On the Exclusion of Assets from Tests of the Mean Variance Efficiency of the Market Portfolio: An Extension
Author(s): Jay Shanken
Source: *The Journal of Finance*, Vol. 41, No. 2 (Jun., 1986), pp. 331-337
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328438
Accessed: 26/11/2009 08:28

THE JOURNAL OF FINANCE • VOL. XLI, NO. 2 • JUNE 1986

On the Exclusion of Assets from Tests of the Mean Variance Efficiency of the Market Portfolio: An Extension

JAY SHANKEN*

ABSTRACT

This paper extends Kandel's [3] analysis of the testability of the mean-variance efficiency of a market index when the return on some component of the index is not perfectly observable. In addition to information about the mean and variance of the missing asset, considered by Kandel, we explore the usefulness of information about the beta of the missing asset on the observed sub-portfolio in an economy with a riskless asset. The results are somewhat more supportive of the notion that mean-variance efficiency is testable on a subset of the assets.

IN HIS IMPORTANT CRITIQUE of tests of the capital asset pricing model, Roll [4] emphasizes that empirical results obtained with market proxies need not be indicative of the results that would be obtained in testing the efficiency of the true market portfolio, were it observable. More recently, Kandel [3] has analyzed the testability of the mean-variance efficiency of a market portfolio when the return on some component of the portfolio is not perfectly observable but partial information about its characteristics is available. He finds that knowledge of the market share and expected return of the missing asset is not sufficient to permit a test. However, bounding the variance of return on the missing asset does make it possible, in principle, to correctly reject the efficiency of the portfolio, provided that the market share of the missing asset is not too large.

Building on the analytical framework established by Kandel, we explore the value of information about the riskless rate of return and the covariance betwen the return on the missing asset and the observed sub-portfolio. Given a riskless asset, the question we address is whether the unobservable market portfolio of risky assets is equal to the tangency portfolio of the efficient frontier. We find that knowledge of the market share and expected return of the missing asset is still not sufficient to permit a test of this more restrictive efficiency condition.

As in the no-riskless asset case, given a bound on the variance of the missing asset, it is sometimes possible to correctly infer that the unobserved portfolio is inefficient. This occurs when the market share of the missing asset or the variance of its return is small and the sub-portfolio is sufficiently far from the (linear) efficient frontier, with respect to the included assets. Information about the expected return on the missing asset, in addition to the variance, sometimes yields a tighter restriction.

* University of Rochester, Graduate School of Management. Thanks to Mike Gibbons and an anonymous referee for helpful comments.

Stronger results are obtained when information about the beta of the missing asset on the observed sub-portfolio is introduced. While such information, by itself, does not permit a test, a lower bound on the missing asset beta and an upper bound on its expected return together yield a testable hypothesis. This hypothesis may, in principle, be rejected even if the market share and variance of the missing asset are both large. It is noteworthy that the condition becomes more restrictive as the correlation between the missing asset and the sub-portfolio increases.

Our results expand the class of circumstances under which mean-variance efficiency may be tested on a subset of the assets. It would be interesting to combine the restrictions considered here with prior distributions on the unobserved parameters of the missing asset, in a Bayesian statistical framework. This and other issues of statistical inference will be considered in future work.

I. The Model and Some Lemmas

Let R be a vector of returns on n risky assets. E is the corresponding expected return vector and V the covariance matrix of returns, assumed to be positive definite. We use the same notation with an asterisk (E^*, V^*) to refer to the parameters of the $n + 1$ assets—the included assets and the missing asset. A portfolio α is a vector such that $\alpha'e = 1$, where e is the unit n-vector $(1, 1, \ldots, 1)$. Shortselling of all risky securities is permitted, as well as borrowing and lending at the riskless rate, r. As usual, an *efficient portfolio* is one which has maximal expected return for its level of variance. The *efficient tangency portfolio* (ETP) is the unique efficient portfolio of the risky assets with zero-beta expected return equal to r.

