a fully observable economy, high interest rate volatility is not necessarily "bad news" (i.e., highly volatile investment opportunity set) and low interest rate volatility is not necessarily "good news" (i.e., low volatility of the investment opportunity set).

In a fully observable economy, investors set market clearing rates of interest to duplicate the changes in the investment opportunity set. By contrast, in a partially observable economy, they do not have this privilege, and market clearing rates reflect the ability of investors to estimate these unobservable changes. Therefore, the mixture of a highly volatile investment opportunity set with low precision of this estimation process might result in low volatility of equilibrium rates of interest. The converse is also true: low volatility of the investment opportunity set together with high estimation error may result in highly volatile equilibrium rates of interest.

Section V introduces markets for contingent claims and focuses on default-free bonds. It demonstrates that by contrast to a fully observable economy where multiperiod bonds on the spot rate are needed to complete the market and ensure efficiency, complex trading strategies in realized outputs provide the multiperiod hedge supplied by multiperiod bonds. In addition, bond prices reflect consumers' ability to hedge the unobservable changes in the stochastic investment opportunity set through the life of the bonds. Section VI concludes the paper and reviews topics for further research.

I. The Model: A Production Exchange Economy

A. Assumptions and Notation

Dynamic economies with partial observability have not been much explored in the finance literature. Williams [40] examined a "Merton [31]-type economy" which assumed individuals did not know constant parameters of the distributions of exogenously determined asset prices. Klein and Bawa [24, 25] and Merton [33] investigated consequences of unknown means generating assets' returns. Feldman [18], Detemple [11], and Gennotte [22] investigated optimal portfolio policies, the term structure of interest rates, and equilibrium prices in a partially observable economy. The focus in this paper, however, is on equilibrium interest rates[1] and the role of default-free bonds.

The following assumptions characterize the economy. Assumptions (A0), (A1), (A4), and (A5) are in the spirit of Cox, Ingersoll, and Ross [8]; assumptions (A2) and (A6) are modifications of assumptions there, and assumption (A3) is new[2].

(A0). Assume a complete probability space (Ω, F, P) with a nondecreasing right continuous family of sub-σ-algebras (F_t), $0 \le t \le T$. Let $W_t = (W_t, F_t)$ be an $(n + 1)$ vector of independent Wiener processes, $W_0 = 0$.

(A1). There is a single numeraire good which may be allocated to consumption or investment.

(A2). There are n observable realized outputs for the good, ξ_t, with constant stochastic returns to scale described by a system of stochastic differential equations:

$$d\xi_t = I(\xi_t)(A_0 + A_1\theta_t)dt + I(\xi_t)B_1 dW_t,$$

with initial condition ξ_0. $I(\xi_t)$ is an $(n \times n)$ diagonal matrix with ξ_{ti}'s as elements,

[1] For other theoretical models examining equilibrium interest rates in dynamic economies, see Vasicek [39], Cox, Ingersoll, and Ross [8, 9] Dothan [12], Dothan and Williams [13], Richard [34], Brennan and Schwartz [4–6], Richard and Sundaresan [35], Cox, Ingersoll, and Ross [7], Ulman and Wood [38], Schaefer and Schwartz [36], Breeden [3], Dunn and Singleton [17], and Sundaresan [37].

[2] Some additional works that contributed to the analysis of equilibrium in a partially observable economy are Duffie [14], Duffie and Huang [15, 16], Kunita and Watanabe [28], and Merton [31, 32].

A_0 and A_1 are $(n \times 1)$ vectors of constants, B_1 is an $[n \times (n + 1)]$ matrix of constants, and $B_1 B_1'$ is positive definite. The values of A_0, A_1, and B_1 are known.

(A3). θ_t is an unobservable state variable (productivity factor) that evolves as the stochastic differential equation:

$$d\theta_t = (a_0 + a_1\theta_t)dt + b_1' dW_t,$$

with initial condition θ_0. The parameters a_0, a_1 are scalar constants, b_1 is an $[(n + 1) \times 1]$ vector of constants, and $b_1' b_1$ is positive. The values of a_0, a_1, and b_1 are known. The choice of a single-state variable is for expositional convenience; all results hold for any finite number of unobservable state variables.

