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The Geometry of the Maximum Likelihood Estimator of the Zero-Beta Return

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ABSTRACT

This paper explores geometric relations, in mean-variance space, among the sample frontier, the maximum likelihood estimator, and two other estimators of the zero-beta return. It is also demonstrated that a partition of the portfolio space is determined by a family of parabolas; the zeros of each parabola are the maximum likelihood estimators associated with all portfolios on the parabola. This observation is the basis for an additional interpretation of the statistic of the Likelihood Ratio Test of portfolio efficiency without a riskless asset.

THE ZERO-BETA RETURN with respect to the market portfolio is one of the two determinants of the expected returns of individual assets in Black's [2] generalized Capital Asset Pricing Model (CAPM). Roll [13] points out that there is an ambiguity in estimating the zero-beta return with respect to a given ex ante efficient portfolio using ex post (sample) parameters. He also presents some geometric relations among the sample frontier, the sample moments of the return on the given portfolio, and two alternative estimators of the zero-beta return introduced by Long [10]. Kandel [8] shows that the ambiguity is eliminated when returns are assumed to be jointly normally distributed and the Maximum Likelihood (ML) approach is used. The exact ML estimator is derived by Kandel [8], who also shows that the ML estimator is a zero of a specific parabola, and that it always lies between the two alternative estimators.

Following Gibbons [5] and Ross [15, 16], ML methods have been applied in tests of the CAPM by Stambaugh [20], Jobson and Korkie [7], Frankel and Dickens [4], Kandel and Stambaugh [9], Shanken [17], Ferson, Kandel and Stambaugh [3], MacKinlay [11], Amseler and Schmidt [1], and others. The small-sample distributions of the statistics of these tests have been investigated analytically and through simulations. The exact distribution is known only for the case in which a risk-free asset exists (or when the zero-beta return is known). Gibbons, Ross, and Shanken [6] show that a transformation of the Likelihood Ratio Test (LRT) has an exact F distribution. Shanken [18] introduces a Cross-Sectional Regression Test (CSRT) of efficiency for the case in which there is no risk-free asset and derives an approximate sampling distribution for the test. He also obtains an upper bound on the exact distribution functions of the CSRT and the LRT. Recently, Shanken [19] has generalized Kandel's [8] method of

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obtaining the exact ML estimator of the zero-beta return. He has also derived a lower bound on the distribution function of the LRT.

LRTs and other tests that use ML methods are not always intuitive. Interpretations of the LRT of efficiency, with and without a risk-free asset, are given by Ross [15, 16] and Kandel [8], respectively. They illustrate the geometry of these tests in mean-variance space. Roll [14] illustrates the geometry of Shanken's [18] CSRT of efficiency.

We will show that a family of infinitely many parabolas partitions the space of portfolios; the zeros of each parabola are the estimators associated with all portfolios on the parabola. This result provides an additional interpretation of the LRT statistic of efficiency without a riskless asset. Thus, this paper improves our understanding of the ML estimator of the zero-beta return by a novel exploration of the geometric relations in mean-variance space. We hope that it will be a contribution toward a better understanding of the exact small-sample properties of the ML estimator of the zero-beta return and the test statistics which use it.

I. The Geometry of the Zero-Beta Return and Its Estimators

We assume that the vector of returns on n assets follows a multivariate normal distribution with stationary mean vector, E, and stationary covariance matrix, V. Following Roll [12], we define the efficient frontier to be the set of portfolios with minimum variance at each level of mean return. An efficient frontier is completely determined by a vector of mean returns and a matrix of the covariances and variances of returns. Let α be an efficient portfolio with respect to the ex ante moments (E, V), but not the global minimum variance portfolio. It is known (e.g., Roll [12]) that all portfolios uncorrelated with α, i.e., the zero-beta portfolios, have the same expected return. We define this expected return as the zero-beta return with respect to α, and denote it by γ_0. Roll [13] proves that all zero-beta portfolios with respect to a given portfolio have the same expected return only if the portfolio is efficient. The ex ante efficient portfolio α is, with probability one, inefficient with respect to the sample mean, $\hat{E}$, and the sample covariance-variance matrix, $\hat{V}$. We further assume that its sample mean differs from the sample mean of the global minimum variance portfolio of the sample. Roll [13] shows that there are portfolios of zero sample correlation with α at all levels of sample mean return, and there is more than one candidate for the estimator of γ_0.

