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Benchmark Portfolio Inefficiency and Deviations from the Security Market Line

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ABSTRACT

This paper theoretically evaluates the robustness of the Security Market Line relationship when the market proxy employed is not mean-variance efficient. The analysis focuses on the behavior of the "benchmark errors," the deviations of assets and portfolios from the Security Market Line. First, we characterize how the location of an asset in mean-variance space determines its benchmark error. Then the continuity properties of the benchmark errors are studied. The results indicate that the magnitudes of the errors exhibit continuous but not uniformly continuous behaviors. The relative rankings based on deviations from the Security Market Line, however, exhibit some severe discontinuities. In fact, these can be exactly reversed for two proxies arbitrarily close in mean-variance space.

THE USE OF THE Security Market Line (SML) relationship in performance measurement and asset pricing tests has been a subject of controversy since the important work of Roll [6, 7]. To some extent, this debate has centered on the robustness of this relationship to misspecifications of the "market portfolio." Roll [7] and others have argued that performance measures based on deviations from the SML in mean-beta space are extremely sensitive to the specification of empirical proxies for the market and can, therefore, be grossly misleading.

The purpose of this paper is to examine further, from a theoretical perspective, the nature and degree of this sensitivity. Expressions are derived for deviations from the SML, or "benchmark errors," which make it easy to relate their magnitude and sign to the asset's position in mean-variance space. Two results are proved in this regard. The first shows that the benchmark errors for portfolios located at a given point in mean-variance space all lie within a particular interval. This interval is centered on the benchmark error for the minimum-variance portfolio with the same mean. The size of this interval depends on the reward to variability ratio for the market proxy, and on how far from the frontier the proxy

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and asset under study are located. Further, for any number within the interval there exists a portfolio with that benchmark error, and the given mean and variance. Thus, the benchmark error for an arbitrary asset is systematically related to the benchmark error of the minimum-variance portfolio with the same expected return. Accordingly, the second theorem characterizes the benchmark errors of these minimum-variance portfolios. They depend linearly on the portfolio's expected return. Together, these results provide a complete description of what benchmark errors occur at each location in mean-variance space, for a given market proxy. Using them, the paper elaborates on the graphical analytics developed in Dybvig and Ross [1].

The paper then applies these results to evaluate the continuity properties of the benchmark errors. An understanding of these properties is important for two reasons. First, economic theory is always concerned with the robustness of its predictions to perturbations. Second, in applications such as performance measurement or capital budgeting, discontinuous behaviors can lead to evaluations and standards which appear arbitrary or whimsical. Divisions, managers, or projects which attain top ratings under one regime should not achieve bottom ratings under a very similar regime. At least evaluators should be aware of the source of such pathologies.

The results on continuity are mixed. Proxies which are close in mean-variance space, *and* highly correlated, will produce benchmark errors which do not differ by much. It is significant, however, that the degree to which they do differ depends on where the proxies are in mean-variance space, as well as on the means and variances of the assets or portfolios being evaluated. Speaking more formally, as the proxies approach each other in mean and variance, and as their correlation approaches unity, the differences between the benchmark errors they produce converge to zero, but this convergence is not uniform.

Closeness in mean and variance need not imply high correlation. Thus, proxies similar only in their first two moments need not produce similar benchmark errors. The potential differences across proxies can, in fact, be decomposed into portions due to imperfect correlation and due to differences in mean and variance, and the paper conducts such an exercise with sample estimates from historical data. One situation, however, where closeness in mean-variance space does imply high correlation is when both proxies are close to the minimum-variance frontier. This is shown to imply that as the proxy portfolio approaches the frontier, the benchmark errors it produces converge to zero, but again, this convergence is not uniform.

Often, in applications, one is interested not just in determining which projects or portfolio managers are good, but also in determining which are better or best. If the benchmark errors themselves generally exhibit continuous behaviors, rankings based on them manifest some striking discontinuities. The situations where these pathologies arise can be characterized very simply in terms of the excess return per unit of variance on the proxy and on the minimum-variance portfolio with the proxy's expected return. When one requires only that the proxies be near each other in mean-variance space, there is enough freedom to show that ranking reversals can occur for all portfolios simultaneously. This analysis allows us to prove a much stronger result concerning reversals of rankings than those in Roll [7] or Dybvig and Ross [1]. We show that, for any inefficient proxy portfolio, there are two other proxies arbitrarily close in mean-variance space, and two associated zero-beta rates arbitrarily close to each other, which exactly reverse the relative rankings of all returns.

The paper is organized as follows. Section I introduces the notation and studies the relation between the locations of the proxy and an asset in mean-variance space, and the asset's benchmark error. Section II considers continuity questions. A brief conclusion is contained in Section III.

I. Benchmark Errors for a Particular Mean and Variance

In this section, we introduce notation and describe a simple and useful way to decompose portfolio returns. Any return can be written as a minimum-variance return plus a "residual." The added residual term has an expected payoff of zero and can be viewed as an "arbitrage" portfolio, since it has weights which sum to zero. Thus, it only adds variance to the position. This decomposition provides a simple way to relate the behavior of an arbitrary portfolio to that of the associated minimum-variance portfolio. A great deal is known about the latter from the classic works of Markowitz [4], Merton [5], Roll [6, 7], and Ross [9]. The remainder of this section uses the decomposition and the classical results to characterize the relationship between the location of an asset in mean-variance space and its benchmark error. The benchmark error is simply the asset's vertical distance, in mean-beta space, from the SML associated with the proxy. We first show that an asset's benchmark error is related to that of the minimum-variance return with the same mean. We then show that the benchmark errors for the minimum-variance returns have a very simple characterization. Finally, a graphical interpretation of the results is provided along the lines suggested in Dybvig and Ross [1].

We will consider the set of portfolio returns which are generated by $n \geq 4$ primitive assets with returns $\tilde{r}_i$, $i = 1, \cdots, n$.[1] To avoid the obvious degeneracies, we assume these returns do not all have the same mean and have a positive definite covariance matrix. Portfolio returns are subscripted with lower case letters (e.g., $\tilde{r}_p$, $\tilde{r}_q$, or $\tilde{r}_\theta$) and are simply linear combinations of the primitive assets where the weights sum to one. The mean return on portfolio p is denoted as $\bar{r}_p$, its variance as σ_p^2, its standard deviation as σ_p, and its covariance with portfolio q as σ_{pq}. The "beta" of portfolio p relative to another portfolio q is given as $\beta_{pq} = \sigma_{pq}/\sigma_q^2$. In general, we will use Greek letters to subscript the returns of minimum-variance portfolios. Thus, $\tilde{r}_\theta$ has the characteristic that $\sigma_\theta^2 \leq \sigma_p^2$ for any feasible portfolio p such that $\bar{r}_p = \bar{r}_\theta$.