(A4). There is free entry to all production processes. Individuals can invest in the n realized outputs through firms or by creating their own firms. Markets are competitive. Investment and trading at equilibrium prices take place continuously in time with no transaction costs.

(A5). There is a market for instantaneous borrowing and lending at an interest rate, r. The market clearing rate, r, is endogenously determined as part of the competitive equilibrium and as a function of the underlying state variables.

(A6). There are a fixed number of individuals, identical in their endowments and preferences. The model's assumptions are common knowledge to them. Each individual seeks to maximize expected lifetime utility of consumption conditional upon individual wealth and the history of the economy:

$$E\left\{\int_t^T h[c(\tau), \tau]d\tau \mid \xi_s, 0 \leq s \leq t; w_t = w_0\right\}.$$

Individual wealth at time t is w_t, henceforth denoted also as w, $c(t)$ is the consumption rate at time t, and h is a concave, twice-differentiable Von-Neumann-Morgenstern utility function.

Additional definitions and notation:

u is an $(n \times 1)$ vector of fractions of wealth allocated to the production processes

c and u are admissible feedback controls, measurable given the observations, $c \geq 0$, $u \geq 0$

$\mathbf{1}$ is an $(n \times 1)$ vector of ones

m_t is the conditional mean of θ_t given all observations, ξ_s, $0 \leq s \leq t$, i.e.,

$$m_t = E(\theta_t \mid \xi_s, 0 \leq s \leq t)$$

γ_t is the second moment of the distribution of θ_t given ξ_s, $0 \leq s \leq t$, i.e.,

$$\gamma_t = E[(\theta_t - m_t)^2 \mid \xi_s, 0 \leq s \leq t].$$

Later in the paper, the path of γ_t will be specified as the path of the filtering error or the estimation error. The path of its reciprocal will be called the path of the precision of the estimate.

$J(w, m_t, t)$	will denote the indirect utility function, i.e., the optimal value function of the utility of a maximization problem
$B \equiv B_1 B_1'$	is the instantaneous variance/covariance matrix of the rates of return on investment in the realized outputs, ξ_t
$b \equiv b_1' b_1$	is the instantaneous variance of θ_t
$D \equiv B_1 b_1$	is the instantaneous covariance between the rates of return on investment in ξ_t and the realizations of the unobservable factor θ_t
$\sigma_m'(\gamma_t) \equiv (D + A_1\gamma_t)' B^{-1/2}$	is the instantaneous standard deviation of m_t as will be demonstrated in Proposition 1
$G \equiv B^{1/2}\, \sigma_m(\gamma_t)$	will denote the instantaneous covariance between the rates of return on investment in ξ_t and the realizations of the estimate, m_t, of the unobservable factor.

Partial derivatives are denoted by subscripts; e.g., f_{rr} is the second partial derivative of $f(r, t)$ with respect to r.

B. Formulation of the Problem

Individuals allocate their wealth among the n realized outputs, borrowing or lending, and consumption. Their problem is:

$$\mathrm{Max}_{c,u}\ E\left\{\int_t^T n[c(\tau), \tau]d\tau \mid \xi_s,\ 0 \le s \le t;\ w_t = w_0\right\},$$

where

$$d\theta_t = (a_0 + a_1\theta_t)dt + b_1' dW_t,$$

$$I^{-1}(\xi_t)d\xi_t = (A_0 + A_1\theta_t)dt + B_1 dW_t,$$

$$dw = w[u' I^{-1}(\xi_t)d\xi_t + (1 - u'\mathbf{1})r] - c\,dt,$$

$$c \ge 0,\ u \ge 0,\ w_0 > 0,\ w_T \ge 0.$$

This formulation of the problem is not Markovian, and, therefore, does not facilitate tracking a solution.