Let μ, σ^2, and u be the parameters of the return on the missing asset—respectively, the expected return, variance, and vector of covariances with the returns on the included assets. Thus,

$$E^* = \begin{bmatrix} E \\ \mu \end{bmatrix} \text{ and } V^* = \begin{bmatrix} V & u \\ u' & \sigma^2 \end{bmatrix}$$

Let $\eta' \equiv [(1 - \gamma)\alpha', \gamma]$, $\alpha \in \mathbb{R}^n$, $0 < \gamma < 1$, be a portfolio that assigns weight γ to the missing asset and weight $1 - \gamma$ to the observed sub-portfolio α. The following lemma simplifies the derivation of our later results.

LEMMA 1. *V^* is positive definite and η is the ETP for (E^*, V^*, r) if and only if there exists a scalar $x > 0$ such that*

$$(1 - \gamma) V\alpha + \gamma u = x(E - re) \tag{1}$$

$$(1 - \gamma)u'\alpha + \gamma\sigma^2 = x(\mu - r) \tag{2}$$

$$u'V^{-1}u < \sigma^2. \tag{3}$$

Proof: Follows as a direct consequence of Lemma 1 in Kandel [3].

Let R^* be the vector of returns on the $n + 1$ assets and let $\lambda \equiv x^{-1}$. Conditions (1) and (2) of Lemma 1 are equivalent to

$$E_i^* - r = \lambda \text{cov}(R_i^*, \eta' R^*) \qquad i = 1, 2, \ldots, n + 1. \tag{4}$$

Thus, x is the reciprocal of the "market price of covariance risk" with respect to the portfolio η.

The statement of our main lemma below requires some additional notation. Let μ_α be the expected return and σ_α^2 the variance of α. We use the fact that θ_T, the slope of the efficient frontier at the ETP for (E, V, r) satisfies[1]

$$\theta_T^2 = (E - re)' V^{-1}(E - re).$$

LEMMA 2. *Let (μ, σ^2, u, x^*) satisfy* (1) *and* (2). *Then $\sigma^2 = g(x^*)$, where*

$$g(x) \equiv x[\gamma(\mu - r) - (1 - \gamma)(\mu_\alpha - r)]/\gamma^2 + \sigma_\alpha^2 (1 - \gamma)^2/\gamma^2. \tag{5}$$

Furthermore, given (1) *and* (2), (3) *is satisfied if and only if x^* is in the open interval $(0, \bar{x})$, where $\bar{x} \equiv [\gamma(\mu - r) + (1 - \gamma)(\mu_\alpha - r)]/\theta_T^2$.*

Proof: Assume (1) and (2) hold. Multiplying each side of (1) on the left by α' and dividing by γ gives

$$\alpha' u = [x(\mu_\alpha - r) - (1 - \gamma)\sigma_\alpha^2]/\gamma. \tag{6}$$

Substituting this expression for $\alpha' u$ in (2) and solving for σ^2, we obtain the right side of (5). Solving for γu in (1), we have

$$\gamma^2 u' V^{-1} u = x^2 \theta_T^2 + (1 - \gamma)^2 \sigma_\alpha^2 - 2x(1 - \gamma)(\mu_\alpha - r).$$

Subtracting this from the expression for $\gamma^2 \sigma^2$ gives

$$\gamma^2(\sigma^2 - u' V^{-1} u) = x[\gamma(\mu - r) + (1 - \gamma)(\mu_\alpha - r) - x\theta_T^2].$$

This is a concave quadratic in x, with zeroes at $x = 0$ and $x = \bar{x}$. Hence, it is positive if and only if x is in the open interval $(0, \bar{x})$. Q.E.D.

Note that $[\gamma(\mu - r) + (1 - \gamma)(\mu_\alpha - r)]$ is the expected excess return on the portfolio η. This must be strictly positive, i.e., $\bar{x}$ must be greater than zero, if η is efficient. Otherwise, there does not exist a positive x satisfying (1) $-$ (3).

E, V, θ_T, μ_α, and σ_α^2 may be consistently estimated from time series of returns on the included assets, given stationarity of the joint return distribution. These parameters are treated as known in order to simplify the presentation. The weight on the missing asset is generally assumed to be known as well, though it should be clear that all results can be modified to incorporate an interval of feasible values of γ.