Consider, then, an arbitrary return, $\tilde{r}_p$, and the minimum-variance return with the same mean, $\tilde{r}_\theta$. If we define $\tilde{\Delta}_p = \tilde{r}_p - \tilde{r}_\theta$, it is obvious that we can write $\tilde{r}_p = \tilde{r}_\theta + \tilde{\Delta}_p$ where $E(\tilde{\Delta}_p) = 0$. As the difference between two returns, $\tilde{\Delta}_p$ is an arbitrage portfolio. It is also uncorrelated with $\tilde{r}_\theta$ and every other portfolio on the minimum-variance frontier. This can be inferred from the fact that the covariance between

[1] The assumption that $n \geq 4$ is needed to apply the Projection Theorem to a set of return differences. See the proof of Theorem 1 in the Appendix.

any minimum-variance portfolio and any other portfolio with the same mean is the former's variance.[2] Thus,

$$\sigma_\theta^2 = \sigma_{p\theta} = \text{Cov}(\tilde{r}_\theta + \tilde{\Delta}_p, \tilde{r}_\theta) = \text{Cov}(\tilde{r}_\theta, \tilde{r}_\theta) + \text{Cov}(\tilde{r}_\theta, \tilde{\Delta}_p) \tag{1}$$

which implies $\text{Cov}(\tilde{r}_\theta, \tilde{\Delta}_p) = 0$. Now, let $\tilde{r}_\gamma$ be any other minimum-variance portfolio. The portfolio return, $\tilde{r}_v = \tilde{r}_\gamma + \tilde{\Delta}_v$, is clearly feasible since $\tilde{\Delta}_v$ has weights which sum to zero. Since $\tilde{r}_v$ and $\tilde{r}_\gamma$ have the same mean, the arguments which led to (1) again imply $\text{Cov}(\tilde{\Delta}_p, \tilde{r}_\gamma) = 0$. This provides the decomposition we need. Any return can be expressed as a minimum-variance return plus a term which has a mean of zero and which is uncorrelated with the frontier portfolios.

Now, suppose $\tilde{r}_p$ is the return on the (inefficient) market proxy employed as a benchmark, and let $\tilde{r}_\theta$ denote the minimum-variance return which has $\bar{r}_\theta = \bar{r}_p$. Similarly, we will use $\tilde{r}_v$ to denote the asset or portfolio return being evaluated and $\tilde{r}_\gamma$ as the frontier return with $\bar{r}_\gamma = \bar{r}_v$. We have just seen we can write $\tilde{r}_p = \tilde{r}_\theta + \tilde{\Delta}_p$ and $\tilde{r}_v = \tilde{r}_\gamma + \tilde{\Delta}_v$. Since $\bar{r}_p = \bar{r}_\theta$ and $\tilde{\Delta}_p$ is orthogonal to $\tilde{r}_\theta$, the variance of the residual term, $\sigma^2(\tilde{\Delta}_p) = (E\,\tilde{\Delta}_p^2) = \sigma_p^2 - \sigma_\theta^2$, is the additional variance associated with moving inside the frontier. It serves as a natural measure of the "inefficiency" of $\tilde{r}_p$. We will see that this distance of the proxy from the frontier is an important determinant of the magnitude of potential benchmark errors.

Let $\bar{r}_{z\theta}$ denote the expected return on the "zero-beta" portfolio associated with $\tilde{r}_\theta$. Since $\tilde{r}_\theta$ is minimum-variance, this expected return is uniquely determined. We will denote as $\bar{r}_{zp}$ the zero-beta rate (arbitrarily) chosen to use with the proxy, $\tilde{r}_p$.[3] Figure 1 illustrates how the various returns we are interested in might plot in mean-standard deviation space.

Finally, we will let ε_{vp} denote the deviation of portfolios v's expected return from the SML generated by proxy portfolio p. To determine the sources of this error, note that since $\tilde{r}_\theta$ is minimum-variance, the SML relation holds exactly[4] and

$$\bar{r}_v = \bar{r}_{z\theta} + \beta_{v\theta}[\bar{r}_\theta - \bar{r}_{z\theta}]. \tag{2}$$

The inefficient proxy, on the other hand, generates benchmark errors:

$$\bar{r}_v = \bar{r}_{zp} + \beta_{vp}[\bar{r}_p - \bar{r}_{zp}] + \varepsilon_{vp}. \tag{3}$$

Subtracting (3) from (2), and adding and subtracting $\bar{r}_{z\theta}$ to $\bar{r}_p - \bar{r}_{zp}$ in (3), gives

$$\varepsilon_{vp} = (\bar{r}_{z\theta} - \bar{r}_{zp})(1 - \beta_{vp}) + (\bar{r}_\theta - \bar{r}_{z\theta})[\beta_{v\theta} - \beta_{vp}]. \tag{4}$$

The benchmark errors due to proxy inefficiency arise from two sources, misspecification of the zero-beta rate and getting the "wrong beta." Each of these, in turn, can be attributed to movement along the frontier and movement inside the frontier, by asking how ε_{vp} is related to the benchmark error for the minimum-variance return with the same mean, $\varepsilon_{\gamma p}$. Define the "market price of

[2] The variance of a portfolio with weights w on $\tilde{r}_p$ and $(1 - w)$ on $\tilde{r}_\theta$ must be minimized at $w = 0$, and the associated first-order condition yields $\sigma_{p\theta} = \sigma_\theta^2$.

[3] Roll [8] and Kandel [3] have emphasized the ambiguity associated with the zero-beta rate when the proxy is inefficient. In fact, there exist zero-beta portfolios with any given expected return.

[4] Roll [6] and Ross [9] emphasize the mathematical equivalence between the exact SML relationship and the position of the proxy on the frontier.

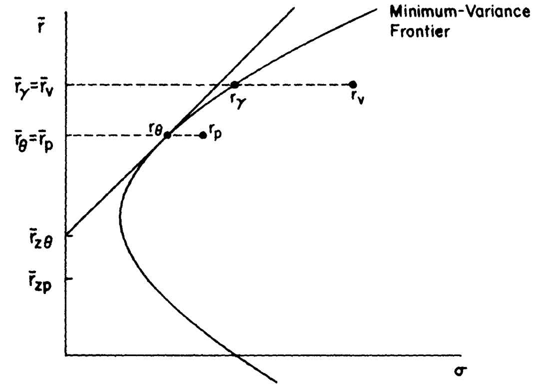

Figure 1. The market proxy being employed, $\tilde{r}_p$, has the same mean return but a larger variance than the frontier portfolio, $\tilde{r}_\theta$. Similarly, the asset being evaluated, $\tilde{r}_V$, has the same mean as $\tilde{r}_\gamma$.

risk" for the frontier portfolio, $\tilde{r}_\theta$, and the proxy, $\tilde{r}_p$, as $\lambda_\theta = (\bar{r}_\theta - \bar{r}_{z\theta})/\sigma_\theta^2$ and $\lambda_p = (\bar{r}_p - \bar{r}_{zp})/\sigma_p^2$, respectively. The following lemma gives a decomposition for the benchmark error which corresponds to the decomposition for the returns themselves. Through it, we can fully characterize which benchmark errors occur at any given point in mean-variance space. The proofs of this, and subsequent lemmas and theorems are in the Appendix.