In order to overcome this obstacle, the following section identifies the existence of an endogenous economy which is equivalent to the exogenous economy presented thus far. The equivalence rests on the fact that consumers possess the same information, and they maximize expected lifetime utility of consumption if and only if they do so in the exogenous economy. There are two main differences between these economies. First, in the endogenous economy, the unobservable factor is replaced by its observable conditional mean which is updated by a new Wiener process and second, this conditional mean is continuously updated as a function of the dynamic conditional variance. Hence, the consumers' problem becomes Markovian. The ability to identify the endogenous economy with an identical informational structure facilitates tracking the solution to this problem.

II. The Model's Solution: Nonlinear Filtering and Stochastic Control

A. The Innovation Process and the Nonlinear Filter

Investors draw inferences about the unobservable factor by observing realized outputs. They form a posterior distribution and continuously update it. Preferably, the formulae defining their estimates are of a recurrent nature; i.e., the estimate of each moment is built from the previous estimate. The following is a key proposition that facilitates solving the problem by suggesting the existence of a recursive estimate, identifying it, and demonstrating that the endogenously determined innovation process that updates this estimate is itself a Wiener process. Additionally, the proposition specifies the posterior distribution by a system of equations known as the nonlinear filter for which the estimate is a sufficient statistic.

PROPOSITION 1. *If the conditional distribution of θ_0 given ξ_0 is Gaussian, $N(m_0, \gamma_0)$, then,*

(a) there exists a vector of independent Wiener processes, $\bar{W}_t$, the innovation process, that generates the economy. That is, the σ-algebras, $F_t^{\xi_0, \bar{W}}$ and F_t^{ξ} are equivalent.

(b) $\bar{W}_t$ innovates the observable conditional mean, m_t, of the unobservable factor θ_t, and $\bar{W}_t$, m_t, and γ_t are the unique, continuous, F_t^{ξ} measurable solutions of the system of equations:

$$dm_t = (a_0 + a_1 m_t)dt + (D + A_1\gamma_t)'B^{-1/2}d\bar{W}_t,$$

$$d\gamma_t = [b + 2a_1\gamma_t - (D + A_1\gamma_t)'B^{-1}(D + A_1\gamma_t)]dt,$$

$$d\bar{W}_t = B^{-1/2}[I^{-1}(\xi_t)d\xi_t - (A_0 + A_1 m_t)dt],$$

where

$$m_0 = E[\theta_0 \mid \xi_0], \quad \gamma_0 = E[(\theta_0 - m_0)^2 \mid \xi_0].$$

Theorem 12.5 in Liptser and Shiryayev [30] contains the proof of Section (a), and Theorem 12.7 in the same work contains the proof of Section (b). All of the technical requirements for the theorems are nonbinding and omitted for brevity. These theorems hold for structures more general than the one used here.

The innovation process describes the path of the deviations of realizations from their expected values. The key feature necessary to the representation of the posterior distribution as a compact Markovian system is the identification of the innovation process, $\bar{W}_t$, as a standard Brownian motion. Combined with the satisfaction of some technical restrictions, a Gaussian distribution of θ_t, conditional on the observations of ξ_t, is sufficient to guarantee the desirable properties of the innovation process.

Since the moments of the conditional distribution of θ_t, given the observations

ξ_t, depend on the observations only through m_t, the latter is a sufficient statistic for the conditional distribution of θ_t given ξ_s, $0 \leq s \leq t$. One can identify the filtering error as

$$\mathrm{Var}(\theta_t - m_t) = E[(\theta_t - m_t)^2] = E\{E[(\theta_t - m_t)^2 \mid \xi_s, 0 \leq s \leq t]\} = E(\gamma_t).$$

However, here, since the parameters b, a_1, D, A_1, and B are not functions of ξ_t, $E\gamma_t = \gamma_t$, and γ_t, the second moment of the conditional distribution—the filtering error—is nonstochastic. In this case, the higher moments of the conditional distribution are determined by the first two. Hence, the process (θ_t, ξ_t) is referred to as a conditionally Gaussian process. The posterior estimate, sufficient for all available information about the unobservable factor, m_t, provides individuals with the best way to estimate the realizations of the unobservable factor, θ_t.