II. Information about the Mean and Variance of the Missing Asset

Following Kandel, we begin by considering information about the expected return on the missing asset.

THEOREM 1. *Assume μ is known and $[\gamma(\mu - r) + (1 - \gamma)(\mu_\alpha - r)] > 0$. There exist $\sigma^2 > 0$ and u in $\mathbb{R}^n$ such that V^* is positive definite and η is the ETP for (E^*, V^*, r).*

Proof: By the assumption on μ, $\bar{x} > 0$. Choose any x in $(0, \bar{x})$ and let u and σ^2 be defined implicitly by (1) and (2), respectively. Thus, these conditions hold

[1] A proof may be found in Jobson and Korkie [2].

automatically. The selection of x guarantees that (3) holds, by Lemma 2. Therefore, $\sigma^2 > 0$ and our conclusion holds by Lemma 1. Q.E.D.

Suppose we are able to empirically reject the null hypothesis that the subportfolio α is efficient with respect to the included assets. Theorem 1 tells us that, even if we know the expected return on the missing asset, we cannot reject the null hypothesis that η is the ETP for (E^*, V^*, r) without further information. The one exception is the unlikely case in which the expected excess return on η is nonpositive. Kandel reaches a similar conclusion with respect to tests of efficiency in the absence of a riskless asset.[2] We now consider information about the variance of the missing asset.

THEOREM 2. *Assume that σ^2 is known. There exist μ in $\mathbb{R}$ and u in $\mathbb{R}^n$ such that V^* is positive definite and η is the ETP for (E^*, V^*, r) only if $\sigma^2 > \Delta$, where*

$$\Delta \equiv (\sigma_\alpha^2 - \sigma_*^2)(1 - \gamma)^2/\gamma^2$$

and σ_*^2 is the variance of the portfolio on the efficient frontier for (E, V, r) with the same expected return as α.

Proof: Assume V^* is positive definite and η is the ETP for (E^*, V^*, r). By Lemmas 1 and 2, there is an x^* in $(0, \bar{x})$ such that $\sigma^2 = g(x^*)$. Since $g(x)$ is monotonic in x, α^2 must lie between $g(0) = \sigma_\alpha^2(1 - \gamma)^2/\gamma^2$ and

$$g(\bar{x}) = [(\mu - r)^2 - (\mu_\alpha - r)^2(1 - \gamma)^2/\gamma^2]/\theta_T^2 + \sigma_\alpha^2(1 - \gamma)^2/\gamma^2.$$

In any event, $\sigma^2 > [\sigma_\alpha^2 - (\mu_\alpha - r)^2/\theta_T^2](1 - \gamma)^2/\gamma^2$. Now observe that $\theta_T^2 = (\mu_\alpha - r)^2/\sigma_*^2$. Q.E.D.

Theorem 2 tells us that the efficiency of η entails an upper bound on the distance of α from the linear efficient frontier for (E, V, r). Kandel obtains a similar result without a riskless asset except that σ_*^2, in his case, is the minimum variance on the "curved frontier" at the expected return level μ_α. Since the linear efficient frontier strictly dominates the curved frontier of the risky assets (except at the point of tangency), incorporating information about the riskless rate leads to a tighter restriction, i.e., a higher lower bound on σ^2.

As Kandel notes, Δ is decreasing in γ. Thus, the efficiency of η may be rejected if

$$\bar{\sigma}^2 \leq (\sigma_\alpha^2 - \sigma_*^2)(1 - \bar{\gamma})^2/\bar{\gamma}^2$$

where $\bar{\gamma}$ and $\bar{\sigma}$ are upper bounds on γ and σ, respectively. With $\bar{\sigma} = \sigma_\alpha$, for example, rejection is impossible if the market share of the missing asset exceeds one-half. This is true regardless of how inefficient α might be. In general, if $\bar{\gamma}$ is large then $\bar{\sigma}$ must be correspondingly small in order for rejection to be possible.

