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Source: *The Journal of Finance*, Vol. 41, No. 2 (Jun., 1986), pp. 347-368
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328440
Accessed: 26/11/2009 08:28

THE JOURNAL OF FINANCE • VOL. XLI, NO. 2 • JUNE 1986

On the Number of Factors in the Arbitrage Pricing Model

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ABSTRACT

Recent theory has demonstrated that the Arbitrage Pricing Model with K factors critically depends on whether K eigenvalues dominate the covariance matrix of returns as the number of securities grows large. The purpose of this paper is to test whether sample covariance matrices can be characterized as having K large eigenvalues. Using all available data on the 1983 CRSP tapes, we compute sample covariance matrices of returns in sequentially larger portfolios of securities. Analyzing their eigenvalues, we find evidence that one eigenvalue dominates the covariance matrix indicating that a one-factor model may describe security pricing. We also find that, for values of K larger than one, there is no obvious way to choose the number of factors. Nevertheless, we find that while only the first eigenvalue dominates the matrix, the first five eigenvalues are growing more distinct.

ARBITRAGE PRICING THEORY (APT) as originally developed by Ross [22, 23] is built on the twin assumptions of no arbitrage opportunities in the capital market and a linear relationship between actual returns and K common factors. The principal hypothesis of the model is that expected returns should be linearly related to the weights of the common factors in the assumed linear process. In every test of the model, the focus has been to use factor analysis to extract K factors from sample covariance matrices and then to test the hypothesis by regressing returns or average returns against the factor loadings of the common factors. Most previous tests have either assumed that the number of factors is equal to some predetermined number or have estimated K simultaneously with estimation of the factor loadings. Thus, existing tests of the APT are joint tests of the APT pricing result and an assumed or estimated number of factors. The question of how many factors are important has not been addressed separately from the pricing tests because the APT pricing result is the significant prediction of a theory which has little to say about the value of K. Nevertheless, cross-section linear models have been a central part of financial economics, and it is clearly important to know at least the approximate number of factors which will be significant in such regressions, particularly given the computational difficulties of large-scale factor analysis. Furthermore, Dhrymes, Friend, and Gultekin [6] report that the number of factors needed for standard chi-squared tests increases

* University at Buffalo, SUNY. I am in debt to Donald Bosshardt, Gary Chamberlain, Thomas Ho, Richard Roll, Lemma Senbet, Mark Zmijewski, and the participants in finance workshops at the Universities of Michigan, Wisconsin, and Buffalo for their extensive comments. Nevertheless, I am solely responsible for what remains. This paper would not have been started without the unselfish cooperation of Jay Leavitt, Director of Academic Computing, University at Buffalo, SUNY, who designed the eigenvalue algorithm which I used. I am also grateful for partial financial support and lavish computer funds from the School of Management, and for the expert programming assistance provided by Ravi Shukla.

as more securities are analyzed, and it is uncertain if this result will still hold with a larger number of securities than they used.

Thus, the purpose of this study is to focus specifically on the number of factors. We will employ the recent extension of the APT by Chamberlain and Rothschild (CR) [4] to provide a basis for our investigation. CR prove that a necessary and sufficient condition for returns to be linearly related to K factors is that K eigenvalues of the covariance matrix of returns grow large as the number of securities increases. This suggests that a natural way to measure the number of factors is to compute the eigenvalues of sequentially larger covariance matrices and determine the number that are increasing. If K is not constant, then the data will not support the *assumption* of a linear relationship between factors and returns. Logically, the APT cannot be rejected by this test because one need only argue that the assumption fits close enough for pricing. However, Ross originally motivated the APT by arguing that the linearity assumption is a plausible model of returns, and it may be worthwhile examining other models if linearity is undermined.[1] On the other hand, if K is constant, we have an estimate of the number of factors which is not based on cross-sectional regressions and consequently, is not subject to their computational and statistical difficulties.

The tests described in this paper use eigenvalues calculated from sequentially larger covariance matrices. The sample covariance matrices are computed from weekly returns of every firm listed for the 1,069 weeks covered by the 1983 CRSP tapes. The largest covariance matrix of 865 securities contains more than three times the number of securities used in any previous study. The expense of the calculations and non-normality of daily data prohibited us from using daily data, and we argue below that the minimal degrees of freedom is a problem for large covariance matrices of monthly returns. We provide a series of straightforward statistical tests using the eigenvalues and conclude that the damaging result found by Dhrymes, Friend, and Gultekin does hold for much larger portfolios if more than one factor is required to explain the variation in returns. However, we do find that at least one eigenvalue is increasing without bound indicating that the appropriate model may be a one-factor model. Finally, we find evidence that the second through the fifth eigenvalues are growing more distinct from each other, while still declining in importance, and we suggest that this may explain why pricing tests often find statistical evidence for more than one factor. Section I briefly discusses the APT and outlines our general approach. Section II describes the data and the specific methodology while Section III provides the empirical results. The paper concludes with a summary and some observations in Section IV.

I. The Arbitrage Pricing Theory

The APT begins with the assumption that the $n \times 1$ vector of one period asset returns, r, is generated by a linear stochastic process with K factors:

$$r = e + Bz + u \tag{1}$$

where z is the $K \times 1$ vector of realizations of K common factors, B is the $n \times k$

[1] Marsh [16] argues that if returns are linearly generated over an infinitestimal time unit, there will be nonlinearities in the generating model for returns measured over discrete intervals and that these nonlinearities may result in extra factors.

matrix of factor weights or loadings, and u is a $n \times 1$ vector of asset-specific risks. It is assumed that z and u have zero expected values so that e is the $n \times 1$ vector of mean returns. It is usually assumed that $E(z_i z_j) = 0$, $E(u_i z_i) = 0$ so that if V is the variance-covariance matrix of returns, it may be written as:

$$V = BB' + W \tag{2}$$

where $E(uu') = W$.

The theory addresses the problem of how expected returns are formed in a market with no arbitrage opportunities and predicts that any asset's expected return is linearly related to the factor loading or

$$e = x_0 + Bx \tag{3}$$

where x_0 is a $n \times 1$ vector of constants representing the risk-free or zero-loading rate, and x is $K \times 1$ vector of factor risk premia. Derivations of (3) from (1) are based on the intuition that firm specific risk represents a diversifiable risk which should have a zero price in a market with no arbitrage opportunities. Recent theoretical papers (Chamberlain and Rothschild [4], Ingersoll [11], and Stambaugh [24]) have significantly weakened the assumptions necessary to derive (3) and essentially show that the quadratic form, $(e - x_0 - Bx)'W^{-1}(e - x_0 - Bx)$ will be bounded. Ingersoll [11] shows that W need not be diagonal, or equivalentally, not all the factors need to be extracted. The pricing bound will hold providing that enough factors are extracted so that each factor is "pervasive," in the sense of affecting a large number of securities and "distinct" relative to other factors. Stambaugh [24] shows that this is equivalent to having K eigenvalues of V increase without bound. CR weaken the assumption of (1) to an "approximate factor structure" which is that (2) holds for all N. They show that a market has an approximate factor structure if and only if K eigenvalues of V increase without bound and all other roots are bounded. Thus, CR prove that assuming a linear generating process is equivalent to assuming that K eigenvalues explode.

CR establish a relationship between the pricing result and the eigenvalues of V by proving that the quadratic form $(e - x_0 - Bx)'W^{-1}(e - x_0 - Bx)$ is bounded if there is an approximate factor structure (and no arbitrage opportunities), and they show that the eigenvectors corresponding to the K exploding eigenvalues converge to factor loadings. Thus, for large enough N, B is simply an $n \times k$ matrix of eigenvectors of V. This pricing result provides a rigorous justification in terms of pricing for our focus on eigenvalues because, in principle, it is only necessary to determine the number of unbounded eigenvalues to estimate how many factors are priced. In empirical work, however, the value of K is usually the number of factors which are sufficient to accept the statistical hypothesis that W is a diagonal matrix, and this number could be larger than the number of exploding eigenvalues. Thus, we are estimating, in Ingersoll's terminology, the "minimal factor representation," that is, the smallest number of factors which can be used to price securities.