At a time t, where $\gamma_t = 0$, individuals know the particular value of θ_t. When $\gamma_t > 0$, they possess a probability distribution over the realization of θ_t. This distribution is normal with mean m_t and variance γ_t. Therefore, the higher the value of γ_t, the more diffuse are the beliefs of individuals about the realization of θ_t, and the lower the value of γ_t, the higher the precision of individuals' estimates of θ_t.

The path of γ_t, therefore, describes the precision path of individuals' estimates of θ_t. γ_t evolves deterministically; hence, individuals know its future values in advance. m_t is stochastic, so they do not know its future values. In other words, individuals know the precisions of the unknown future estimates in advance. Section III will delineate and interpret the possible paths of γ_t.

The following section uses Proposition 1 to restate the individuals' problem as a Markovian one.

B. Optimality Conditions

Using the results of Proposition 1 to express the changes in ξ_t in terms of the innovation process, $\bar{W}_t$, and the conditional mean, m_t, one can write the following proposition.

PROPOSITION 2. *The consumers' problem can be restated as:*

$$\mathrm{Max}_{c,u}\, E\left\{\int_t^T h[c(\tau), \tau]d\tau \mid w_t = w_0\right\},$$

where

$$\begin{aligned} dm_t &= (a_0 + a_1 m_t)dt + (D + A_1\gamma_t)'B^{-1/2}d\bar{W}_t, \\ d\gamma_t &= [b + 2a_1\gamma_t - (D + A_1\gamma_t)'B^{-1}(D + A_1\gamma_t)]dt, \\ I^{-1}(\xi_t)d\xi_t &= (A_0 + A_1 m_t)dt + B^{1/2}d\bar{W}_t, \\ dw &= w[u'I^{-1}(\xi_t)d\xi_t + (1 - u'\mathbf{1})r] - cdt, \\ & c \geq 0, \quad u \geq 0, \quad w_0 > 0, \quad w_T \geq 0. \end{aligned}$$

This formulation indicates that it is optimal for individuals to proceed in their maximization problem in two stages: first, to estimate θ_t, then to use the filter equations in their maximization problem. This result resembles the Linear Quadratic Gaussian (LQG) separation result in the stochastic control literature.[3] The structure here is more general than the LQG since it is not linear (θ_t given ξ_t is linear and ξ_t given θ_t is locally linear), and it is not Gaussian (it is conditionally Gaussian), but it is a special case of the LQG since the criterion is not a function of the unobservable state variable. This separation-type result holds for any finite number of unobservable state variables and for coefficients a_0, a_1, b_1, A_0, A_1, and B_1, which are functions of ξ_t and t. However, realized outputs must maintain the linear structure in θ_t to guarantee the Gaussian distribution of θ_t given ξ_t.

Procedures developed by Merton [31], Breeden [2], and Cox, Ingersoll, and Ross [8, 9] identify the necessary and sufficient conditions which determine equilibrium interest rates and optimal portfolio rules as,

$$r = u'(A_0 + A_1 m_t) + \frac{wJ_{ww}}{J_w} u'Bu + \frac{J_{wm}}{wJ_w} u'G,$$

or,

$$r = (\mathbf{1}'B^{-1}\mathbf{1})^{-1}[\mathbf{1}'B^{-1}(A_0 + A_1 m_t) + \frac{wJ_{ww}}{J_w} + \frac{J_{wm}}{J_w} \mathbf{1}'B^{-1}G],$$

and,

$$u = \frac{-J_w}{wJ_{ww}} B^{-1}[(A_0 + A_1 m_t) - r\mathbf{1}] + \frac{-J_{wm}}{wJ_{ww}} B^{-1}G.$$

For a logarithmic utility function,

$$h[c(t), t] = e^{-\zeta t}\log c(t),$$

the necessary and sufficient conditions for the solution of the problem become:

$$r\mathbf{1} = A_0 + A_1 m_t - Bu,$$

$$r = u'(A_0 + A_1 m_t) - u'Bu,$$

$$u'B^{1/2}\beta = \alpha - r.$$

Henceforth, the paper refers to those conditions which evolve from logarithmic preferences.