LEMMA 1. *The benchmark error for an arbitrary return, $\tilde{r}_v$, is related to the benchmark error for the minimum-variance return with the same mean, $\tilde{r}_\gamma$, as follows:*

$$\varepsilon_{vp} = \varepsilon_{\gamma p} + \lambda_p \mathrm{Cov}(\tilde{\Delta}_v, \tilde{\Delta}_p). \tag{5}$$

In general, there is no reason for $\tilde{\Delta}_p$ and $\tilde{\Delta}_v$ to covary in any particular way. However, the absolute magnitude of $\mathrm{Cov}(\tilde{\Delta}_v, \tilde{\Delta}_p)$ is restricted by the need for the correlation coefficient to lie between −1 and +1. Given that $\tilde{r}_v$ lies $\sigma^2(\tilde{\Delta}_v)$ units of variance within the frontier, perfect negative correlation between $\tilde{\Delta}_v$ and $\tilde{\Delta}_p$ would imply

$$\varepsilon_{vp} = \varepsilon_{\gamma p} - \lambda_p \sigma(\tilde{\Delta}_v)\sigma(\tilde{\Delta}_p). \tag{6}$$

Perfect positive correlation would result in

$$\varepsilon_{vp} = \varepsilon_{\gamma p} + \lambda_p \sigma(\tilde{\Delta}_v)\sigma(\tilde{\Delta}_p). \tag{7}$$

Putting these expressions together, we have:

$$\varepsilon_{\gamma p} - \lambda_p \sigma(\tilde{\Delta}_v)\sigma(\tilde{\Delta}_p) \le \varepsilon_{vp} \le \varepsilon_{\gamma p} + \lambda_p \sigma(\tilde{\Delta}_v)\sigma(\tilde{\Delta}_p). \tag{8}$$

Thus, the benchmark errors for portfolios at a given point in mean-variance space lie in an interval centered on the benchmark error for the minimum-variance return with that mean. What is perhaps more surprising is that, for any value within this interval, there will be a feasible portfolio having that benchmark error and occupying the given location in mean-variance space. The proof of this exploits our freedom to choose $\tilde{\Delta}_v$ to have any correlation with $\tilde{\Delta}_p$ between one and negative one.

THEOREM 1. *At any feasible point in mean-variance space, $[\bar{r}_\gamma, \sigma_\gamma^2 + \sigma^2(\tilde{\Delta}_v)]$, there exists a portfolio with benchmark error, ε, for any ε in the closed interval $\varepsilon_{\gamma p} \pm \lambda_p \sigma(\tilde{\Delta}_v)\sigma(\tilde{\Delta}_p)$, and there are no portfolios with benchmark errors outside this interval.*

This theorem generalizes and clarifies a result in Dybvig and Ross [1] which showed that, for variances beyond a certain (unspecified) distance from the frontier, the benchmark error could be either positive or negative. Closer to the frontier, they showed that the errors had to be one sign or the other depending on the sign of the error for the minimum-variance portfolios nearby. Theorem 1 makes clear where these restrictions come from. We can choose the residual term, $\tilde{\Delta}_v$, of an asset so that it covaries either positively or negatively with the proxy portfolio's residual component, $\tilde{\Delta}_p$, but we cannot make this covariance large in absolute value without making the variance of the residual term large and increasing the variance of the return.

Theorem 1 would be more useful if we could characterize the benchmark errors for minimum-variance portfolios. The next result accomplishes this. Together, Theorems 1 and 2 provide a complete description of what benchmark errors occur at each point in mean-variance space. The intuition behind the result is quite simple. By the "Two-Fund Theorem" (Merton [5, p. 1858]), the minimum-variance frontier is spanned by two portfolios. As noted in Roll [7], both expected returns and covariances combine linearly, so portfolios of these two portfolios must line up in mean-beta space, whether or not the proxy is efficient. We will refer to this line as the Minimum-Variance Line. The benchmark errors for the minimum-variance portfolios are simply the distances between this line and the SML. These distances are, in turn, linear functions of expected return. The relative slopes of the two lines are determined by λ_p and λ_θ, the "market prices of risk."[5]

THEOREM 2. *The deviations from the SML for the minimum-variance portfolios are given by the following linear equation in expected return:*

$$\varepsilon_{\gamma p} = \bar{r}_\gamma[1 - (\lambda_p/\lambda_\theta)] + \bar{r}_{z\theta}(\lambda_p/\lambda_\theta) - \bar{r}_{zp}. \tag{9}$$

From Equation (9), it is clear that as we move along the mean-variance frontier in the direction of higher expected return, we increase the benchmark error if $\lambda_p < \lambda_\theta$ and decrease it if $\lambda_p > \lambda_\theta$. The reasons for this are illustrated in mean-beta space in Figure 2. The Minimum-Variance Line must lie to the left of the SML at the mean of the proxy.[6] The case $\lambda_p > \lambda_\theta$ corresponds to SML_A in Figure 2, where the slope of the SML is steeper than that of the Minimum-Variance Line. SML_C, on the other hand, rises less steeply, and $\varepsilon_{\gamma p}$ is increasing in $\bar{r}_\gamma$. When the Minimum-Variance Line and the SML are parallel, as with SML_B in Figure 2, all frontier portfolios plot the same distance above the SML. In (9), $\lambda_p = \lambda_\theta$ and the coefficient on $\bar{r}_\gamma$ is zero. Notice that as we move the zero-beta rate through

[5] We focus, for expositional purposes, on the case where the market prices of risk, λ_p and λ_θ, are both positive. The case where $\lambda_\theta < 0$ yields results which simply mirror these with the signs reversed, as is clear from Dybvig and Ross [1].

[6] This can be shown by substituting $\bar{r}_\theta$ into (9) and reducing $\varepsilon_{\theta p}$ to $\lambda_p(\sigma_p^2 - \sigma_\theta^2) > 0$. It also follows from Theorem 1 of Dybvig and Ross [1].

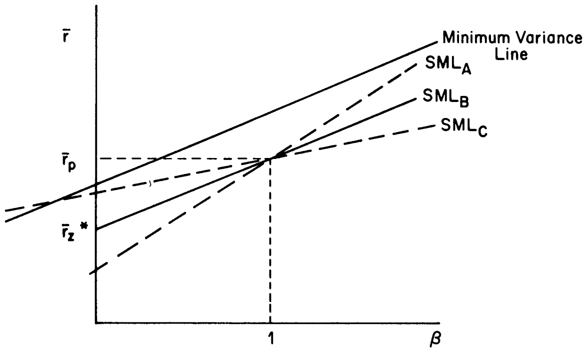

Figure 2. The relative slopes of the SML and minimum-variance line determine the relationship between mean (or beta) and benchmark error for frontier portfolios. Using zero-beta, $\bar{r}_z^*$, the lines are parallel.

the point, $\bar{r}_z^*$ in Figure 2, where the two lines are parallel, we reverse the relation between expected return (or beta) and the benchmark error. This leads to the discontinuities in rankings based on deviations from the SML which are discussed in Section II. Also, note that as we pass through the parallel case where $\lambda_\theta = \lambda_p$, the intersection between the SML and the Minimum-Variance Line jumps from a very low to a very high expected return.

This intersection locates a portfolio which plays a central role in the analysis in Dybvig and Ross [1]. It is the minimum-variance return which plots on the SML. Theorems 1 and 2 allow us to generalize their graphical analysis and divide up mean-variance space into areas with different benchmark errors. The minimum-variance frontier for the subset of portfolios that plot on the SML will be a hyperbola enveloped by the frontier for all assets. The point at which the two hyperbolas are tangent gives the mean at which the SML and Minimum-Variance Lines intersect in $\bar{r} - \beta$ space which we will denote as $\bar{r}_0$. Figure 3 illustrates three possible cases from Figure 4 in Dybvig and Ross [1].[7] These correspond to the three cases depicted in Figure 2. In Case A, where the tangency occurs at $\bar{r}_0 > \bar{r}_p$, $\lambda_\theta < \lambda_p$ so Equation (9) implies the benchmark errors will be increasing as we move down the frontier. By Theorem 2, the benchmark errors must also be increasing in that direction for portfolios close to the frontier [i.e., where $\sigma^2(\tilde{\Delta}_v)$ is small] because they have benchmark errors close to those of the minimum-variance portfolio with the same mean. Thus, portfolios between the two frontiers and below the tangency must, in Case A, have positive benchmark errors. The situation is reversed in Case C, and Case B is the "razor's edge" situation where we pass between A and C, $\lambda_\theta = \lambda_p$, and the two lines in $\bar{r} - \beta$ space are parallel.