II. Methodology and Hypotheses

A. Data

We use weekly data from the 1983 Daily CRSP tapes to estimate the covariance matrices. There are 1,069 weeks of data on the tape and 865 firms listed for all

20 years having greater than 865 observations. The minimum number of observations is 906, and only nine firms have between 906 and 1,000 observations. We could have expanded the number of observations and firms by using daily data but recent evidence suggests that there are a variety of problems with using daily data for our purpose. First, Roll and Ross [20] found the results of their APT tests improved when every other observation was used, and they argued that it was due to skewness in daily data and the effects of nonsynchronous trading. Second, Perry [18] suggests that daily data is best modeled by an unstable variance process, and it is well known that daily returns are not well described by the normal distribution (Fama [8]). Thus, we felt that the gain in observations and firms was not worth the exponential increase in cost of the calculations. Conversely, we did not feel monthly data would provide us with enough degrees of freedom as there are only 696 months on the 1983 Monthly CRSP tape. The chi-squared statistics used below are only asymptotically chi-squared so that the larger the matrix is, the smaller would be the degrees of freedom. With weekly data we have a minimum of 264 degrees of freedom while with monthly data we could have as little as one. Since the purpose of this study is to use very large matrices, we settled on weekly data.

The theoretical papers mentioned above which derive asymptotic results each rely on sequences of nested matrices. Therefore, we constructed a nested sequence of sample covariance matrices by starting with a covariance matrix of 50 securities and adding rows and columns randomly but without altering the previous rows and columns. For example, $n = 100$ contains the original 50 securities plus 50 additional randomly selected securities. The complete nested sequence is $n = 50$, 100, 200, 300, 400, 500, 600, 700, 800, and 865. However, the results based on this sequence will not necessarily be an adequate test of the APT. The cited papers study the behavior of eigenvalues and pricing bounds by looking at the nested sequence of population covariance matrices. While our matrices are large, we certainly do not claim to have the population of assets, and it is questionable whether a nested sequence of sample covariance matrices necessarily represents a nested population sequence. Thus, we will also employ a "random" sequence in our tests where, in this sequence, $n = 50$ is not necessarily a subset of $n = 100$. Constructing a sequence in this manner allows us to examine the point raised by Dhrymes, Friend, and Gultekin of whether the number of significant factors is stable if a researcher factor analyzes larger and larger randomly selected portfolios. While we will not be able to determine the result of using maximum likelihood factor analysis on sequentially larger matrices, we can examine the behavior of the sample variance of the principal components which CR suggest converge to factor loadings when the number of securities is large. The random sequence begins with a sample covariance matrix of size 30, and we increase each sample by 10 percent so that $n = 30, 33, 36, \cdots, 722, 794$, and 865. There are a total of 36 randomly selected matrices.[2]

[2] Performing our tests for monthly, weekly, and daily data was judged to be prohibitively expensive. This is in spite of the efficiency of the algorithm which is designed to optimize the trade-off between I/0 and the core capacity of our hardware (at the time a CYBER 730). The basic computational approach is to tridiagonalize the matrix with the Householder method applied to partitions of the upper diagonal half. Once the matrix is reduced, the QR method is used to solve for the eigenvalues. We tested the algorithm extensively, and it is as accurate as any IMSL subroutine. In addition, we

B. Chi-Squared Tests

Let $L_j(n)$ be the j[th] largest eigenvalue of the population covariance matrix of n securities where $L_1(n) \geq L_2(n) \geq \cdots \geq L_n(n)$. Let $l_j(n)$ be the j[th] largest eigenvalue of the sample covariance matrix of size n. The hypotheses we wish to test is that for some K, $L_1(n)$ through $L_K(n)$ increase without bound when n increases. We still test this hypothesis by using $l_1(n), \cdots, l_n(n)$ as sample statistics for the population eigenvalues, and n will increase as specified by either the nested sequence or the random sequence.

Eigenvalues are roots of polynomials whose coefficients are computed by using the elements of the covariance matrix. The coefficients are complicated functions of these elements and, consequently, statistical inference is difficult but not impossible. Asymptotic results have been derived by Girshick [9] and Anderson [1] using the assumption that the observations are drawn from a multivariate normal population.[3] Based on these results, we will employ a chi-squared statistic which tests the hypothesis that eigenvalues L_{K+1} through L_{K+r} are equal. If $K + r = n$, the test is whether the distribution is "spherical" around the $n - k$ dimensions, implying that not one of the $n - k$ dimensions can explain more than any of the others. While the CR results do not require equality of the last $n - k$ roots, we expect that if k eigenvalues are exploding, $n - k$ will be negligible, and for large enough n, the statistical hypothesis of equality may be accepted.[4]

All previous researchers have used, in combination with other tests, a similar chi-squared statistic to determine the value of k by simply computing the statistic for larger and larger values of k until it accepts the null hypothesis.[5] We will follow this standard approach, and as detailed below, we will examine the behavior of the roots as n increases but it is worthwhile noting the statistical limitations

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used the IMSL EISPACK routine to compute eigenvalues for matrices of size 596 in the random sequence and 600 in the nested sequence. For each matrix in each sequence, the results were identical to our algorithm.

[3] Anderson's basic confidence intervals and chi-square tests are described in most multivariate textbooks such as Morrison [17, pp. 292–9]. The sampling theory of roots of correlation matrices is not as well developed, although Lawley [12] has established an asymptotic test for equality of roots. Hence, our focus in this paper is on covariance matrices. Furthermore, Anderson suggests that if all variables are measured on the same scale, analysis of covariance matrices will be equivalent to analysis of correlation matrices.

[4] Formally, the quantity χ^2 defined as

$$\chi^2 = -(T-1)\textstyle\sum_{j=k+1}^{n} \mathrm{Ln}\, l_j + (T-1)\,\mathrm{Ln}[r \textstyle\sum_{j=k+1}^{n} l_j]$$

has the chi-squared distribution with $\frac{1}{2}(r+1)r$ degrees of freedom for large T. See Mauchley [16] for a justification of the "hypothesis of sphericity."

[5] The factor analysis chi-squared statistic used in previous studies tests the same hypothesis in a factor analysis model rather than in our principal components setting. Formally, their statistic is:

$$\chi^2 = m\ \mathrm{Ln}\frac{|\hat{B}\hat{B}' + W|}{|S|}$$

where m is constant, $|\cdot|$ indicates a determinant, S is the sample covariance matrix, and $\hat{B}\hat{B}' + W$ is the maximum likelihood estimate of the covariance matrix. The difference between the statistic in footnote 4 and the one above is that factor analysis places less constraints on the decomposition of the covariance matrix than principal component analysis. Thus, we should find at least as many factors as those studies which use a factor analysis approach.