Roll [13] presents some geometric relations among the sample frontier, the sample moments of the return on α, and two estimators of the zero-beta return introduced by Long [10]. The two alternative estimators are:

$\hat{\gamma}_0 =$ the sample mean return on the portfolio with the minimum variance among all ex post (sample) zero-beta portfolios to α, and

$\hat{\hat{\gamma}}_0 =$ the sample mean return on the unique portfolio which is both sample efficient and ex post (sample) zero-beta with respect to α.

Let $e' = (1, 1, 1, \cdots, 1)$, $\hat{M} = e'\hat{V}^{-1}\hat{E}$, $\hat{N} = \hat{E}'\hat{V}^{-1}\hat{E}$, $\hat{L} = e'\hat{V}^{-1}e$, and $\hat{D} = \hat{N} \cdot \hat{L} - \hat{M}^2$. Denote by $\hat{\sigma}^2(m)$ the minimum sample variance for sample mean return m. Then

$$\hat{\gamma}_0 = \frac{\alpha'\hat{E} - \hat{M}\cdot\hat{\sigma}^2(\alpha'\hat{E})}{1 - \hat{\sigma}^2(\alpha'\hat{E})\cdot\hat{L}},$$

and

$$\hat{\hat{\gamma}}_0 = \frac{\hat{N} - \hat{M}\cdot(\alpha'\hat{E})}{\hat{M} - \hat{L}\cdot(\alpha'\hat{E})}.$$

Figure 1 illustrates the geometric relations, in mean-variance space, among the above estimators and the sample parameters. It displays the parabola $\hat{\sigma}^2(m)$ which corresponds to the sample efficient frontier. Since, in almost all textbooks and journal articles, the variance is on the horizontal axis and the expected return is on the vertical axis, we "turn the figures on their side," so that the parabola $\hat{\sigma}^2(m)$ is the graph of a quadratic function of the variable on the vertical axis. Special attention is given to the following five points:

Point I = $(\alpha'\hat{V}\alpha, \alpha'\hat{E})$—the point which corresponds to the sample variance and mean return on α
Point II = $(1/\hat{L}, \hat{M}/L)$—the sample global minimum variance point
Point III = $(0, \hat{\gamma}_0)$
Point IV = $(0, \hat{\hat{\gamma}}_0)$
Point V = $(\hat{\sigma}^2(\alpha'\hat{E}), \alpha'\hat{E})$—the point on the sample frontier which corresponds to the portfolio with the same sample mean return as α.

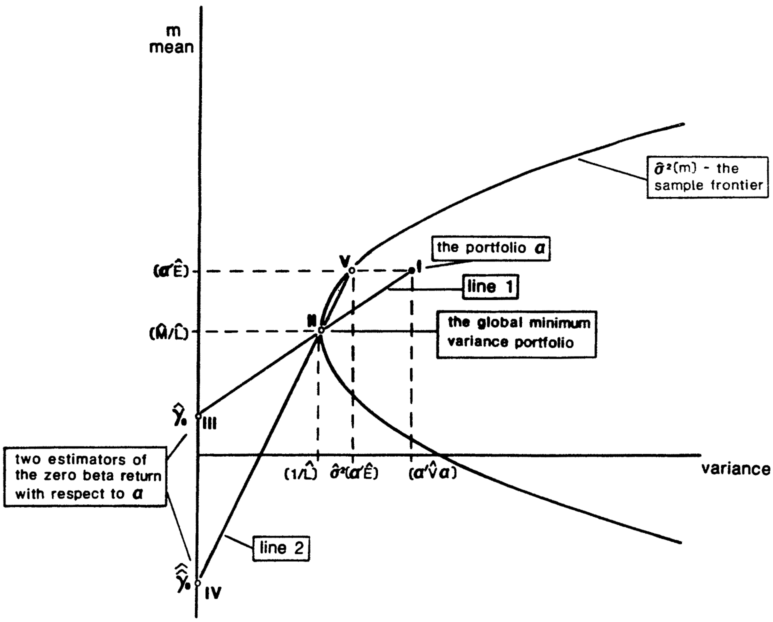

Figure 1. The geometric relations among the sample frontier, the portfolio α (point I), the global minimum-variance point (point II), two estimators of the zero-beta return with respect to α (points III and IV), and the efficient portfolio with sample mean $\alpha'\hat{E}$ (point V). Points I, II, and III are collinear. Points V, II, and IV are collinear.

Proposition 1 (Roll, 1980). *In the mean-variance space of Figure* 1, *points* I, II, *and* III *are collinear* (*line* 1), *and points* V, II, *and* IV, *are collinear* (*line* 2).