Figure 4 generalizes this analysis for the case $\lambda_p < \lambda_\theta$. The three interior hyperbolas represent, from top to bottom, the minimum-variance frontiers for portfolios with benchmark errors of ε, 0, and $-\varepsilon$, respectively. Each has a single tangency. Since (9) is linear in $\bar{r}_\gamma$, there will be only one frontier portfolio with any given benchmark error. The portfolios that generate the interior hyperbolas are already familiar to us from the construction of the bounds in Theorem 1.

[7] Case C in Figure 4 of Dybvig and Ross [1] does not differ from Case A for our purposes.

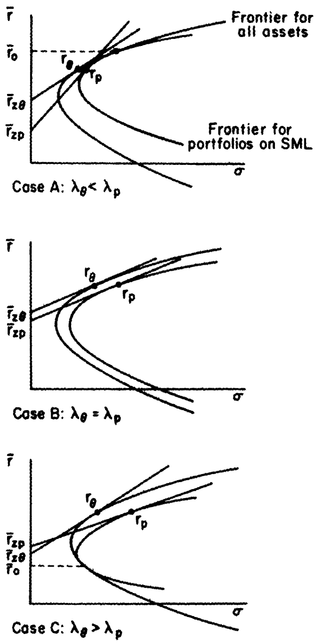

Figure 3. The relative positions of the frontiers for all portfolios, and for the subspace of portfolios with zero benchmark errors, depends on the reward to variability ratio for the proxy and the frontier return with the same mean. The mean of the frontier portfolio which plots on the SML is $\bar{r}_0$.

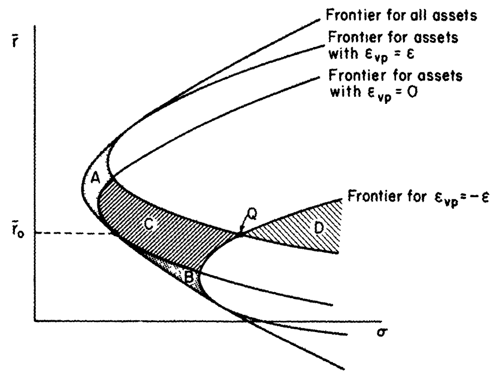

Figure 4. The minimum-variance frontiers for portfolios with given benchmark errors subdivide the feasible region.

They are the portfolios which have residual terms with the lowest variance for a given covariance with $\tilde{\Delta}_p$. Such portfolios will have residual terms which are perfectly correlated with the residual for the proxy. These are also the portfolios which bound the benchmark errors at any point in mean-variance space, since they have residual terms with maximal or minimal covariance for a given variance. Thus, at point Q in Figure 4, where the hyperbolas for $\varepsilon_{vp} = +\varepsilon$ and $\varepsilon_{vp} = -\varepsilon$ intersect, one could find a portfolio with every benchmark error in the interval $[-\varepsilon, +\varepsilon]$. Portfolios close to the frontier and between tangency points have benchmark errors that lie between those of the tangency portfolios. For example, area A contains portfolios with $0 \leq \varepsilon_{vp} \leq +\varepsilon$ and in area B are portfolios with $-\varepsilon \leq \varepsilon_{vp} \leq 0$. Further away, in area C, we have portfolios with either positive or negative benchmark errors, but all of them within the interval $-\varepsilon$ to $+\varepsilon$. Only in area D, where the interiors for all three hyperbolas intersect, would we find portfolios with $\varepsilon_{vp} > \varepsilon$ or $\varepsilon_{vp} < -\varepsilon$ occupying the same location in mean-variance space. How quickly the interior hyperbolas "bend back" from the minimum-variance frontier is determined, according to Theorem 1, by the magnitude of $\sigma(\tilde{\Delta}_p)$, the distance the proxy lies inside the frontier, and the proxy's market price of risk, λ_p. The larger these quantities, the "wider" the hyperbolas are in Figure 4.

II. Continuity Properties of Benchmark Errors

This section applies the results above to evaluate the continuity properties of the benchmark errors. Do "similar" proxies generate "similar" benchmark errors? The answer depends, perhaps not surprisingly, on how one formalizes the notion of "similar."

The results obtained can be useful in several ways. Some of the bounds derived involve such quantities as "the distance from the minimum-variance frontier," which are unobservable in the context of testing the CAPM. Nevertheless, an understanding of the sources of discontinuous or pathological behaviors in terms of these quantities may still be of use to practitioners and empiricists in evaluating, based on informal or nonstatistical information, what measures or conclusions are likely to be robust. Other results involve quantities which can be estimated. Thus, these bounds could be useful in ensuring consistency in applications like performance evaluation.

Consider two candidate proxy portfolios, $\tilde{r}_p$ and $\tilde{r}_q$. With each proxy, we can associate a zero-beta rate, $\bar{r}_{zp}$ and $\bar{r}_{zq}$, respectively. Given these, the means, and the variances, we can calculate $\lambda_p = (\bar{r}_p - \bar{r}_{zp})/\sigma_p^2$ and similarly, λ_q. Using Equation (3) to solve for the difference in the benchmark errors, ε_{vp} and ε_{vq}, gives:

$$\begin{aligned}\varepsilon_{vq} - \varepsilon_{vp} &= (\bar{r}_{zp} - \bar{r}_{zq}) + \lambda_p\sigma_{vp} - \lambda_q\sigma_{vq} \\ &= (\bar{r}_{zp} - \bar{r}_{zq}) + (\lambda_p - \lambda_q)\sigma_{vp} + \lambda_q(\sigma_{vp} - \sigma_{vq}). \qquad (10)\end{aligned}$$

The second term in this expression can be rewritten as

$$\begin{aligned}\sigma_{vp}(\lambda_p - \lambda_q) &= \sigma_{vp}\left[\frac{\sigma_q^2(\bar{r}_p - \bar{r}_{zp}) - \sigma_p^2(\bar{r}_q - \bar{r}_{zq})}{\sigma_p^2\sigma_q^2}\right] \\ &= \frac{\sigma_{vp}}{\sigma_q^2}\left[\lambda_p(\sigma_q^2 - \sigma_p^2) + (\bar{r}_p - \bar{r}_q) + (\bar{r}_{zq} - \bar{r}_{zp})\right]. \qquad (11)\end{aligned}$$

Thus, the first two terms in (10) can be expressed as the sum of terms which are proportional to the differences between the means, variances, and zero-beta rates for the two proxies. If the proxies are close in mean-variance space, and if the zero-beta rates are close, these terms will be small.