of our tests. First, in using a sequence of chi-squared statistics to estimate k, there is a nested hypothesis problem. If χ_1^2 is the chi-squared statistic for one factor or one-component model ($k = 1$), then its outcome will be known when χ_2^2 is computed for a two-component model. The density of χ_2^2 is conditional on the outcome of the first test and even if the unconditional density is chi-squared, the conditional density need not be.[6] Second, little is known about the effect of departures from normality on the sampling distribution of the chi-squared statistics or of the set, $\{l_1(n), \cdots, l_n(n)\}$, of estimates. While recent theoretical work demonstrates that, asymptotically, multivariate tests on means are not altered by violations of normality, tests on variances appear sensitive to nonnormality, particularly when the kurtosis of the sampling distribution is affected.[7] Thus, departures from normality may affect both multivariate analysis of covariance (e.g., principal components and factor analysis) and the tests with eigenvalues proposed below. Simulation could potentially determine the statistical power of our tests but we simply do not have the resources to simulate. However, there is no reason to expect that one statistic in a sequence as n increases will be more subject to such errors than another statistic, especially since our degrees of freedom are always large. Therefore, we will focus more on the behavior of the sequence of statistics than any single number because it is the behavior of the sequence that is the concern of the theory.[8]

C. Eigenvalue Tests

In addition to the tests discussed above, we will examine the behavior of the eigenvalues in each sequence by performing a series of regressions using the sample roots, their rank, and the number of securities. This is not meant to imply that the theoretical relationship between L_j and n is linear but that the simple regressions can be used to summarize differences between sequences. We will present graphs in those cases where the relationships are not linear. A direct test of whether L_j is unbounded for $j \leq k$ is to use both the random sequence and the nested sequence to regress l_j on n:

$$l_j(n) = a_{0j} + a_{1j}n + e_n$$

$$\text{for } n = 30, 33, \cdots, 794, 865 \text{ (random sequence)} \qquad (4)$$

$$n = 50, 100, \cdots, 800, 865 \text{ (nested sequence).}$$

[6] See Lawley and Maxwell [13, pp. 36–39] for a discussion of the problem. They claim that the value of K determined by the sequential process will exceed the true value with no more than probability α, where $1 - \alpha$ is the nominal significance of the test.

[7] See Huber [10] and Arnold [2]. Note that if multivariate tests on means are not sensitive to departures from normality, the pricing tests may be more statistically powerful than previous researchers have suggested.

[8] Anderson [1, pp. 125–27] shows that the sample roots will be independent if some transformation exists which diagonalizes S. This result is not dependent on normality. It is worth noting that some researchers, Luedke [14], e.g., have treated the sample roots as if they are from a population covariance matrix. While we disagree with this approach because realized returns form only one of many possible sample paths over time, we do have nearly all the data available and any large-scale factor analysis will use matrices very similar, if not identical, to our own.

We should find that, for some K, $a_{1j} > 0$ for $j \leq K$. Similarly if K eigenvalues are unbounded, they will eventually dominate the trace of the matrix. Let $t_j(n) = l_j/\sum_{i=1}^{n} l_j$ *be* l_j's percent of the trace of a matrix of size n and let $T_K(n) = \sum_{j=1}^{k} t_j(n)$ and $T_{n-k}(n) = 1 - T_K(n) = \sum_{i=K+1}^{n} t_i(n)$. Thus, the generating process can be tested by choosing K and regressing

$$T_K(n) = b_0 + b_1 n + U_n \quad \text{for } n = 30, 33, \cdots, 865 \quad n = 50, 100, \cdots, 865. \tag{5}$$

The trace of the matrix will grow with the number of eigenvalues, and if K eigenvalues are unbounded, they will eventually become a constant percent of the trace with the remaining eigenvalues becoming negligible.[9] Finally, consider the set of eigenvalues, $l_1(n) \geq l_2(n) \geq \cdots \geq l_n(n)$, as similar to a "time series" and perform the following regressions:

$$\begin{aligned} l_j(n) &= c_0 + c_1 j + V_j \\ t_j(n) &= d_0 + d_1 j + W_j \end{aligned} \tag{6}$$

where $j = 1, \cdots, n$. This "within portfolio" regression is intended to provide information about the functional relationship between eigenvalues and their rank, and how this relationship changes with n. For a given n, this test cannot reject the APT because the theory has no implications about how eigenvalues decrease with their rank. However, if $c_1(n)$ and $d_1(n)$ are the slope coefficients when the covariance matrix is of size n, then the arbitrage pricing matrix clearly predicts that, in absolute value, $c_1(n) > c_1(m)$ and $d_1(n) > d_2(m)$ for $n > m$. If some eigenvalues are growing larger and some smaller, the relationship should grow more divergent as n grows larger. Furthermore, as n increases, the APT

[9] Intuitively, the eigenvalue as a percent of trace is equivalent to the percent of variation that the factor explains. If k eigenvalues decline as a percent of trace (even though growing large), the k factors will explain nothing in the limit. Thus, either the k roots are a constant percent of trace or they grow to one. To clarify this, let $k = 1$, $W = I_n$ and $L_1(A) \geq L_2(A) \geq \cdots > L_n(A) > 0$ be the roots of any positive definite matrix, A. Then, by Rao [19, p. 191] we have:

$$L_1(V) \leq L_1(BB') + L_1(W) = (\textstyle\sum_{i=1}^{n} b_1^2 + n) + 1$$

because B is assumed to have two columns, one for factor loadings and a column of ones. For the remaining roots, note that the rank of BB' is one so that

$$L_i(V) \leq 1$$

but since Trace (V) = Trace (BB') + Trace $(I_n) = \sum_{i=1}^{n} b_1^2 + n + n = \sum_{i=1}^{n} L_i(V)$, we have

$$L_1(V) = \textstyle\sum_{i=1}^{n} b_1^2 + n + 1 \quad \text{and} \quad L_i(V) = 1 \quad \text{for} \quad i > 1.$$

Thus, a percent of trace

$$t_1 = \frac{L_1}{\mathrm{Tr}\ V} = \frac{\sum b_1^2 + n}{b_1^2 + 2n}, \qquad t_i = \frac{L_i}{\mathrm{Tr}\ V} = \frac{1}{\sum b_1^2 + 2n}.$$

It is easy to see that t_1 converges to a constant and that each t_i will become negligible with n. However, the sum of the $n - 1$ t_i will not necessarily become zero unless the first K roots go to one, and there is no guarantee that will happen in the general case.

predicts that, for some large n, c_1 and d_1 will be subject to a structural change at point K. The next section gives the details of the empirical findings.

III. Empirical Results

A. Chi-Squared Tests

Table I reports the results of computing the chi-squared statistic which tests the hypothesis that roots $K + 1$ to n are equal. This is equivalent to the hypothesis that K components "explain" the covariance matrix. The table reports the findings for the nested sequence with the results for the random sequence being similar. It is clear from the table that the statistics never accept the null. None of the statistics are less than four times their associated 5 percent critical value, and most of the statistics are on the order of five to ten times their critical value. (Note that all numbers in the table have been divided by 10^3). Thus, the hypothesis that K principal components dominate the matrix for some size n is overwhelmingly rejected by this test for values of K commonly estimated in the past or even used as "safely" representing the number of factors.[10]

In addition to convincingly rejecting the null, there is an obvious relationship between the size of the statistics and the portfolio size. This was evident in Dhrymes, Friend and Gultekin's Table 2 and is clearly evident here for covariance matrices over three times as large. There appears no tendency for the number of factors to stabilize as the number of securities grows large. Furthermore, there is no tendency for the chi-squared statistic to grow closer to its 95 percent critical value.[11] In fact, the critical value and the statistic grow further apart as n grows large for any value of K in either sequence. The random sequence produces a similar result. Thus, there appears to be little support for the hypothesis that the statistic will "accept" for some n, using previously suggested values of K.