III. The Equilibrium Spot Rate of Interest and the Best Estimate of the Unobservable Productivity Factor

PROPOSITION 3. *The equilibrium, market clearing, instantaneous, spot rate of interest, r, is a linear function of m_t, the conditional mean of the unobservable*

[3] One of the first references for the separation between filtering and control is Wonham [41]; a review can be found in Davis [10], and a more general treatment in Fleming and Rishel [21, Chapter VI.]

factor. In particular,

$$r = \frac{\mathbf{1}'B^{-1}A_0 - 1}{\mathbf{1}'B^{-1}\mathbf{1}} + \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}} m_t.$$

To attain this expression, one could left multiply the first equation of the necessary and sufficient conditions by $\mathbf{1}B^{-1}$, and then divide the equation by $\mathbf{1}'B'\mathbf{1}$ (B is a nonsingular matrix).

Note that the coefficients of the linear equation for r are normalized, weighted, average returns on investment in the realized outputs. Applying Ito's formula to the stochastic differential equation that describes the evolution of m_t from Proposition 1 and using the linear relation of Proposition 3, one attains the following corollary.

COROLLARY. *The instantaneous spot rate of interest, r, evolves like the stochastic differential equation:*

$$dr = (\bar{a}_0 + \bar{a}_1 r)dt + \sigma_r'(\gamma_t)d\bar{W}_t,$$

with initial conditions at $t = 0$:

$$r = \frac{\mathbf{1}'B^{-1}A_0 - \mathbf{1}}{\mathbf{1}'B^{-1}\mathbf{1}} + \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}} m_0,$$

where

$$\bar{a}_0 \equiv \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}} a_0 + \frac{1 - \mathbf{1}'B^{-1}A_0}{\mathbf{1}'B^{-1}\mathbf{1}} a_1, \quad \bar{a}_1 \equiv a_1,$$

and,

$$\sigma_r(\gamma_t) \equiv \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}} \sigma_m(\gamma_t) = \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}B^{-1}\mathbf{1}} B^{-1/2}(D + A_1\gamma_t).$$

Since r is a one-to-one sure function of m_t, and since m_t is a sufficient statistic for all information available at time t about θ_t, r represents the best estimate of realizations of this factor.

The paper proceeds by specifying two of the determinants of the prices of multiperiod claims: the paths of the estimation error and the paths of the instantaneous variance of the spot rate. Bond prices are affected by the path of the future volatility of the spot rate which, in turn, is a function of the path of the precision of future estimates. This precision—which is proportional to the reciprocal of the filtering error—evolves deterministically through time.

IV. Equilibrium in a Partially Observable Economy

Two different sources give rise to the covariance between the unobservable productivity factor and realized outputs. The first, the source of the instantaneous covariance, characterizes the impact of the underlying Wiener process which generates the uncertainty in the economy. Changes in the components of this Wiener process affect both the factor and realized outputs. If, on average, a change in the values of the Wiener process that causes the value of the factor to increase causes realized outputs to increase (decrease), then the instantaneous covariance, D, is positive (negative). In addition to the instantaneous covariance, or the "nonsystematic source" of covariance, that relates to the "diffusion component" of the observations, the unobservable factor's impact on the instantaneous expected returns on the realized outputs gives rise to the "drift source" of covariance or equivalently, the "systematic source" of covariance. If higher realizations of the factor result in higher realized outputs, the "systematic source" generates positive covariance, and the factor is a "good factor." The instantaneous covariance and the "drift source" of covariance combine to build the covariance between realized outputs and the unobservable productivity factor.