As an illustration, let α be a value-weighted portfolio of all corporate securities and let the missing asset be real estate. We wish to determine whether the market portfolio η of corporate securities and real estate is efficient. Suppose the market

[2] Note that Theorem 1 does not follow from Kandel's corresponding Theorem 2. The latter asserts the existence of a portfolio on the curved efficient frontier but does not place any restriction on the zero-beta rate of that portfolio. In this sense, Theorem 1 is a stronger result. The proofs are similar, however.

value of real estate is about twice that of all corporate securities. The data of Ibbotson and Fall [1] indicate that the standard deviation of annual return on real estate was about one-quarter that of all corporate securities over the period 1947–1978. Ignoring the ex post nature of the data and other measurement problems, $\Delta = (\sigma_\alpha^2 - \sigma_*^2)(1/3)^2/(2/3)^2$ and $\sigma^2 = (\sigma_\alpha/4)^2$. Theorem 2 requires that $\sigma_\alpha^2/4$ exceed $\sigma_\alpha^2 - \sigma_*^2$, if η is efficient.

In general, $\sigma_*^2/\sigma_\alpha^2 = \theta_\alpha^2/\theta_T^2$, where $\theta_\alpha \equiv (\mu_\alpha - r)/\sigma_\alpha$ is the Sharpe measure of performance for α. Thus, in the example above, the necessary condition is equivalent to $1/4 > 1 - \theta_\alpha^2/\theta_T^2$ or $\theta_T < 1.15\theta_\alpha$. The efficiency of η may be rejected, therefore, if there exists a portfolio of corporate securities and the riskless asset with Sharpe performance at least 15% higher than that of α. Of course, in reality, the returns on all corporate securities and real estate are not perfectly observable and ex post estimates are not equal to ex ante expectations. This illustration, as well as others below, is intended only to convey the general nature of our restrictions.

We now consider the implications of joint information about μ and σ^2.

THEOREM 3. *Assume μ and σ^2 are known. There exists u in $\mathbb{R}^n$ such that V^* is positive definite and η is the ETP for (E^*, V^*, r) only if σ^2 lies between $g(0)$ and $g(\bar{x})$, where g and $\bar{x}$ are defined in Lemma 2 and $\bar{x} > 0$.*

Proof: Immediate from Lemma 2 and the monotonicity of $g(x)$ in x.

Partial information about the expected return on the missing asset is still useful, in conjunction with information about the variance, if it enables us to determine whether $\gamma(\mu - r)$ is greater or less than $(1 - \gamma)(\mu_\alpha - r)$. Suppose the correct relation is "greater than." In this case, $g(x)$ is increasing in x, hence $\sigma^2 > g(0) = \sigma_\alpha^2(1 - \gamma)^2/\gamma^2$. This is a sharper (higher) lower bound on σ^2 than that given in Theorem 2, in the absence of information about μ.

For example, the mean return on real estate over the period 1947–1978 was not much less than the return on all corporate securities. With $\gamma = 2/3$, therefore, Theorem 3 requires that the standard deviation for real estate (3.53%) exceed one-half the standard deviation for corporate securities (13.84%). As the condition is violated, we conclude that the corresponding portfolio η was not ex post efficient over this period.

III. Information about the Beta of the Missing Asset on the Observed Portfolio α

The conditions for efficiency of η in Lemmas 1 and 2 do not impose an upper bound on the market price of covariance risk, λ, as values of x arbitrarily close to zero are permitted. It will be shown that bounds on λ may be derived from bounds on the beta of the missing asset relative to α.

Assume η is efficient. Since α is a portfolio of assets in (4), its excess return, $\mu_\alpha - r$, is given by the product of λ with the covariance between α and η. Equivalently, $\theta_\alpha/\sigma_\alpha$ equals λ times the beta of η on α. Thus,

$$\theta_\alpha/\sigma_\alpha = \lambda[\gamma\beta + (1 - \gamma)], \tag{7}$$

where β is the beta of the missing asset on α (the beta of α on itself is one).