It, however, is clear from a comparison of these results with similar chi-squared statistics from maximum likelihood factor analysis that factor analysis allows each factor to explain a much greater percentage of the common variation in the covariance matrix than does principal component analysis. Roll and Ross [20], Brown and Weinstein [3], and Dhrymes, Friend, and Gultekin report some chi-squared statistics which accept the null for low values of K and n (e.g., $K = 5$, $n = 30$), while we never accept the null for any reasonable K. As observed in footnote 5, the two chi-squared statistics are not strictly comparable because principal components analysis places more restrictions on the estimation procedure than does factor analysis. Thus, it is not surprising that more factors are needed to "significantly" factor the sample covariance matrix when components analysis is used rather than factor analysis. What is surprising is the degree of the difference. As is customary in factor analysis, we computed chi-squared statistics until we accepted the null, and except for the smaller portfolios, we

[10] Roll and Ross [20] estimate that no more than five factors are needed to explain security returns, while Chen [5] uses up to ten factors to form his portfolios.

[11] The ninety-fifth percentile of the chi-squared distribution with V degrees of freedom for large T given by $0.5[1.645 + (aV - 1)^{0.5}]^2$ and $V = 0.5(n + k + 1)(n - k + 2) - 0.5$, where n is the portfolio size, and k is the number of factors.

Table I

Chi-Squared Statistics for Test of Distinct Roots Using Sample Ocvariance Matrices from the Nested Sequence

H_0: $L_{K+1} = L_{K+2} = \cdots = L_n$

Value of n	Value of K: 1	2	3	4	5	6	7	8	9	10	11	12
50	11.95*	10.15	9.13	8.46	7.76	7.21	6.73	6.33	5.95	5.59	5.19	4.88
100	28.66	26.46	25.16	23.89	22.87	21.82	21.02	20.20	19.45	18.67	17.99	17.37
200	80.52	77.31	75.05	73.11	71.41	69.70	68.01	66.46	64.90	63.41	61.99	60.57
300	142.59	137.95	134.94	132.76	130.63	128.76	126.94	125.13	123.35	121.58	119.89	118.19
400	214.90	208.98	205.08	202.44	199.89	197.64	195.49	193.31	191.20	189.13	187.08	185.14
500	307.49	299.41	293.58	289.50	286.40	283.63	281.11	278.65	276.18	273.76	271.50	269.25
600	413.38	404.84	398.03	393.43	389.70	386.49	383.32	380.56	377.80	375.16	372.51	369.97
700	574.26	536.61	529.39	524.22	519.58	515.65	512.04	508.57	505.45	502.43	499.51	496.58
800	705.99	693.90	685.53	697.71	674.18	670.00	666.01	662.40	659.00	655.80	652.66	649.53
865	832.05	818.76	809.24	802.83	796.78	792.40	788.20	786.44	780.82	777.37	773.96	770.62

* All numbers have been divided by 10^3. There are no chi-squared statistics in the table *less* than four times the critical value using a normal approximation to the 95 percentile of the chi-squared distribution: $1/2[1.645 + (2V - 1)^{1/2}]^2$ where $V = 1/2(n - k + 1)(n - k + 2) - 1$.

always needed over half the n eigenvalues to accept the null, that is, K needed to be at least ½ n in most cases. This difference suggests that the estimates of idiosyncratic error terms, the "specificities" of the variables, play a critical role in factor analysis of security returns. As Lawley and Maxwell [13] show, the two methods differ only by a constant if all variables have the same specificity. Thus, the inability of a reasonable number of components to capture the common variation in the covariance matrix when, for some samples, a relatively small number of factors are adequate, suggests that there are large differences between the specific error terms of securities. This is because, if idiosyncratic risks were constant, the two techniques would be identical. It also suggests that at least some of these firm-specific error terms are large relative to the variation explained by the common factors.[12] It is, therefore, unlikely that any principal component model will provide better pricing results, given K, than a maximum likelihood factor analysis model. The results of Table 1 are very similar to the findings of Dhrymes, Friend, and Gultekin and the positive relationship between the number of factors and the number of securities may also occur with a factor analysis model. Chi-squared statistics, however, say nothing about whether the factors are priced, and while these results demonstrate that the number of statistical factors may be unbounded, the number of economic (i.e., priced) factors may be constant.[13] Thus, the table shows that there is no obvious way to choose K principal components using standard chi-squared tests. While this finding does not directly test the CR result, it does indicate that principal components analysis may be difficult to apply in practice.

B. Eigenvalue Size Tests

To recapitulate, if one assumes a linear-generating model with K constant factors, then this is mathematically equivalent to assuming that K eigenvalues of V explode. This result, proved by CR, can be used to estimate the number of priced factors. If K does not exist, the theory is not formally rejected because one can always argue that a K factor model may still be a good enough approximation to price securities. However, given that the original motivation for the model was that a linear-generating process is widely acceptable, the nonexistence of K would suggest that nonlinear models may more reasonably describe security price behavior. On the other hand, if K eigenvalues explode, then we have an estimate of the number of factors which is not dependent on cross-sectional regressions. Table II reports the results of regressing the eigenvalues against the number of

[12] Lawley and Maxwell [13] show that factor analysis is equivalent to applying principal component analysis to $V^* - I$ where V^* is the "standardized" covariance matrix $W^{-1/2}VW^{1/2}$. Principal components analysis is simply taking the eigenvectors associated with the K largest eigenvalues of V to form $V = GG'$ where G is the matrix of K eigenvalues. The two procedures will be identical when $V = V^*$, which occurs if W is a constant times I. The fact that, for small n, the chi-square statistics imply such differences between the two models suggests that there are large differences between the elements of W and that some of the elements of W are not negligible relative to the elements of V.

[13] Roll and Ross [21] make this point, and in a recent paper, DFG [7] report that priced factors also tend to increase as the number of securities rises from 80 to 90 but at a much slower rate than the statistical factors. Whether this increase in the number of factors continues or stops for larger covariance matrices remains a question for future research.

Table II

Regressions of Eigenvalues on Size of Portfolio*

Eigenvalue	Nested Sequence			Random Sequence		
Rank	Intercept	Slope	R-Square	Intercept	Slope	R-Square
1	0.448 (2.8)	0.048 (16.2)	0.9997	−0.034 (0.5)	0.049 (25.8)	0.9995
2	0.704 (21.3)	0.004 (6.3)	0.9982	0.567 (27.0)	0.004 (65.6)	0.9926
3	0.514 (11.7)	0.003 (35.7)	0.9942	0.554 (18.5)	0.003 (35.7)	0.9683
4	0.522 (11.1)	0.002 (22.5)	0.9867	0.576 (57.6)	0.002 (69.7)	0.9928
5	0.463 (10.8)	0.002 (24.5)	0.9870	0.552 (34.5)	0.002 (42.5)	0.9775
6	0.524 (10.3)	0.002 (20.9)	0.9689	0.548 (45.7)	0.001 (28.6)	0.9812
7	0.481 (8.6)	0.001 (9.5)	0.9609	0.512 (42.7)	0.001 (30.3)	0.9815
8	0.479 (7.7)	0.001 (8.5)	0.9426	0.491 (28.9)	0.001 (20.8)	0.9583
9	0.474 (7.4)	0.001 (8.3)	0.9340	0.464 (25.8)	0.001 (19.4)	0.9500
10	0.468 (7.2)	0.001 (8.1)	0.9254	0.436 (22.9)	0.001 (18.3)	0.9426
11	0.443 (7.3)	0.001 (8.6)	0.9325	0.405 (20.3)	0.001 (17.4)	0.9368
12	0.418 (6.6)	0.001 (8.3)	0.9287	0.379 (18.9)	0.001 (17.7)	0.9384
13	0.403 (6.3)	0.001 (8.2)	0.9232	0.352 (17.6)	0.001 (17.3)	0.9367
14	0.383 (6.3)	0.001 (8.6)	0.9308	0.319 (15.2)	0.001 (16.9)	0.9360
15	0.370 (6.2)	0.001 (8.8)	0.9325	0.297 (14.8)	0.001 (17.8)	0.9416
16	0.355 (6.0)	0.001 (8.9)	0.9332	0.280 (14.0)	0.001 (17.1)	0.9367
17	0.338 (5.8)	0.001 (9.2)	0.9357	0.257 (12.2)	0.001 (16.7)	0.9357
18	0.325 (5.6)	0.001 (9.6)	0.9406	0.240 (12.2)	0.001 (16.5)	0.9338
19	0.312 (5.7)	0.001 (9.6)	0.9405	0.228 (10.9)	0.001 (16.8)	0.9357
20	0.301 (5.5)	0.001 (9.6)	0.9395	0.215 (10.2)	0.001 (16.8)	0.9352