The portfolio α is ex ante efficient. The constrained ML estimators of the ex ante parameters (E, V) are the vector, E^*, and the matrix, V^*, which maximize the likelihood function subject to the constraint that the portfolio α is efficient with respect to (E^*, V^*). In this case, the constrained ML estimator of the ex ante γ_0 is well defined and unique. We denote this ML estimator of the zero-beta return by γ_0^*. Kandel [8] shows that γ_0^* is a zero of a specific parabola. Here, we show how this parabola may be constructed using the sample parameters. In Figure 2, we add points VI and VII to points I–V and the sample frontier of Figure 1.

Point VI $= (1/\hat{L} + \hat{D}/\hat{L}^2, \hat{M}/\hat{L})$—A point on the horizontal line through point II (the sample global minimim-variance point). Point VI depends on the sample moments but not on the particular choice of the portfolio α.

Point VII $= \left(\dfrac{\hat{D} \cdot (\hat{\sigma}^2(\alpha'\hat{E}) - \alpha'\hat{V}\alpha)}{\hat{M} - \hat{L} \cdot \alpha'\hat{E}}, \hat{\hat{\gamma}}_0\right)$—The point of intersection of line 1 (the line which passes through points I, II, and III; see explanation of Figure 1) and the horizontal line through point IV.

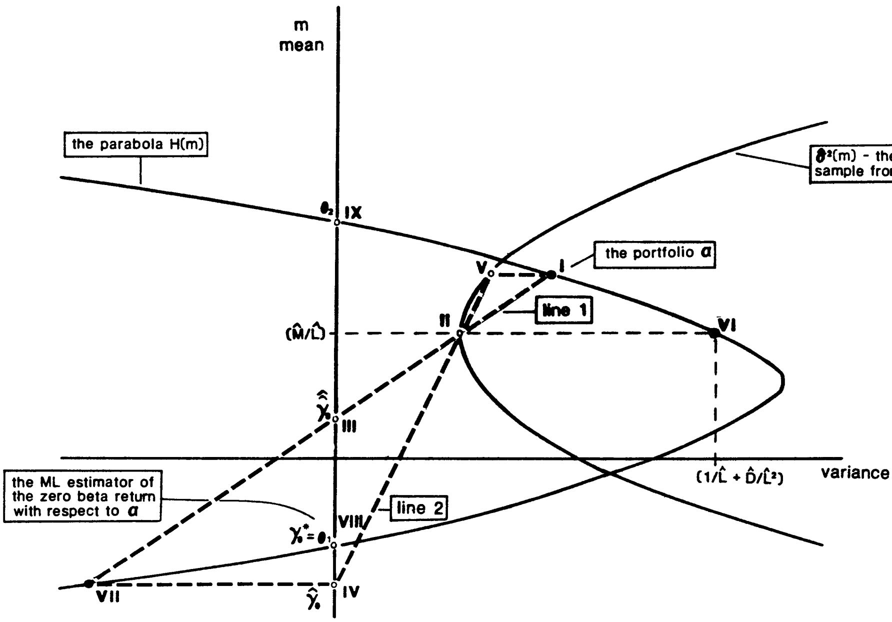

Figure 2. The ML estimator of the zero-beta return, γ_0^* (point VIII), is a zero of the parabola determined by points I and VII, which depend on α, and point VI, which is independent of α. The ML estimator always lies between the other two estimators (points III and IV).

A parabola is uniquely determined by any three of its points. Proposition 2 asserts that a specific parabola whose zero is γ_0^* can be determined by points I and VII, which depend on α, and point VI, which is independent of α. The parabola is the graph of a quadratic function of m. Since m is the variable on the vertical axis, the zeros of this parabola are also on the vertical axis.

PROPOSITION 2. *Let $H(m)$ be the quadratic function satisfying*

$$H(\alpha'\hat{E}) = \alpha'\hat{V}\alpha \qquad \text{(point I)} \tag{i}$$

$$H(\hat{\gamma}_0) = \frac{\hat{D}\cdot(\hat{\sigma}^2(\alpha'\hat{E}) - \alpha'\hat{V}\alpha)}{\hat{M} - \hat{L}\cdot(\alpha'\hat{E})} \qquad \text{(point VII)} \tag{ii}$$

$$H(\hat{M}/\hat{L}) = (1/\hat{L}) + (\hat{D}/\hat{L}^2). \qquad \text{(point VI)} \tag{iii}$$

The equation $H(m) = 0$ has two real solutions, $\theta_1 < \theta_2$. If $\alpha'\hat{E} > \hat{M}/\hat{L}$, then $\gamma_0^* = \theta_1$, and if $\alpha'\hat{E} < \hat{M}/\hat{L}$, then $\gamma_0^* = \theta_2$. (Points VIII *and* IX *of Figure* 2 *correspond to θ_1 and θ_2, respectively.)*