The third term in (10) cannot be expressed in terms of the means and variances alone. We can, however, bound this term with an expression that depends on the standard deviations and the correlation between the proxies ρ_{pq}. Since $\sigma^2(\tilde{r}_p -$

$\tilde{r}_q) = \sigma_p^2 + \sigma_q^2 - 2\sigma_{pq}$,

$$\begin{aligned}\lambda_q \,|\, \sigma_{vp} - \sigma_{vq} \,| &= \lambda_q \,|\, \mathrm{Cov}(\tilde{r}_v, \tilde{r}_p - \tilde{r}_q) \,| \\ &\leq \lambda_q \sigma_v \sigma(\tilde{r}_p - \tilde{r}_q) \\ &= \lambda_q \sigma_v \left\{ \sigma_p \sigma_q \left[\frac{\sigma_p}{\sigma_q} + \frac{\sigma_q}{\sigma_p} - 2\rho_{pq} \right] \right\}^{1/2}. \end{aligned} \tag{12}$$

As σ_p approaches σ_q, and ρ_{pq} approaches unity, the expression in square brackets, $[\sigma_p/\sigma_q + \sigma_q/\sigma_p - 2\rho_{pq}]$, goes to zero.

Thus, taken together, expressions (10), (11), and (12) are a demonstration that the benchmark errrors are continuous in the proxy, for any norm which is strong enough to imply closeness in mean, variance, and correlation. Examples of such norms are mean-squared difference, $E\{(\tilde{r}_p - \tilde{r}_q)^2\}$, and the Euclidean norm in portfolio space, $\sum_{i=1}^{n} (w_{pi} - w_{qi})^2$ where w_{pi} and w_{qi} are the weights on asset i in portfolios p and q, respectively. As these measures of distance between the proxies approach zero, the benchmark errors they generate will converge. They will not, however, converge uniformly, and the sources of this lack of uniform convergence can be instructive about potential problems in applications. First, from Equation (11), we see that the differences between the benchmark errors, ε_{vp} and ε_{vq}, has a term proportional to σ_{vp}. Thus, as we move across similar proxies, we are more likely to see big changes in benchmark errors for assets with extreme betas (or expected returns). In a performance measurement or capital budgeting context, this clearly puts certain types of managers or projects at greater risk of inconsistent evaluations. The same holds for assets with high "residual" risk, since the bound in (12) is proportional to the standard deviation of the portfolio being evaluated.

Equation (10), along with the bound in (12), are simple to implement in

Table I

Descriptive Statistics for Market Proxies

	VWNYSE	EWNYSE	Stocks, Bonds, and Bills	S&P 500
1/72–12/76				
Mean[a]	0.49272	0.71489	0.54235	0.5155
Standard deviation[a]	5.11	7.22	1.96	4.93
Correlation with VWNYSE	1.0	0.873	0.873	0.996
Mean TBILL return[a] 0.48067				
1/77–12/81				
Mean[a]	0.8625	1.5724	0.4009	0.7365
Standard deviation[a]	4.37	5.32	2.64	4.20
Correlation with VWNYSE	1.0	0.932	0.696	0.997
Mean TBILL return[a] 0.77383				

[a] Monthly returns multiplied by 100.

Table II

Benchmark Errors for Alternative Market Proxies

			EWNYSE		Stocks, Bonds, and Bills		S&P 500	
	Beta on VWNYSE	Benchmark Error Using VWNYSE[a]	Change in Benchmark Error[a]	Due to Corr.	Change in Benchmark Error[a]	Due to Corr.	Change in Benchmark Error[a]	Due to Corr.
1/72–12/76								
VWNYSE	1.000	0.00000	−0.1053 ± 0.085	19%	−0.4072 ± 0.300	5%	−0.0257 ± 0.004	8%
Small Stocks	1.240	0.34759	−0.1303 ± 0.133	22%	−0.5038 ± 0.451	5%	−0.3139 ± 0.005	1%
GM	0.809	0.28776	−0.0852 ± 0.113	25%	−0.3295 ± 0.384	6%	−0.0205 ± 0.005	12%
IBM	0.883	−0.03040	−0.0930 ± 0.109	23%	−0.3596 ± 0.369	6%	−0.0224 ± 0.005	11%
ATT	0.563	0.72900	−0.0593 ± 0.072	24%	0.2292 ± 0.244	6%	0.0143 ± 0.003	11%
1/77–12/81								
VWNYSE	1.000	0.0000	−0.4500 ± 0.249	19%	1.1100 ± 0.740	18%	0.1290 ± 0.003	1%
Small Stocks	1.350	1.4770	−0.6096 ± 0.389	21%	1.5040 ± 0.156	4%	0.1748 ± 0.005	1%
GM	0.538	−1.1660	−0.2379 ± 0.319	30%	0.5870 ± 0.948	28%	0.0682 ± 0.004	3%
IBM	0.700	−0.6495	−0.3151 ± 0.286	25%	0.7774 ± 0.850	24%	0.0904 ± 0.004	4%
ATT	0.119	−0.1586	−0.0534 ± 0.183	41%	0.1316 ± 0.543	36%	0.0153 ± 0.003	9%

[a] Mean monthly return multiplied by 100.

practice. They translate the potential changes in benchmark errors into the kinds of summary statistics which are routinely produced for widely used market indices; namely, correlations, means, and standard deviations. Suppose, for example, we have estimated betas for a set of assets with respect to a particular proxy and have used them to calculate "abnormal performance" measures. We wish to know how different these measures might be with an alternative market index. The first two terms of (10) can be calculated directly, and the absolute value of the final term can be bounded using (12). Tables I and II report the results of such an exercise using four different market indices. The first two indices are the value-weighted (VWNYSE) and equally weighted (EWNYSE) portfolios of common stocks from the CRSP data base. The third index is an equally weighted portfolio of four monthly return series, the value-weighted CRSP portfolio along with the treasury bill, long-term government bond, and long-term corporate bond series from Ibbotson and Sinquefield [2]. The fourth index is a dividend adjusted return series based on the Standard and Poor's 500, also from Ibbotson and Sinquefield [2]. It is included because performance ratings for portfolio managers are often tied to the S&P 500.

Table I gives summary statistics for these indices over two recent five-year periods. The means and standard deviations are nominal monthly percentage returns. Table II then describes how the benchmark errors for five different "assets" vary across the four market proxies. The first of these assets is the value-weighted stock index itself. The second is the small stock portfolio from Ibbotson and Sinquefield [2]. The last three are the common stocks of large corporations from the CRSP monthly returns data base. We use the value-weighted stock series as our base index [portfolio p in (10)]. The first two columns of Table II report betas calculated over the subperiods, along with the benchmark errors, using the VWNYSE as a proxy. For a zero-beta rate, we use the mean return on the Ibbotson and Sinquefield [2] T-bill portfolio, and we keep this fixed across all the proxies. Thus, the first term in (10) is always zero.

The remaining columns of Table II report, for each alternative market index, two quantities. The first of these is an interval describing the possible changes in the value of the benchmark error. The first term in this column is the second term in (10), the change in the benchmark error which is proportional to the difference in the market prices of risk for the two proxies. This term is of known sign and magnitude. The second term is the bound given in (12) on the third term of (10). For each proxy, Table II then gives the percentage by which the maximum possible change in the benchmark error would be reduced if the proxies were perfectly correlated. As is clear from the table, the contribution of the imperfect correlation tends to be fairly small.