* All coefficients have been multiplied by 100. The numbers in parentheses are t-ratios. The independent variable for the nested sequence is 50, 100, 200, 300, 400, 500, 600, 700, 800, and 865 while the independent variable for the random sequence is 30, 33, $\cdots$, 794, and 865 (36 portfolios).

securities in the covariance matrix as described by Equation (4). The table shows the regressions for the largest twenty eigenvalues. The results certainly support the hypothesis that there is a positive relationship between eigenvalues and portfolio size but the regressions do not support any previously hypothesized value of K because the relationship is always strongly positive for the first twenty roots. If CR are correct, eigenvalues of rank less than K should decline with n or at least be non-increasing but this does not occur for K less then twenty and in fact, it does not occur for any rank. All eigenvalues, including the smallest, increase with the size of the portfolio, and the relationship is monotonic for any rank that is chosen. It does appear, however, that the first eigenvalue, the largest, has a stronger relationship with n than any of the others and for the random sequence, it has an insignificant intercept indicating there are few deviations from the regression line. This fact is confirmed by the graph (not shown) which appears almost perfectly linear. The graphs of the remaining eigenvalues are not nearly as linear as the largest but the relationship is clearly strong.

Table III shows the actual values of the first twelve eigenvalues along with their percent of trace for the nested sequence. The relationships found in Table II are evident but the table also shows that the first eigenvalue is always much larger than the remaining eigenvalues, growing from about three times as large as the eigenvalue for the fifty security portfolio to about ten times as large for the largest portfolio. The next section will discuss the relationship between the eigenvalue as a percent of trace and the number of securities but it is clear from this table that while the largest eigenvalues' percent of trace is declining, it is doing so at a much slower rate than the remaining eigenvalues. Thus, while Tables II and III do not appear to support the hypothesis that K eigenvalues will dominate the matrix as n grows large, the tables provide some evidence that the largest eigenvalue is growing large relative to the remaining eigenvalues.

To test this hypothesis statistically, we computed the chi-squared statistic of Table I for adjacent eigenvalues. The hypothesis is that $L_j(n) = L_{j+1}(n)$ and Table IV reports the statistics for $j = 1, \cdots, 11$ using the random sequence. Similar statistics for the nested sequence would not be independent as one portfolio is a subset of another. The table certainly supports the hypothesis that the largest eigenvalue is growing large relative to the remaining eigenvalues. There is a strong relationship between the size of the chi-squared statistic testing $L_1(n) = L_2(n)$ and the number of securities. This relationship is not as strong for the test of $L_2(n) = L_3(n)$ and is weaker the lower the rank of the eigenvalues selected. Interestingly, the hypothesis of $L_j(n) = L_{j+1}(n)$ is always accepted for $j \geq 6$ with significant differences between the first five eigenvalues existing for the larger covariance matrices. Thus, it appears that if the patterns of Tables II to IV persist for larger covariance matrices, the largest eigenvalue may eventually dominate the matrix, and there is a smaller possibility that the second largest through the fifth largest may also grow large enough to dominate the matrix. There appears to be little chance that any of the lower ranks (sixth through n^{th}) will grow large relative to its adjacent eigenvalues. These results suggest that the findings of cross-sectional pricing tests that five factors may be important is plausibly due to the first through fifth roots being more distinct than the

Table III

Twelve Largest Eigenvalues and their Percent of Trace Using the Nested Sequence

Portfolio Size	1	2	3	4	5	6
50	0.02717 (0.2024)	0.00947 (0.0705)	0.00660 (0.0492)	0.00524 (0.0390)	0.00513 (0.0382)	0.00449 (0.0335)
100	0.05189 (0.1922)	0.01094 (0.0405)	0.00779 (0.0289)	0.00754 (0.0279)	0.00657 (0.0244)	0.00650 (0.0240)
200	0.10057 (0.1739)	0.01519 (0.0262)	0.01187 (0.0205)	0.01064 (0.0184)	0.00966 (0.0167)	0.00954 (0.0165)
300	0.15388 (0.1735)	0.01995 (0.0225)	0.01460 (0.0165)	0.01173 (0.0132)	0.01142 (0.0129)	0.01042 (0.0117)
400	0.19697 (0.1723)	0.02304 (0.0202)	0.01682 (0.0147)	0.01283 (0.0112)	0.02139 (0.0108)	0.01134 (0.0099)
500	0.24445 (0.1716)	0.02910 (0.0204)	0.02226 (0.0156)	0.01700 (0.0119)	0.01365 (0.0096)	0.01314 (0.0092)
600	0.29310 (0.1723)	0.03176 (0.0186)	0.02365 (0.0139)	0.01851 (0.0093)	0.01582 (0.0093)	0.01420 (0.0083)
700	0.34533 (0.1723)	0.03653 (0.0182)	0.02640 (0.0132)	0.02033 (0.0101)	0.01861 (0.0093)	0.01639 (0.0082)
800	0.39270 (0.1722)	0.04040 (0.0177)	0.02951 (0.0129)	0.02211 (0.0097)	0.02105 (0.0092)	0.01707 (0.0075)
865	0.42032 (0.1708)	0.04353 (0.0177)	0.03261 (0.0133)	0.02369 (0.0096)	0.02246 (0.0092)	0.01759 (0.0072)

Portfolio Size	7	8	9	10	11	12
50	0.00410 (0.0305)	0.00369 (0.0275)	0.00357 (0.0266)	0.00341 (0.0254)	0.00338 (0.0252)	0.00307 (0.0228)
100	0.00558 (0.0207)	0.00553 (0.0205)	0.00522 (0.0194)	0.00518 (0.0192)	0.00480 (0.0178)	0.00453 (0.0168)
200	0.00933 (0.0161)	0.00872 (0.0151)	0.00864 (0.0149)	0.00827 (0.0143)	0.00797 (0.0138)	0.00781 (0.0135)
300	0.01018 (0.0115)	0.01000 (0.0113)	0.00979 (0.0110)	0.00965 (0.0109)	0.00931 (0.0105)	0.00922 (0.0104)
400	0.01095 (0.0096)	0.01094 (0.0096)	0.01059 (0.0093)	0.01038 (0.0091)	0.01023 (0.0089)	0.00979 (0.0086)
500	0.01200 (0.0084)	0.01171 (0.0082)	0.01163 (0.0082)	0.01139 (0.0080)	0.01085 (0.0076)	0.01070 (0.0075)
600	0.01392 (0.0082)	0.01266 (0.0074)	0.01252 (0.0073)	0.01211 (0.0071)	0.01202 (0.0071)	0.01161 (0.0068)
700	0.01536 (0.0077)	0.01480 (0.0074)	0.01370 (0.0068)	0.01332 (0.0066)	0.01294 (0.0065)	0.01286 (0.0064)
800	0.01638 (0.0072)	0.01524 (0.0067)	0.01447 (0.0064)	0.01385 (0.0061)	0.01356 (0.0060)	0.01344 (0.0059)
865	0.01695 (0.0069)	0.01557 (0.0063)	0.01512 (0.0062)	0.01451 (0.0059)	0.01430 (0.0058)	0.01399 (0.0057)