Therefore, information about β generally yields information about λ, since θ_α and σ_α are known. The next result indicates that this information is not useful by itself, however, in the present context.

THEOREM 4. *Let β be the beta of the missing asset on α. Assume beta is known and that $[\gamma\beta + (1 - \gamma)]$ has the same sign as $\mu_\alpha - r$.*[3] *There exist $\sigma^2 > 0$, μ in $\mathbb{R}$ and u in $\mathbb{R}^n$ such that V^* is positive definite and η is the ETP for (E^*, V^*, r).*

Proof: If $\mu_\alpha \neq r$, let $x^* \equiv \lambda^{-1}$, where λ is the unique solution to (7). Our assumptions imply that $x^* > 0$. If $\mu_\alpha = r$, let x^* be any positive number. Choose a value of μ large enough to ensure that $x^* < \bar{x}$. Given x^* and μ, let u and σ^2 be defined implicitly by (1) and (2), respectively. Thus, (1) and (2) hold automatically and (3) holds by Lemma 2. Therefore, our conclusion holds by Lemma 1. The covariance between α and the missing asset is given in (6). Using this, it is easily verified that our specification of x^* and u is consistent with the assumed missing asset beta. Q.E.D.

While knowledge of μ or β separately is not helpful, information about both may be sufficient to reject the efficiency of η.

THEOREM 5. *Let $\mu_\alpha > r$ and assume β and μ are known.*[4] *There exist $\sigma^2 > 0$ and u in $\mathbb{R}^n$ such that V^* is positive definite and η is the ETP for (E^*, V^*, r) only if $\bar{x} > 0$ and*

$$[\gamma\beta + (1 - \gamma)]\sigma_\alpha\theta_T^2 < [\gamma(\mu - r) + (1 - \gamma)(\mu_\alpha - r)]\theta_\alpha. \tag{8}$$

Proof: Assume V^* is positive definite and η is the ETP. Let x^* satisfy (1) – (3) in Lemma 1. By (7), $x^* = \lambda^{-1} = [\gamma\beta + (1 - \gamma)]\sigma_\alpha/\theta_\alpha$. By Lemma 2, $0 < x^* < \bar{x}$. Substituting the definition of $\bar{x}$ given in Lemma 2 and rearranging yields the required result. Q.E.D.

The necessary condition of Theorem 5 is clearly rejected if it is violated with β and μ replaced by lower and upper bounds, respectively. In contrast to Theorem 2, rejection is possible even if the market share and variance of the missing asset are both large. In particular, since $\theta_\alpha^2 \leq \theta_T^2$ and the expressions in (8) are linear in γ, a sufficient condition for rejecting efficiency regardless of the value of γ is[5]

$$(\beta - 1)\sigma_\alpha\theta_T^2 \geq (\mu - \mu_\alpha)\theta_\alpha. \tag{9}$$

Intuitively, α is a better proxy for η when the portfolios are highly correlated. Given σ_α and σ, higher correlation between α and the missing asset implies a higher value of β, which increases the left sides of (8) and (9). We may conclude that, other things equal, a given level of performance for α is more likely to imply rejection of the efficiency of η if the correlation between α and the missing asset

[3] If η is efficient, (7) holds with $\lambda > 0$. Since $\theta_\alpha \equiv (\mu_\alpha - r)/\sigma_\alpha$, $\mu_\alpha - r$ must have the same sign as $[\gamma\beta + (1 - \gamma)]$, in this case.

[4] If $\mu_\alpha < r$, the inequality in (8) is reversed since $\theta_\alpha < 0$. If $\mu_\alpha = r$, efficiency of η implies $\beta = (\gamma - 1)/\gamma$, by (7). No information about λ is provided by (7), in this case.