In general, closeness of the proxies in mean and variance is not a strong enough notion of "similarity" to yield similar benchmark errors. This is because portfolios can be close in mean-variance space, and not be highly correlated. Suppose, for example, that both have means of $\bar{r}_\theta$ and lie $\sigma^2(\tilde{\Delta}_p)$ units of variance within the frontier. Thus, $\tilde{r}_p = \tilde{r}_\theta + \tilde{\Delta}_p$ and $\tilde{r}_q = \tilde{r}_\theta + \tilde{\Delta}_q$ where $E(\tilde{\Delta}_p^2) = E(\tilde{\Delta}_q^2) = \sigma^2(\tilde{\Delta}_p)$ and $\mathrm{Cov}(\tilde{\Delta}_p, \tilde{r}_\theta) = \mathrm{Cov}(\tilde{\Delta}_q, \tilde{r}_\theta) = 0$. Then

$$\rho_{pq} = \frac{\sigma_{pq}}{\sigma_p \sigma_q} = \frac{\sigma_\theta^2 + \mathrm{Cov}(\tilde{\Delta}_p, \tilde{\Delta}_q)}{\sigma_\theta^2 + \sigma^2(\tilde{\Delta}_p)} \tag{13}$$

since $\tilde{\Delta}_p$ and $\tilde{\Delta}_q$ both have variance $\sigma^2(\tilde{\Delta}_p)$, the largest the covariance in the numerator can be is $\sigma^2(\tilde{\Delta}_p)$ and the smallest (i.e., for $\tilde{\Delta}_q = -\tilde{\Delta}_p$) would be $-\sigma^2(\tilde{\Delta}_p)$. For proxies well within the interior of the feasible region, the lower bound on the correlation $[\sigma_\theta^2 - \sigma^2(\tilde{\Delta}_p)]/[\sigma_\theta^2 + \sigma^2(\tilde{\Delta}_p)]$ could be considerably less than unity.

Proxies close in mean and variance will be highly correlated, however, if they are close to the minimum-variance frontier. The correlation between any proxy, $\tilde{r}_p$, and the minimum-variance portfolio with the same mean is

$$\rho_{\theta p} = \frac{\sigma_\theta^2}{[\sigma_\theta^2(\sigma_\theta^2 + \sigma^2(\tilde{\Delta}_p))]^{1/2}} \tag{14}$$

which approaches unity as $\sigma^2(\tilde{\Delta}_p)$ goes to zero.

This suggests that we can bound the benchmark errors in terms of $\sigma(\tilde{\Delta}_p)$, the standard deviation of the residual. This quantity measures the proxy's distance from the frontier. To focus on the effect of the proxy's inefficiency alone, set $\bar{r}_{zp} = \bar{r}_{z\theta}$. Then Equation (A3) in the Appendix will yield, for this case:[8]

$$\varepsilon_{vp} = \frac{\bar{r}_\theta - \bar{r}_{z\theta}}{\sigma_p^2} \{\beta_{v\theta}\sigma^2(\tilde{\Delta}_p) - \operatorname{Cov}(\tilde{\Delta}_v, \tilde{\Delta}_p)\}. \tag{15}$$

The Cauchy-Schwartz inequality allows us to bound this expression as:[9]

$$|\varepsilon_{vp}| \leq |\lambda_\theta| \{|\beta_{v\theta}|\, \sigma(\tilde{\Delta}_p) + \sigma(\tilde{\Delta}_v)\}\sigma(\tilde{\Delta}_p). \tag{16}$$

Thus, as $\sigma(\tilde{\Delta}_p) \to 0$, the benchmark errors approach zero in a continuous fashion. "Almost efficient" proxies do, as one might expect, lead to very small deviations from the SML for all assets. Once again, however, the convergence to zero is not uniform across either proxies or the assets being evaluated. Multiplying the right-hand side of (16) is the reward to variability ratio, λ_θ, for the minimum-variance portfolio. It becomes unbounded for expected returns close to that of the global minimum-variance return, where the slope of the efficient frontier is infinite. Since there have been substantial subperiods where market proxies fail to outperform the "riskless asset," these pathologies may be important when working with *ex post* returns. When the proxy performs poorly, its mean may be close to that of the global minimum-variance return, and even if it is "almost efficient" it can produce large benchmark errors. The bound in (16) also suggests that certain types of assets and portfolios are likely to produce large benchmark errors even if the proxy is close to the frontier; namely, those with extreme betas or those which lie far away from the frontier. It may be possible to exploit this fact in designing tests for mean-variance efficiency with more power.[10]

[8] The first expression on the right side of (A3), with $\bar{r}_{z\theta} = \bar{r}_{zp}$ gives

$$\varepsilon_{vp} = \frac{(\bar{r}_\theta - \bar{r}_{z\theta})}{\sigma_p^2}\left[\frac{\sigma_{v\theta}}{\sigma_\theta^2}[\sigma_\theta^2 + E(\tilde{\Delta}_p^2)] - \sigma_{\gamma p} - E(\tilde{\Delta}_v\tilde{\Delta}_p)\right].$$

Noting that, since $\tilde{r}_\gamma$ and $\tilde{r}_\theta$ are minimum-variance, $\sigma_{\gamma p} = \sigma_{\gamma\theta} = \sigma_{v\theta}$ yields (15).

[9] We have also exploited the fact that $\sigma_p^2 > \sigma_\theta^2$, in deriving (16). This bound is similar to one derived by Shanken [10], through a different approach. The two bounds are non-nested, however. This can be seen by applying them to the proxy portfolio itself, where Shanken's bound is tighter, and to the minimum-variance portfolio with mean $\bar{r}_{z\theta}$, where (16) is tighter.

[10] I thank one of the reviewers for pointing this out.

Subject to these qualifications, the benchmark errors do behave continuously across inefficient proxies and as the proxy approaches the frontier. A somewhat different question involves how the benchmark errors behave relative to each other. Do proxy portfolios which are similar tend to rank assets in the same order? This question is particularly important in performance measurement, where one may be attempting to evaluate not only which managers are doing well, relative to the SML, but also which managers are doing better than others.

Theorem 3 shows that for any given inefficient proxy portfolio, there exists another proxy arbitrarily close in mean-variance space, and associated zero-beta rates which are arbitrarily close to each other, such that the relative rankings provided by the two proxies exactly reverse each other. This result is stronger than similar results in Roll [7] and Dybvig and Ross [1]. The former argues that there is always a proxy which exactly reverses the benchmark errors. The reversing portfolio, however, will in general be very far away from the original proxy, and in many cases one must choose a zero-beta rate which reverses the market price of risk. Dybvig and Ross [1] point out these problems and also illustrate a degenerate case where no reversal is possible. They suggest that an affine reversal of the benchmark errors is always possible, but again, the reversing portfolio may be very far away from the original proxy. Since reversals across two proxies which are dramatically different in mean and variance seem unlikely to mislead or confuse a reasonably circumspect practitioner, it may be of interest to ask when these behaviors are local properties.

Theorem 2 suggests a situation where such a discontinuity may arise. It implies a "reversal" in the ordering of the benchmark errors for minimum-variance portfolios, at the point where $\lambda_p = \lambda_\theta$. There is a corresponding discontinuity in the unique minimum-variance portfolio which lies on the SML. It jumps from a position on the frontier below the proxy portfolio's mean to one above the proxy. For a given proxy portfolio, $\tilde{r}_p$, the market price of risk, λ_p, is determined by the choice of the zero-beta rate. The zero-beta rate which sets $\lambda_p = \lambda_\theta$ will be denoted as $\bar{r}_z^*$:

$$\bar{r}_z^* = \bar{r}_{z\theta} - (\bar{r}_\theta - \bar{r}_{z\theta}) \frac{E(\tilde{\Delta}_p^2)}{\sigma_\theta^2}. \tag{17}$$

If we begin with a zero-beta rate close to $\bar{r}_z^*$, and then perturb it, the SML pivots through the position of the proxy in $\bar{r} - \beta$ space. As Figure 2 illustrates, this may change the relative slopes of the SML and the Minimum-Variance Line, and the distance between them can change from increasing to decreasing with expected return, or vice versa. Thus, the entire relationship between expected return and benchmark error is reversed for portfolios on the frontier. Theorem 1 tells us that these benchmark errors are, in turn, important determinants of the errors for interior portfolios.