Proof: Kandel (1984) considers the quadratic function

$$H(m) = Am^2 + Bm + C,$$

where

$$A = -1,$$

$$B = -\frac{\hat{L}(\alpha'\hat{V}\alpha) - \hat{L}(\alpha'\hat{E})^2 + 1 + \hat{N}}{\hat{M} - \hat{L}(\alpha'\hat{E})},$$

$$C = \frac{\hat{M}(\alpha'\hat{V}\alpha) + \hat{M}(\alpha'\hat{E})^2 - (\alpha'\hat{E}) - \hat{N}(\alpha'\hat{E})}{\hat{M} - \hat{L}(\alpha'\hat{E})},$$

and shows that: (a) equation $H(m) = 0$ has two real solutions ($\theta_1 < \theta_2$) and (b) the constrained ML estimator of the zero-beta return with respect to α, γ_0^*, is a zero of $H(m)$:

if $\alpha'\hat{E} > \hat{M}/\hat{L}$, then $\theta_1 < (\hat{M}/\hat{L}) < (\alpha'\hat{E}) < \theta_2$ and $\gamma_0^* = \theta_1$, and
if $\alpha' < \hat{M}/\hat{L}$, then $\theta_1 < (\alpha'\hat{E}) < (\hat{M}/\hat{L} < \theta_2$ and $\gamma_0^* = \theta_2$.

It is easy to check that this quadratic function satisfies conditions (i) to (iii). Q.E.D.

The next proposition, Proposition 3, shows that there is an infinite family of parabolas in mean-variance space which determine a partition of the portfolio space: the zeros of each parabola are the ML estimators associated with all portfolios on this parabola. All the parabolas have one point of intersection: point VI. Clearly, the portfolios corresponding to this point cannot be included in the set of portfolios on each parabola. For these portfolios, and for the other portfolios on the horizontal line through point II, there exists no ML estimator of the zero-beta return.

PROPOSITION 3. *Let $p'\hat{E}$ and $p'\hat{V}p$ be the sample mean and variance, respectively, of the return on a portfolio p. Let $H(m)$ be the quadratic function introduced in Proposition 2. Assume that $p'\hat{E} \neq \hat{M}/\hat{L}$. Then*

$$p'\hat{V}p = H(p'\hat{E})$$

if and only if the ML estimator of the zero-beta return with respect to p is a zero of $H(m)$.

Proof of Necessity: Consider a portfolio p whose sample parameters satisfy

$$p'\hat{V}p = H(p'\hat{E}), \quad p'\hat{E} \neq \hat{M}/\hat{L}$$

namely, it lies on the parabola associated with α. Suppose we wish to find the ML estimator of the zero-beta return with respect to p. Obviously, the estimator is a zero of the parabola associated with p. By Proposition 2, this parabola is uniquely determined by the following three points:

$$(p'\hat{V}p, p'\hat{E}) \tag{i}$$

$$\left(\frac{\hat{D}}{\hat{L}^2} + \frac{1}{\hat{L}}, \frac{M}{\hat{L}}\right) \tag{ii}$$

$$\left(\frac{\hat{D} \cdot (\hat{\sigma}^2(p'\hat{E}) - p'\hat{V}p)}{\hat{M} - \hat{L} \cdot (p'\hat{E})}, \frac{\hat{N} - \hat{M} \cdot (p'\hat{E})}{\hat{M} - \hat{L} \cdot (p'\hat{E})}\right). \tag{iii}$$

The first two points belong to the parabola, $H(m)$, which is the parabola associated with α. In order to show that this is also the parabola associated with p, it is sufficient to show that the third point also belongs to this parabola. An investigation of the function $H(m)$ shows that it satisfies the following equation for all $m \neq \hat{M}/\hat{L}$:

$$H\left(\frac{\hat{N} - \hat{M} \cdot m}{\hat{M} - \hat{L} \cdot m}\right) = \frac{\hat{D} \cdot (\hat{\sigma}^2(m) - H(m))}{(\hat{M} - \hat{L} \cdot m)}.$$

This equation and the assumption that $H(p'\hat{E}) = p'\hat{V}p$ complete the argument that the parabolas associated with α and p coincide.