The instantaneous variance of the spot rate can be explained as follows. The stochastic changes of the spot rate are due to changes in realizations of the productivity factor which, in turn, change the investment opportunity set. In a fully observable economy, the variance of the interest rate is a sure linear function of the variance of the factor. This would have been the result here if θ_t were observable, and this is the result in Cox, Ingersoll, and Ross [8, 9]. By contrast, in a partially observable economy, where the factor is unobservable, investors do not have the privilege to set the equilibrium rate of interest to duplicate the changes in the investment opportunity set. Realized outputs are used to make inferences about the factor's realizations. Now, the stochastic changes of the spot rate are due to changes in the unobservable productivity factor as seen through the changes in realized outputs. Moreover, the covariance between realized outputs and the factor takes over the role that the factor's variance played in a fully observable economy.

During a transient stage, the filtering error moves from its initial condition to a steady state. Then, depending on its steady-state value and on the relative signs and sizes of the two sources of covariance, the instantaneous variance of the estimated investment opportunities can be lower than, equal to, or higher than the instantaneous variance of the true investment opportunities. In a special case, when the two sources of covariance offset each other, the estimated investment opportunities will not be updated stochastically although investors know that the true investment opportunities continue to fluctuate randomly. These relationships lead to the following insight. While low (high) interest rate volatility in a fully observable economy implies low (high) volatility of the investment opportunity set and, therefore, "good news" ("bad news"), in a partially observable economy, low (high) volatility of the spot rate is not necessarily "good news" ("bad news"). Low volatility of the interest rate might now imply low learning ability about the changes in the investment opportunity set (i.e., low precision of the estimates of θ_t) rather than low volatility of it, and high volatility of the spot rate might imply high estimation error of the changes in the investment opportunity set rather than high volatility of it. In the first case, the two sources of covariance have opposite signs and in the second, they have the same sign.

V. The Introduction of Default-free Bonds and their Role as the Best Hedge against the Unobservable Changes in the Investment Opportunity Set

The need for a multiperiod hedge against the unobservable changes in the investment opportunity set triggers the development of new trading strategies. This section identifies these hedging strategies as multiperiod claims to the amount of the good, contingent on the best estimate of the unobservable factor or on the realized spot rate of interest. Multiperiod default-free bonds emerge as the optimal multiperiod hedge against the unobservable changes in the investment opportunity set for two reasons. First, they are claims on the spot rate which is a sufficient statistic for available information about the unobservable factor. Second, their values reflect the impact of future changes in the investors' hedging ability, changes that are not reflected in the spot rate and in realized output.

If there are n unobservable state variables, the equilibrium rate of interest is a function of the n conditional means which are, respectively, sufficient statistics for the n posterior distributions. Except for changes in dimensions of matrices, this function is identical to that of a single unobservable state variable. It is linear, and the sufficient statistics are weighted by the effects of the state variables they represent on investment opportunities. In fact, the coefficients of the linear function are normalized average expected returns. Consequently, multiperiod bonds become a function of these sufficient statistics, and consumers' hedging ability is a function (again of identical structure) of the estimation errors.

In a fully observable economy, the assumption of logarithmic preferences is sufficient for no hedging of the changes in the investment opportunity set. This result is not maintained in a partially observable economy. Since demands generated in a certain partially observable economy are (quantitatively and qualitatively) different from those generated in a similar economy when all state variables are observable, hedging demands are indicated. Second, the equivalent Markovian economy can be used to identify an exogenous fully observable economy where portfolio rules and prices of contingent claims are quantitatively identical. In this latter observable economy, there are no hedging demands, but in the given unobservable economy, the qualitatively different interpretation of the quantitatively identical rules implies that there are hedging demands (see Feldman [20]).