In fact, the "reversal" which occurs at $\bar{r}_{zp} = \bar{r}_z^*$ is an affine reversal of the benchmark errors of the minimum-variance portfolios. All of the differences, $\varepsilon_{\gamma p} - \varepsilon_{\delta p}$, for all pairs of minimum-variance returns, $\tilde{r}_\gamma$ and $\tilde{r}_\delta$, can be exactly reversed. Our final result shows that this reversal can be extended to all of the returns without moving very far from the proxy portfolio in mean-variance space, and without changing the sign of the "market price of risk."

THEOREM 3. *For any inefficient proxy, $\tilde{r}_p$, with a mean greater than that of the global minimum-variance return, there exist two portfolios, $\tilde{r}_q$ and $\tilde{r}_s$, arbitrarily close in variance and with mean, $\bar{r}_p$, and two zero-beta rates, $\bar{r}_{zq}$ and $\bar{r}_{zs}$, which are arbitrarily close and imply λ_q, $\lambda_s > 0$, such that: $\varepsilon_{uq} - \varepsilon_{vq} = -[\varepsilon_{us} - \varepsilon_{vs}]$ for any feasible returns, $\tilde{r}_u$ and $\tilde{r}_v$.*

The reversal of relative rankings described in Theorem 3 is an affine reversal of the benchmark errors. That is, $\varepsilon_{vq} = a\varepsilon_{vs} + b$ where $a = -1$ and b is a constant. Dybvig and Ross [1] argue that some affine reversal always exists with $a < 0$, but they note that it may not be possible to solve their system of equations for $a = -1$. Our result shows such a solution can be found, and further that the reversing portfolio can be found arbitrarily close to the original one in mean and variance. We focus on $\bar{r}_{zq}$ and $\bar{r}_{zs}$ close to $\bar{r}_z^*$, but there appears to be nothing "wrong" with this choice since all three zero-beta rates give positive market prices of risk. In fact, $\bar{r}_z^*$ delivers $\lambda_p = \lambda_\theta$, the "true" premium per unit variance earned by the frontier portfolio.

There is at least one version of the CAPM under which this situation of $\lambda_p \approx \lambda_\theta$ would seem likely to arise in applications. Suppose that the CAPM holds with a riskless asset but no personal borrowing at the riskless rate. The market portfolio, $\tilde{r}_M$, will lie on the frontier above the point where the borrowing and lending line is tangent. If the market proxy is within the frontier close to $\tilde{r}_M$, and the risk-free rate, r_f, is used as the zero-beta rate, the line between r_f and $\tilde{r}_p$ in mean-variance space, which has slope λ_p, will lie below the line between $\tilde{r}_\theta$ and its unique zero-beta rate $\bar{r}_{z\theta}$, and may well have about the same slope.

The construction in the proof of Theorem 3 requires freedom to choose the correlation between $\tilde{\Delta}_q$ and $\tilde{\Delta}_s$. Thus, the reversing proxies may not be highly correlated. This might seem to suggest that ranking reversals will not be a problem if investigators are careful to look beyond mean and variance. Two cautions are in order, however. First, if the original proxy is close to mean-variance efficient, the reversing portfolio will be highly correlated with it, as the discussion below Equation (13) indicates. Second, any pair of proxies, $\tilde{r}_s$ and $\tilde{r}_q$, which deliver market prices of risk that straddle the true market price of risk, i.e., $\lambda_s > \lambda_\theta > \lambda_q$ or $\lambda_s < \lambda_\theta < \lambda_q$, rank at least some assets very differently. The arguments in the theorem would still lead to a reversal of the rankings for portfolios on the mean-variance frontier, even if we required $\tilde{r}_q$ and $\tilde{r}_s$ to be highly correlated. Assets off the frontier might not have their relative rankings exactly reversed, but the order of their rankings would no doubt change dramatically because their benchmark errors are tied to those of the minimum-variance portfolios through Theorem 1.

III. Conclusions

This paper has theoretically evaluated the robustness of the SML relationship when proxy portfolios are employed which are not mean-variance efficient. The focus of the analysis has been on where, in mean-variance space, particular "benchmark errors" occur, and on their continuity properties. Our results suggest that, in terms of their overall magnitude, these errors exhibit the types of continuous behavior one might intuitively expect. In terms of relative rankings, however, there are some severe discontinuities. One interpretation of these results would be that performance evaluation is particularly sensitive to the choice of the proxy or benchmark, since how managers perform relative to each other is often of importance to the evaluator.

Appendix

Proof of Lemma 1: The SML relationship implies the covariance between all portfolios with a given mean and a minimum-variance portfolio is a constant, so that $\sigma_{v\theta} = \sigma_{\gamma\theta} = \sigma_{\gamma p}$. Also, since $\tilde{\Delta}_p$ and $\tilde{\Delta}_v$ are orthogonal to $\tilde{r}_\theta$ and $\tilde{r}_\gamma$, we can write

$$\sigma_{vp} = \sigma_{v\theta} + E(\tilde{\Delta}_v\tilde{\Delta}_p) = \sigma_{v\theta} + \mathrm{Cov}(\tilde{\Delta}_v, \tilde{\Delta}_p) \tag{A1}$$

$$\sigma_p^2 = \sigma_\theta^2 + E(\tilde{\Delta}_p^2) = \sigma_\theta^2 + \sigma^2(\tilde{\Delta}_p). \tag{A2}$$

Noting, finally, that by assumption $\bar{r}_p = \bar{r}_\theta$, we can rewrite the benchmark error for $\tilde{r}_v$ from (4) as follows:

$$\begin{aligned}
\varepsilon_{vp} &= (\bar{r}_{z\theta} - \bar{r}_{zp})\left[1 - \frac{\sigma_{\gamma p}}{\sigma_p^2} - \frac{E(\tilde{\Delta}_v\tilde{\Delta}_p)}{\sigma_p^2}\right] \\
&\quad + (\bar{r}_\theta - \bar{r}_{z\theta})\left[\beta_{v\theta} - \frac{\sigma_{\gamma p}}{\sigma_p^2} - \frac{E(\tilde{\Delta}_v\tilde{\Delta}_p)}{\sigma_p^2}\right] \\
&= (\bar{r}_{z\theta} - \bar{r}_{zp})(1 - \beta_{\gamma p}) \\
&\quad + (\bar{r}_\theta - \bar{r}_{z\theta})(\beta_{\gamma\theta} - \beta_{\gamma p}) + \frac{\bar{r}_\theta - \bar{r}_{zp}}{\sigma_p^2} E(\tilde{\Delta}_v\tilde{\Delta}_p) \\
&= \varepsilon_{\gamma p} + \lambda_p \mathrm{Cov}(\tilde{\Delta}_v, \tilde{\Delta}_p) \quad \text{Q.E.D.}
\end{aligned} \tag{A3}$$