Proof of Sufficiency: Let $\hat{\gamma}_0(q)$, $\hat{\hat{\gamma}}_0(q)$, and $\gamma_0^*(q)$ be estimators of the zero-beta return with respect to a portfolio q. From the construction of these estimators, it is clear that, if q is sample-efficient, then

$$\hat{\gamma}_0(q) = \gamma_0^*(q) = \hat{\hat{\gamma}}_0(q).$$

It is also clear from this construction that, if q is sample efficient, then the following three points are collinear:

$(q'\hat{V}q, q'\hat{E})$	(The portfolio q).	(i)
$(0, \gamma_0^*(q))$	(The zero-beta return with respect to q).	(ii)
$\left(\frac{1}{\hat{L}}, \frac{\hat{M}}{\hat{L}}\right)$	(The global minimum-variance point).	(iii)

This last result implies that there exists a unique portfolio, q, on the sample efficient frontier such that $\gamma_0^*(q) = \gamma_0^*(\alpha) = \gamma_0^*$.

Suppose there exists a portfolio, p, which is not on the parabola associated with α such that $\gamma_0^*(p) = \gamma_0^*(\alpha)$. The parabola associated with p will intersect the efficient frontier at a point different from the intersection with the parabola associated with α. By the proof of necessity, there will be two portfolios on the frontier with the same estimator of the zero-beta return. Contradiction: hence, if $\gamma_0^*(p) = \gamma_0^*(\alpha)$, then the parabolas associated with p and α coincide. Q.E.D.

II. An Interpretation of the LRT Statistic

The geometry of the ML estimator of the zero-beta return provides an additional interpretation of the statistic of an LRT discussed in Kandel [8]. Two more points on the parabola $H(m)$ are labeled in Figure 3.

Points X *and* XI—The two points of intersection of the parabola $H(m)$ with the sample frontier.

PROPOSITION 4. *In the mean-variance space of Figure* 3, *points* X, II, *and* IX *are collinear* (*line* 3) *and points* XI, II, *and* VIII *are collinear* (*line* 4).

Proof: A straightforward implication of the proof of sufficiency in Proposition 3. Q.E.D.

Kandel [8] derives an LRT of the efficiency of α without a riskless asset. The test statistic is, in essence, a measure of the distance between point I, which

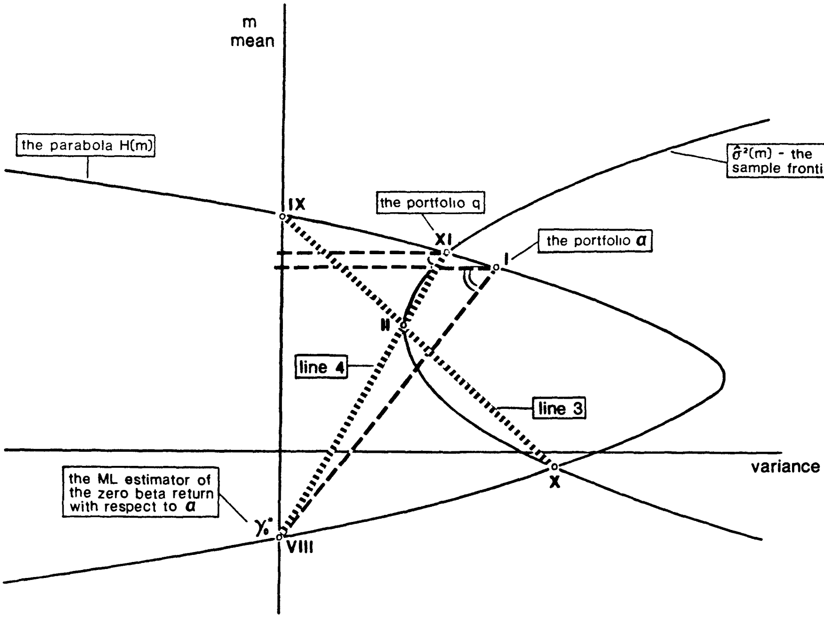

Figure 3. The zeros of the parabola $H(m)$ (points VIII and IX), are the ML estimators of all portfolios on the parabola, including the sample efficient portfolio q (point XI). When the efficiency of α is tested, the LRT statistic is the ratio of ratios of excess mean return to variance of the portfolios α and q, respectively.

corresponds to α, and point XI, which corresponds to the unique sample efficient portfolio whose zero-beta return is γ_0^*. If we denote this portfolio by q, it follows that

$$\text{the LRT statistic} = T \log \left[\frac{q'\hat{E} - \gamma_0^*}{q'\hat{V}q} \Big/ \frac{\alpha'\hat{E} - \gamma_0^*}{\alpha'\hat{V}\alpha} \right]$$

where T is the number of periods. The term in the numerator is the ratio of the excess return to variance of the portfolio q (point IX) and the term in the denominator is the ratio of excess return to variance of the portfolio α (point I).

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