Table IV

Chi-Squared Tests for Differences Between Adjacent Eigenvalues in the Random Sequence*

Adjacent Eigenvalues	No. of Securities 30	33	36	40	44	49	54	59	65	72	79	87
1 vs. 2	153.09**	166.21**	176.01**	186.44**	200.06*	231.66*	283.40**	321.09**	342.46**	379.51**	410.32**	455.80**
2 vs. 3	1.61	1.71	1.62	4.02	4.20	4.43	4.92	4.94	1.75	2.01	2.17	2.34
3 vs. 4	7.70	7.93*	5.17	1.95	1.80	1.43	0.91	0.77	2.28	2.09	1.91	1.92
4 vs. 5	0.35	0.37	0.94	4.50	0.49	0.76	1.25	1.24	0.88	0.64	0.43	0.32
5 vs. 6	1.08	0.94	0.20	0.92	3.39	3.12	1.77	1.60	0.75	1.23	1.67	1.77
6 vs. 7	5.86	2.83	4.39	0.23	0.34	0.38	0.68	0.01	0.93	0.58	0.60	0.47
7 vs. 8	13.87**	12.27**	7.92*	0.56	0.27	0.14	0.05	0.65	0.05	0.18	0.17	0.28
8 vs. 9	0.55	1.70	1.98	15.02*	0.40	0.22	0.37	0.35	0.65	0.57	0.51	0.39
9 vs. 10	1.62	0.22	1.04	0.21	15.67*	18.38*	0.93	0.67	0.62	0.60	0.43	0.26
10 vs. 11		1.56	2.24	2.71	0.22	0.19	12.48**	0.15	0.27	0.10	0.07	0.11
11 vs. 12	1.15	0.04	0.61	2.78	1.99	0.41	2.38	0.25	0.41	0.79	1.11	1.29
Adjacent Eigenvalues	**96**	**106**	**117**	**129**	**142**	**156**	**172**	**189**	**208**	**229**	**252**	**277**
1 vs. 2	508.60**	529.76**	562.63**	605.68**	654.94**	700.51**	728.27**	732.47**	769.93**	794.28**	826.10**	869.64**
2 vs. 3	2.61	2.90	3.23	3.94	5.02	6.32*	6.30*	8.06*	13.08**	17.30**	20.21**	22.52**
3 vs. 4	1.37	2.17	1.85	1.73	1.38	1.22	2.66	2.54	1.59	1.97	1.70	2.46
4 vs. 5	0.69	0.40	0.02	0.01	0.05	0.11	0.08	1.74	2.40	0.19	0.06	0.03
5 vs. 6	1.64	1.48	1.98	1.37	1.24	0.77	0.54	0.06	0.10	1.87	3.46	2.67
6 vs. 7	0.49	0.99	0.67	1.02	1.16	1.50	1.79	0.51	0.06	0.23	0.21	0.30
7 vs. 8	0.10	0.15	0.88	0.91	0.61	0.57	0.39	0.36	0.59	0.05	0.01	0.04
8 vs. 9	0.63	0.32	0.20	0.31	0.41	0.57	0.71	0.84	0.32	0.56	0.50	0.03
9 vs. 10	0.06	0.51	0.06	0.01	0.02	0.02	0.05	0.52	0.22	0.35	0.25	0.88
10 vs. 11	0.29	0.28	1.43	1.48	1.22	0.67	0.31	0.29	0.05	0.01	0.42	0.38
11 vs. 12	1.29	0.22	0.14	0.05	0.11	0.09	0.25	0.17	0.44	0.11	0.18	0.21

Adjacent Eigenvalues	305	336	370	407	448	493	542	596	656	722	794	865
1 vs. 2	885.74**	909.87**	926.40**	924.45**	937.18**	464.13**	968.23**	1008.90**	992.89**	975.41**	985.16**	972.94**
2 vs. 3	23.53*	22.20**	24.67**	32.40**	39.61**	32.57**	30.43**	27.09**	21.07**	19.96**	22.59**	18.93**
3 vs. 4	1.29	2.98	3.58	3.46	2.00	1.82	3.86	5.85	16.56**	21.22**	18.07**	23.04**
4 vs. 5	0.98	0.45	0.69	0.77	2.55	4.93	5.82	7.27*	0.61	0.25	0.48	0.66
5 vs. 6	0.34	0.26	0.17	0.15	0.11	1.29	2.44	1.65	2.41	6.74*	9.42**	13.37**
6 vs. 7	1.97	1.25	0.84	0.57	0.60	0.23	0.03	0.22	2.33	0.53	0.40	0.31
7 vs. 8	0.06	0.38	0.19	0.34	0.26	0.34	0.64	0.51	0.60	0.78	1.23	1.64
8 vs. 9	0.08	0.03	0.39	0.23	0.04	0.03	0.10	0.08	1.02	0.68	0.52	0.21
9 vs. 10	0.07	0.16	0.30	0.62	0.47	0.10	0.11	0.13	0.08	0.53	0.48	0.36
10 vs. 11	1.13	0.06	0.03	0.06	0.17	0.89	0.73	0.53	0.01	0.18	0.09	0.05
11 vs. 12	0.05	1.38	1.31	0.36	0.18	0.03	0.06	0.17	0.20	0.31	0.05	0.11

* Statistic is significant at the 5 percent level.
** Statistic is significant at the 1 percent level.

remaining roots. It is clear, however, that the first eigenvalue is far more important (in the sense of being distinct) than the second through the fifth.

C. Eigenvalue Trace Tests

The significance of an eigenvalue may be interpreted in the sense of "explained" variation rather than in terms of being statistically distinct from another eigenvalue as tested by the chi-squared statistics. Whether a factor analysis or principal components model is used, the trace of S is the total variation of the matrix, and the j[th] eigenvalue as a percent of trace can always be interpreted as the percent of total variation explained by the j[th] component or factor.[14] Thus, observing the relationship between the eigenvalue as a percent of trace and portfolio size shows how much additional variation a principal component explains of sequentially larger portfolios. As discussed in the methodology section, if t_j is the j[th] largest eigenvalue's percent of trace, then the sum of t_1 through t_k should grow to a constant as n grows large.

Table V tests this prediction by computing the sum for various K. Denoting the sum as T_K, Part A of the table reports the behavior of T_K for the nested sequence and values of K from one to ten. Parts B and C report the results of regressing T_K on the number of securities in the covariance matrix for both sequences. The hypothesized relationship is not supported for K greater than one. T_K falls monotonically for both sequences as n increases, showing no tendency to converge, and the negative relationship is stronger the larger the value of K. Previously hypothesized values of K have all been greater than one but none of these values are supported by this table. Another way of looking at these results is to ask what number of components will explain X percent of the variation in the matrix. For example, if one wishes to explain 30 percent of the variance, the number of components will vary from three to nineteen (Table VI).

This is the finding of the Dhrymes, Friend, and Gultekin study and is suggested by the chi-squared. test. The results of Table VI do, however, provide relatively strong support for a "minimal factor representation" with $k = 1$; that is, a one-factor generating process may safely be assumed. This is because of the obvious convergence of the first (largest) eigenvalue's percent of trace. After the first two nested covariance matrices, the largest eigenvalue is about 17¼ percent of the trace and appears constant. This is also true of the random sequence except that the largest eigenvalue is about 17¾ percent of trace for portfolios with larger

[14] In a factor analysis model, $S = \hat{B}\hat{B}' + \hat{W}$ and Trace (S) = Trace $(\hat{B}\hat{B}')$ + Trace $(\hat{W})$ where $s_{ii} = b_{ii}^2 + b_{i2}^2 + \cdots + b_{ik}^2 + w_{ii}$. With the standard assumptions of factor analysis, b_{ij} is the correlation of the i[th] variable and j[th] factor. Thus, b_{ij}^2 is the variation of i explained by the j[th] factor, and s_{ii} is the sum of total explained variation plus w_{ii}, which is the unexplained variance. This means that Trace $(\hat{B}\hat{B}')$ is the total explained variation while Trace W is the total unexplained variance. In a factor analysis model, $\sum_{i=1}^{n} b_{ij}$ will be proportional to the j[th] eigenvalue of $S - W$ (see Lawley and Maxwell [13, Chapter 2]). Thus, $\sum_{i=1} b_{ij}$/Trace (S) is the percent of variance explained by the j[th] factor and if $W = I$, as assumed by principal components analysis, $L_j = \sum_{i=1}^{n} b_{ij}$ so that L_j/Trace (S) is the percent of variance explained by the j[th] factor. Of course, if $W \neq I$, one cannot interpret the eigenvalue's percent of trace as the measure of what the j[th] factor can explain, but it is the measure of that j[th] component's explanatory power.