[5] $\theta_\alpha^2 \leq \theta_T^2$ ensures that (8) is violated at $\gamma = 0$. If the inequality in (9) is reversed then efficiency may be rejected regardless of the true value of γ, provided (8) is violated at $\gamma = 1$. I am grateful to an anonymous referee for pointing out this case to me.

is high.[6] When the correlation is close to one, efficiency of η requires that α be nearly efficient with respect to the included assets.[7]

Suppose, for example, that γ equals one-half and μ and σ are bounded above by the corresponding parameters of α. One can easily verify that Theorems 2 and 3 are not useful in this context. Let 0.6 be a lower bound on β. Theorem 5 implies that

$$[(0.5)(0.6) + (0.5)]\sigma_\alpha \theta_T^2 < (\mu_\alpha - r)\theta_\alpha$$

or $\theta_T < 1.12\theta_\alpha$, if η is efficient. The efficiency of η is rejected if there exists a portfolio of the included assets with Sharpe performance 12% higher than that of α. Using zero as a lower bound for beta, the necessary condition is $\theta_T < 1.41\theta_\alpha$. While rejection is now less likely, it is still quite possible.

Finally, we consider joint information about μ, σ^2, and β.

THEOREM 6. *Let $\underline{\beta} < \beta < \overline{\beta}$. Assume μ and σ^2 are known and $\mu_\alpha \neq r$. There exists u in $\mathbb{R}^n$ such that V^* is positive definite and η is the ETP for (E^*, V^*, r) only if σ^2 lies between $g(x_1)$ and $g(x_2)$ where*

$$x_1 \equiv [\gamma\underline{\beta} + (1 - \gamma)]\sigma_\alpha/\theta_\alpha \qquad \text{and} \qquad x_2 \equiv [\gamma\overline{\beta} + (1 - \gamma)]\sigma_\alpha/\theta_\alpha.$$

Proof: Assume V^* is positive definite and let η be efficient. By Lemmas 1 and 2, there is a positive x^* such that $\sigma^2 = g(x^*)$. By (7), $x^* = \lambda^{-1}$ lies between x_1 and x_2. The conclusion above now follows from the monotonicity of $g(x)$ in x. Q.E.D.

Information about β enables us to bound the value of the market price of covariance risk, limiting the feasible values of x and σ^2 in Lemma 2. As with Theorem 3, partial information about μ is still useful if it allows us to determine whether $\gamma(\mu - r)$ is greater or less than $(1 - \gamma)(\mu_\alpha - r)$, i.e., whether $g(x)$ is increasing or decreasing in x. Note that the necessary condition of Theorem 6 is more restrictive than that of Theorem 3 when x_1 and x_2 lie in the interval $(0, \bar{x})$.[8]

[6] Shanken [5] analyzes the role of correlation between observables and unobservables in a more general asset pricing context.

[7] Let the correlation between α and the missing asset be close to one. In this case, the return on the missing asset is nearly equal to its linear projection on α and a constant. Thus, the covariance between the missing asset and η is approximately equal to β times the covariance between α and η. By (4), $\mu - r \approx \beta(\mu_\alpha - r)$. Substituting this in (8) and simplifying gives $\theta_T^2 < \theta_\alpha^2$, approximately. However, by definition, $\theta_T^2 \geq \theta_\alpha^2$. It follows that $\theta_T \approx \theta_\alpha$ when the correlation between the missing asset and α is close to one.

[8] Mathematically, our results only require that the zero-beta rate for η is known. This circumstance arises most naturally in the riskless asset case discussed in the paper.

REFERENCES

1. R. Ibbotson and C. Fall. "The United States Market Wealth Portfolio." *Journal of Portfolio Management* 6 (Fall 1979), 82–92.
2. J. Jobson and B. Korkie. "Portfolio Performance and Tests of Portfolio Efficiency." *Journal of Financial Economics* 10 (December 1982), 433–66.
3. S. Kandel. "On the Exclusion of Assets from Tests of the Mean Variance Efficiency of the Market Portfolio." *Journal of Finance* 39 (March 1984), 63–75.
4. R. Roll. "A Critique of the Asset Pricing Theory's Tests." *Journal of Financial Economics* 4 (March 1977), 129–76.
5. J. Shanken. "Proxies and Asset Pricing Relations: Living with the Roll Critique." Working Paper, Graduate School of Management, University of Rochester, 1985.