Proof of Theorem 1: The final statement has been demonstrated in the derivation of (8).Then let ε lie within the admissible interval. To construct a portfolio $\tilde{r}_v = \tilde{r}_\gamma + \tilde{\Delta}_v$ with this benchmark error, we must choose $\tilde{\Delta}_v$ with the required variance and such that $\varepsilon_{\gamma p} + \lambda_p E(\tilde{\Delta}_v\tilde{\Delta}_p) = \varepsilon$. Let Z denote the set of all differences between portfolios with the same means and all linear combinations of these differences. This set will be a finite dimensional linear (or vector) space. We can always construct a feasible portfolio by adding an element of Z to any other portfolio. By Equation (1), therefore, the elements of Z are orthogonal to the minimum-variance portfolios. It also has more than one dimension since the $n \geq 4$ primitive assets are linearly independent. Choose c_1 and c_2 so that $\tilde{\Delta}_c = c_1(\tilde{r}_1 - \tilde{r}_2) + c_2(\tilde{r}_2 - \tilde{r}_3)$ has mean zero and construct $\tilde{\Delta}_b = b_1(\tilde{r}_3 - \tilde{r}_4) + b_2(\tilde{r}_1 - \tilde{r}_3)$ similarly. Both $\tilde{\Delta}_c$ and $\tilde{\Delta}_b$ are in Z, and their linear independence is assured by that of the four assets.

Since Z has more than one dimension, the projection theorem implies there exists a nontrivial $\tilde{\Delta}_p^*$ in Z which is orthogonal (under the covariance norm) to $\tilde{\Delta}_p$. Now choose $\tilde{\Delta}_v = a_1\tilde{\Delta}_p + a_2\tilde{\Delta}_p^*$. Then $E(\tilde{\Delta}_v\tilde{\Delta}_p) = a_1 E(\tilde{\Delta}_p^2)$, and $a_1 = -[\varepsilon -$

$\varepsilon_{\gamma p}]/\lambda_p E(\tilde{\Delta}_p^2)$ delivers the correct benchmark error. Since $E(\tilde{\Delta}_v) = 0$, $\tilde{r}_v$ has the correct mean. By the orthogonality of $\tilde{\Delta}_p$ and $\tilde{\Delta}_p^*$, all that remains is to solve for a_2 in the equation:

$$\sigma^2(\tilde{\Delta}_v) = a_1^2\sigma^2(\tilde{\Delta}_p) + a_2^2\sigma^2(\tilde{\Delta}_p^*). \tag{A4}$$

Then, choosing

$$a_2 = \left[\frac{\sigma^2(\tilde{\Delta}_v) - a_1^2 E(\tilde{\Delta}_p^2)}{E(\tilde{\Delta}_p^{*2})}\right]^{1/2} \tag{A5}$$

completes the proof since the bracketed quantity on the right-hand side is positive when ε is in the admissible interval. Q.E.D.

Proof of Theorem 2: From the SML relation for minimum-variance returns, it follows that $\sigma_{\theta\gamma} = \dfrac{\bar{r}_\gamma - \bar{r}_{z\theta}}{\bar{r}_\theta - \bar{r}_{z\theta}}\,\sigma_\theta^2$ and if $\tilde{r}_\gamma$ is minimum-variance $\sigma_{\theta\gamma} = \sigma_{p\gamma}$ as $\bar{r}_\theta = \bar{r}_p$. Substituting for the covariance in (3), therefore, yields:

$$\begin{aligned}\varepsilon_{\gamma p} &= \bar{r}_\gamma - \bar{r}_{zp} - (\bar{r}_\gamma - \bar{r}_{z\theta})\left[\frac{(\bar{r}_p - \bar{r}_{zp})}{\sigma_p^2}\,\frac{\sigma_\theta^2}{(\bar{r}_\theta - \bar{r}_{z\theta})}\right] \\ &= \bar{r}_\gamma[1 - (\lambda_p/\lambda_\theta)] + \bar{r}_{z\theta}(\lambda_p/\lambda_\theta) - \bar{r}_{zp} \quad \text{Q.E.D.}\end{aligned} \tag{A6}$$

Proof of Theorem 3: For any proxy portfolio, $\tilde{r}_p$, (5) and (9) give,

$$\varepsilon_{up} - \varepsilon_{vp} = (\bar{r}_u - \bar{r}_v)\left(1 - \frac{\lambda_p}{\lambda_\theta}\right) - \lambda_p E[\tilde{\Delta}_p(\tilde{\Delta}_u - \tilde{\Delta}_v)] \tag{A7}$$

where $\tilde{\Delta}_u$ and $\tilde{\Delta}_v$ are the residual terms of the two returns. Let $\bar{r}_z^*$ be chosen as in (17) for the proxy, $\tilde{r}_p$. Choose $\bar{r}_q = \bar{r}_s = \bar{r}_p = \bar{r}_\theta$. Let $\tilde{\Delta}_s = \alpha\tilde{\Delta}_p$ and $\tilde{\Delta}_q = -(1/\alpha)\tilde{\Delta}_p$. For α very close to one, these portfolios will be arbitrarily close in mean-variance space. We will also choose $\bar{r}_{zq} = \bar{r}_z^* + \pi_q$ and $\bar{r}_{zs} = \bar{r}_z^* + \pi_s$. To reverse all the rankings, (A7) tells us we must solve two equations for α, π_q, and π_s. We must have $(1 - \lambda_q/\lambda_\theta) = -(1 - \lambda_s/\lambda_\theta)$ which implies $\lambda_\theta - \lambda_q = \lambda_s - \lambda_\theta$. We also need $-\lambda_q E(\tilde{\Delta}_q(\tilde{\Delta}_u - \tilde{\Delta}_v)) = +\lambda_s E(\tilde{\Delta}_s(\tilde{\Delta}_u - \tilde{\Delta}_v))$, or given our choice of $\tilde{\Delta}_q$ and $\tilde{\Delta}_s$, $\lambda_q = \lambda_s\alpha^2$. Together, these conditions yield $\lambda_s = 2\lambda_\theta/(1 + \alpha^2)$. We now need to solve this for small π_s and π_q with α close to unity. Substituting in the definition of λ_s, and minor algebraic manipulations give:

$$\frac{\pi_s}{\sigma_\theta^2 + \alpha^2 E(\tilde{\Delta}_p^2)} = \frac{2}{1 + \alpha^2}\,\lambda_\theta - \frac{(\bar{r}_\theta - \bar{r}_z^*)}{\sigma_\theta^2 + \alpha^2 E(\tilde{\Delta}_p^2)}. \tag{A8}$$

As $\alpha \longrightarrow 1$, $\pi_s \longrightarrow 0$, because by the definition of $\bar{r}_z^*$, $(\bar{r}_\theta - \bar{r}_z^*)/\sigma_p^2 = \lambda_\theta$. It remains to find π_q which solves $\lambda_q = 2\alpha^2\lambda_\theta/(1 + \alpha^2)$. The solution is given by

$$\frac{\pi_q}{\sigma_\theta^2 + (1/\alpha^2)E(\tilde{\Delta}_p^2)} = \frac{2\alpha^2}{1 + \alpha^2}\,\lambda_\theta - \frac{(\bar{r}_\theta - \bar{r}_z^*)}{\sigma_\theta^2 + (1/\alpha^2)E(\tilde{\Delta}_p^2)}. \tag{A9}$$

This also goes to zero as $\alpha \longrightarrow 1$, and hence completes the proof. Q.E.D.

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