Table V

Eigenvalues as a Percent of Trace

Part A—Sum of Eigenvalues as a Percent of Trace (T_K) for the Nested Sequence

Portfolio	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
50	0.2024	0.2729	0.3221	0.3611	0.3994	0.4328	0.4633	0.4908	0.5174	0.5429
100	0.1922	0.2327	0.2616	0.2895	0.3139	0.3380	0.3587	0.3792	0.3986	0.4178
200	0.1739	0.2001	0.2206	0.2390	0.2557	0.2722	0.2883	0.3034	0.3813	0.3326
300	0.1735	0.1960	0.2125	0.2257	0.2386	0.2503	0.2618	0.2731	0.2841	0.2950
400	0.1723	0.1925	0.2072	0.2184	0.2292	0.2391	0.2487	0.2583	0.2676	0.2767
500	0.1716	0.1920	0.2076	0.2195	0.2291	0.2383	0.2467	0.2549	0.2631	0.2711
600	0.1720	0.1906	0.2045	0.2154	0.2246	0.2330	0.2411	0.2486	0.2559	0.2630
700	0.1723	0.1906	0.2038	0.2139	0.2232	0.2309	0.2382	0.2451	0.2517	0.2582
800	0.1722	0.1900	0.2029	0.2126	0.2218	0.2293	0.2365	0.2432	0.2495	0.2556
865	0.1711	0.1888	0.2018	0.2115	0.2207	0.2278	0.2348	0.2411	0.2473	0.2532

Part B—Regression of T_K on n: Nested Sequence*

$\hat{\alpha}$	0.189	0.236	0.269	0.298	0.324	0.349	0.372	0.393	0.414	0.435
	(39.1)	(19.6)	(16.0)	(14.5)	(13.3)	(12.8)	(12.4)	(12.2)	(12.1)	(12.0)
$\hat{\beta}$	−0.003	−0.007	−0.010	−0.013	−0.015	−0.018	−0.020	−0.022	−0.024	−0.026
	(−3.3)	(−3.5)	(−3.3)	(−3.3)	(−3.0)	(−3.6)	(−3.3)	(−3.7)	(−4.0)	(−3.7)
R^2	0.517	0.536	0.545	0.568	0.577	0.594	0.608	0.622	0.634	0.646

Part C—Regression of T_K on n: Random Sequence

$\hat{\alpha}$	0.181	0.233	0.280	0.322	0.361	0.398	0.432	0.464	0.494	0.522
	(11.7)	(45.3)	(32.1)	(27.5)	(25.0)	(23.4)	(22.5)	(22.1)	(22.9)	(21.8)
$\hat{\beta}$	−0.001	−0.008	−0.014	−0.014	−0.025	−0.030	−0.034	−0.039	−0.043	−0.046
	(−2.3)	(−8.0)	(−4.7)	(−6.3)	(−6.3)	(−6.0)	(−5.7)	(−6.5)	(−7.2)	(−6.6)
R^2	0.245	0.443	0.471	0.500	0.513	0.542	0.535	0.549	0.562	0.573

* $\hat{\alpha}$ and $\hat{\beta}$ are the intercept and slope estimates, respectively. $\hat{\beta}$ has been multiplied by 10^2. All parentheses contain t-ratios. There are 8 degrees of freedom for the nested regression and 34 degrees of freedom for the random.

than 60 securities. In fact, eliminating the first several covariance matrices in each sequence results in an insignificant slope coefficient and a higher r-square.[15] Interestingly, Dhrymes, Friend, and Gultekin found that the factor loadings for the first (i.e., largest eigenvalue) factor remain constant as n grows larger, and it appears that this finding may be the result of the largest eigenvalue's constant percent of trace. Thus, this finding may be viewed as supporting the assumption of a one-factor model, and there appears to be no obvious way to choose a value of K greater than one. Given the previous result, that the second through the fifth eigenvalues are growing more distinct, it may be that assuming, for example, a five-factor generating model will be a good enough approximation for pricing securities. However, it appears that more than a one-factor model cannot be justified by exploding eigenvalues.[16]

D. Eigenvalues and Rank

To study the behavior of the eigenvalues within a covariance matrix, we rank the eigenvalues from smallest to largest and regress the eigenvalue, l_j, and its percent of trace, t_j, against the rank in separate regressions. As outlined in the methodology section, as n increases the slope of the regression should be increasing because K eigenvalues are growing larger and $n - k$ are going to zero. In fact, as Table VII shows, the relationship is the reverse with the slope coefficient growing smaller with n.[17] This is due to the fact that, for large n, either the last (i.e., the largest) or the last few eigenvalues are outliers in the sense that the $n = 5$ eigenvalues increase very gradually until the last few. The $n - 1$ (or perhaps $n - 5$) eigenvalues are virtually constant with a very gradual increase as the rank moves from smallest to largest. The slope of regression of eigenvalue against rank is dependent on the last few eigenvalues which are decreasing in their ability to influence the regression line as n grows larger. These largest eigenvalues would have needed to be a growing percentage of the trace to influence the slope, and we know this does not occur from the previous section. These relationships are shown dramatically in Figure 1. The fall in slope is obvious as the relationship is virtually constant for $n = 865$ except for the last few eigenvalues, the last of which, the largest, is far larger than the others. This is contrasted to $n = 50$ which has eigenvalues as small as the 25th smallest making a substantial contribution to the total. These two graphs summarize the tendency revealed by the previous empirical findings that the largest eigenvalue appears to be growing more important as the number of securities increases, with the remaining eigenvalues becoming less important.

[15] For the nested sequence, eliminating the first matrix results in an insignificant slope coefficient ($t = 0.018$). For the random sequence, eliminating the first three ($n = 30, 33, 36$) results in an insignificant slope ($t = 0.048$). The r^2 of the former regression is about 0.61 while the r^2 of the later is 0.33.

[16] Luedke [14] conducts extensive pricing tests using eigenvectors of portfolios up to 400 securities. His results with eigenvalues are similar to our using daily and monthly data. The pricing tests he conducts with eigenvectors tend to confirm the conclusion that there is one large factor and then no obvious way to choose the next several factors due to the relationship with the number of securities.

[17] We reversed the ranking simply to present a clearer picture in Figure 1.