The procedure of introducing markets for multiperiod claims and evaluating them is similar to that of Cox, Ingersoll, and Ross [8] and will not be repeated here. The resulting general valuation equation for a claim, f, contingent on the interest rate is[4]:

$$\frac{1}{2}\,\sigma_r^2(\gamma_t)f_{rr} + [\bar{a}_0 + \bar{a}_1 r - u'B^{1/2}\sigma_r(\gamma_t)]f_r - rf + f_t + \delta(r,\,t) = 0,$$

for any boundary conditions γ_0 and $f(r,\,\bar{t}) = \bar{f}$. The necessary and sufficient

[4] See Kreps [26], Cox, Ingersoll, and Ross [8], Kreps [27], Bick [1], and Huang [23].

conditions define u as:

$$u = B^{-1}A_0 + B^{-1}A_1 m_t - B^{-1}\mathbf{1}\, r,$$

and δ is the payout received from the claim. The general valuation equation with payout function $\delta = 0$, terminal condition $f(r, \bar{t}) = 1$, and initial condition γ_0, prices a default-free discount bond.

The valuation equation demonstrates that, in addition to the variance of the rate of interest, the market risk premium is a function of the path of the estimation error. Hence, prices of multiperiod default-free bonds are a function of the path of the estimation error through the life of the bond.

Multiperiod claim prices reflect the effect of future changes in the estimation error and the resulting changes in investors' hedging ability. In the absence of multiperiod claims, investors can compute the path of hedging ability using: (1) realized outputs; (2) the spot rate; (3) all parameters of all probability distributions of economic variables; and (4) the solution to their stochastic optimal control problem of a partially observable system.

After computing the path of the estimation error, investors can use trading strategies in the realized outputs to achieve a multiperiod hedge against changes in the investment opportunity set. These trading strategies are equivalent to holding multiperiod claims on the spot rate. In other words, in a frictionless partially observable economy where consumers have perfect information about all parameters of all probability distributions of economic variables, the market is complete in relation to multiperiod claims on the interest rate.

When considering market completeness, where the estimation error of a certain unobservable state variable converges to a zero steady state, it might seem that an additional state variable is added to the economy and, hence, markets become incomplete. However, examination of the necessary and sufficient conditions for a zero steady-state filtering error suggests that the "revealed" state variable must be perfectly correlated with the observations; hence, the "revelation" of the state variable does not change the martingale multiplicity of the economy.

VI. Conclusions

In the context of a production and exchange economy with a single unobservable source of nondiversifiable risk, this paper presents a separation-type result necessary for solving the consumers' problem. As long as the economy has a conditionally Gaussian information structure, consumers first estimate the realizations of the unobservable productivity factor and then use the estimates and their precisions in their optimization problem. The paper describes the emergence of bonds as the optimal hedge for the unobservable factor and analyzes the resulting equilibrium spot rate of interest. It demonstrates that the paths of the spot rate's instantaneous variance evolve as a function of the paths of consumers' ability to hedge the unobservable changes in the investment opportunity set.

The paper further distinguishes a partially observable economy from a fully observable economy. First, in a partially observable economy, high (low) volatility of the interest rate is not "bad" ("good") news. Second, in contrast to a fully observable economy where multiperiod claims on the spot rate are needed to complete the market, a frictionless unobservable economy where consumers have perfect information of all economic parameters is complete in relation to multiperiod bonds.

These results can be further extended. First, a new term structure theory in a partially observable economy can be developed (see Feldman [19]). Second, a new interpretation of myopic decisions and hedging demands is necessary (see Feldman [20]). Third, while the argument for the emergence of bonds as a hedge of the unobservable factor is made in a heterogeneous economy, the equilibrium prices of bonds are obtained only in a homogeneous economy without trade. Extensions can include heterogeneous preferences and trade, heterogeneous beliefs, or both. Finally, equilibrium bond prices are derived in this paper only when the estimate of the unobservable factor is stochastic and the estimation error is a sure function of time. Intuition suggests that investigating an economy where the estimation error is a stochastic function of time will generate additional insights into the formation of equilibrium demands and prices in a partially observable economy.

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