Table VI

Number of Components Needed to Explain 30 Percent of the Variance

Nested Sequence	50	100	200	300	400	500	600	700	800	865
No. of Components	3	5	8	11	13	14	16	17	18	19
Random Sequence	49	96	208	305	407	493	596	722	794	865
No. of Components	3	5	9	11	13	15	16	17	18	19

Table VII

Regression of Eigenvalue on Rank

	Nested Sequence Eigenvalue			Eigenvalue as a Percent of Trace		
	Intercept	Slope	*R*-Squared	Intercept	Slope	*R*-Squared
50	−0.00136 (−1.46)	0.00016 (5.33)	0.3443	−0.01010 (−1.46)	0.00118 (4.92)	0.3443
100	−0.00158 (−1.66)	0.00008 (4.00)	0.2145	−0.00586 (−1.66)	0.00031 (5.17)	0.2145
200	−0.00212 (−2.21)	0.00005 (6.01)	0.1539	−0.00367 (−2.20)	0.00009 (9.00)	0.1539
300	−0.00240 (−2.40)	0.00004 (6.95)	0.1140	−0.00271 (−2.40)	0.00004 (6.17)	0.1140
400	−0.00249 (−2.58)	0.00003 (7.18)	0.0928	−0.00218 (−2.56)	0.00002 (5.47)	0.0928
500	−0.00266 (−2.74)	0.00002 (5.98)	0.0799	−0.00187 (−2.75)	0.00002 (8.52)	0.0799
600	−0.00279 (−2.87)	0.00002 (7.16)	0.0701	−0.00164 (−2.88)	0.00001 (6.10)	0.0701
700	−0.00298 (−3.04)	0.00002 (8.25)	0.0634	−0.00149 (−3.04)	0.832* (6.88)	0.0634
800	−0.00309 (−3.15)	0.00001 (4.73)	0.0581	−0.00136 (−3.16)	0.651* (7.02)	0.0581
865	−0.00316 (−3.25)	0.00001 (5.20)	0.0559	−0.00129 (−3.31)	0.560* (7.16)	0.0559

* Coefficient has been multiplied by 10^5.

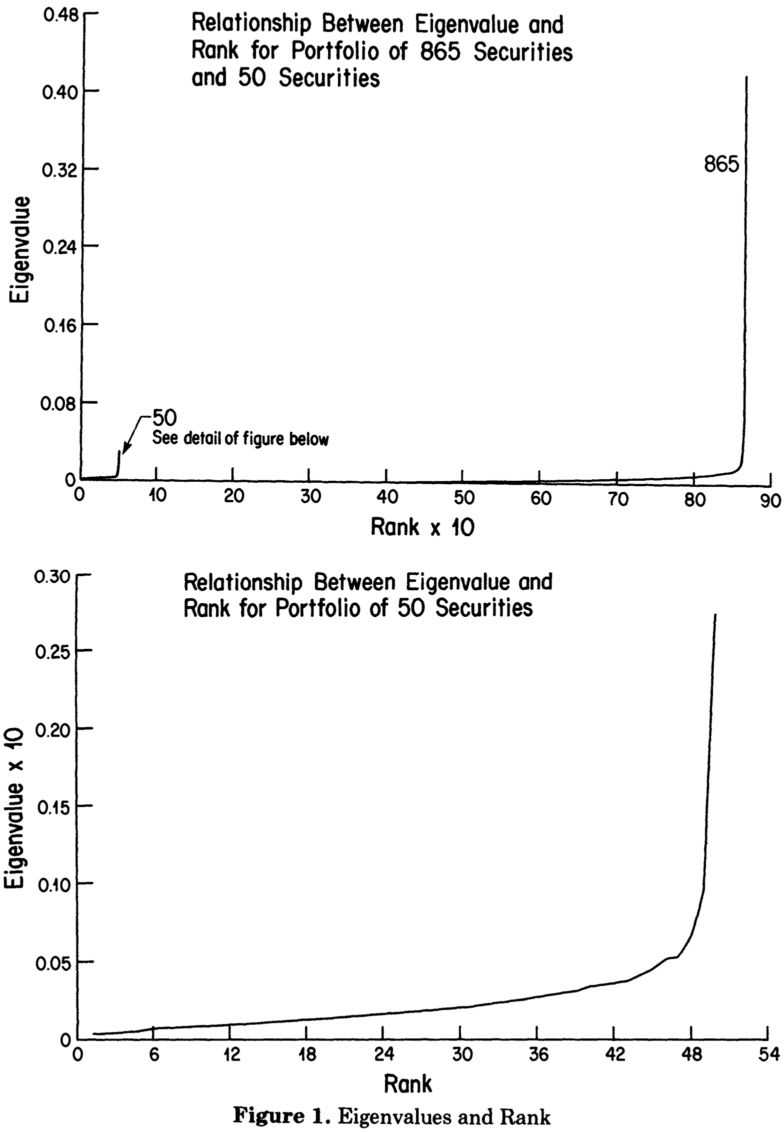

Figure 1. Eigenvalues and Rank

IV. Summary, Limitations, and Conclusions

A critical assumption of Arbitrage Pricing Theory is that K factors generate security returns. Recent theoretical results show that this assumption is equivalent to assuming that K eigenvalues of the covariance matrix of returns increase as the number of securities increases. This theoretical relationship between eigenvalues and the number of securities provides a natural method for estimating the number of factors in the APT. Furthermore, if no eigenvalues are dominant, then the validity of the theory should be questioned because the APT was originally motivated by arguing that a K factor generating model is a plausible assumption. Thus, the purpose of this paper is to examine if, in fact, K eigenvalues dominate the covariance matrix of returns as securities increase.

Using all of the data on the 1983 CRSP tapes to form two sequences of sample covariance matrices of weekly returns, we find that at most, one eigenvalue dominates the covariance matrix. The largest eigenvalue as a percentage of the trace converges to a constant but it is apparent that no other eigenvalue converges. However, we find some evidence that the second through the fifth eigenvalues are growing more distinct from each other, and this is not true of the remaining eigenvalues. This may explain why some pricing tests find more than one factor significant. In terms of the absolute size of the eigenvalues, all eigenvalues are growing larger with the number of securities, and our chi-squared tests show that we can never accept the hypothesis that a constant K principal components can adequately capture the variation in security returns. We point out, however, that principal components analysis is quite different from factor analysis, and the behavior of our chi-squared statistics when compared with those from factor analysis studies demonstrates that idiosyncratic risks are large and varied.

One should not conclude from this evidence that only one factor is important for security pricing. We have found evidence that there is one large factor and no obvious way to choose more than one. Recall, however, that we are testing only an assumption of the APT, and we provide no evidence on how many factors are needed to price securities. For example, it is possible that, say, a five-factor model would provide an accurate and stable model of security pricing even if the data are consistent with only one exploding eigenvalue. If we had found *no* exploding eigenvalues, then one could reasonably argue that another theoretical approach may be needed because of the role that the eigenvalue assumption plays in motivating the theory. Finding that the assumption of a one-factor model is exactly supported by the data does not, however, rule out a larger number of factors in a pricing test. While Chamberlain and Rothschild do prove that K factor pricing and K exploding eigenvalues are equivalent, their definition of a factor is not the same as the definition commonly used in the factor analysis pricing tests. They define factors in terms of a covariance matrix that has K exploding eigenvalues, and the result is that their covariance matrix of firm-specific risks need not be diagonal. The pricing tests which use factor analysis typically choose the number of factors necessary to force the residual covariance matrix to be diagonal. Consequently, the number of factors in a factor analysis approach will obviously be greater than or equal to the number factors defined by exploding eigenvalues.

Furthermore, this study has assumed that the factor structure is constant over twenty years of data, and this is certainly a very rough approximation, justifiable only by the need for large covariance matrices. It is possible that a factor structure extracted from estimates based on such a long period would be so unstable that it would have little to do with security pricing. Indeed, this may be the reason why we found only one eigenvalue exploded. However, the fact that one factor, and possibly more, emerges even with this strong assumption is encouraging for tests of the model using more stable data.

In conclusion, it is clear from our analysis of the eigenvalues and our chi-squared tests that the question of how many factors is far from settled. Our partial and unsatisfying answer to this question is "at least one." In spite of this ambiguity, it is useful to know that the number probably is not zero.

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