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Testing Portfolio Efficiency when the Zero-Beta Rate is Unknown: A Note

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ABSTRACT

A lower bound on the distribution function of the likelihood ratio test of portfolio efficiency is derived. An empirical application demonstrates that the bound may sometimes be used to infer rejection of the null hypothesis without appeal to asymptotic statistical approximations. A procedure for incorporating partial information about the zero-beta intercept, in the multivariate framework, is also developed and applied.

A FUNDAMENTAL PROBLEM IN empirical finance is that of determining whether a given portfolio is on the mean-variance efficient frontier. From a normative perspective, a solution to this problem has relevance for any investor wishing to maximize expected portfolio return for a given level of risk. Since a number of equilibrium models, including the well-known Capital Asset Pricing Model (CAPM), yield predictions that particular portfolios are efficient, the problem is also important from a positive economic perspective.

Fama [3], Roll [9], and Ross [10] observe that the efficiency of a portfolio is equivalent to the existence of a positive linear relation between the expected returns of the component securities and their betas computed relative to the portfolio. If borrowing and lending at a known riskless rate, r, is assumed, then the "zero-beta" intercept in the linear relation must equal r, and the econometric analysis is greatly simplified. Gibbons, Ross, and Shanken [5] show that a transformation of the likelihood ratio test (LRT) statistic has an exact F distribution in this case, assuming joint normality of returns.

The analysis is more complicated when the zero-beta intercept is unknown and must be treated as an additional parameter in the econometric specification. Shanken [11] introduces a multivariate "cross-sectional regression test" (CSRT) of efficiency for this case and derives an approximate sampling distribution for the test. An upper bound on the exact distribution functions of the CSRT and the LRT is also obtained.[1] Under certain circumstances, this bound enables the researcher to infer, without appeal to asymptotic approximations, that the null hypothesis of efficiency cannot be rejected on the basis of the given statistics. In this paper, we present a complementary small-sample result. A lower bound on the distribution function of the LRT is derived which, in some cases, permits the researcher to infer that the null hypothesis should be rejected. An application involving the CRSP equal-weighted stock index illustrates the usefulness of the result.

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[1] LRT's of portfolio efficiency in the absence of a riskless asset were introduced by Gibbons [4] and Stambaugh [12].

As a by-product, our analysis suggests a simple procedure for incorporating *partial* information about the zero-beta intercept in a LRT of portfolio efficiency. This topic has not previously been addressed in the multivariate empirical literature. An obvious practical application is one where an investor is interested in testing the efficiency of a given portfolio of risky assets over a range of possible riskless rates. In an equilibrium context, Black [1] considers a model with risk-free lending but not borrowing. This leads to a CAPM in which the zero-beta rate of the market portfolio exceeds the riskless lending rate. Similarly, Brennan [2] allows for risk-free borrowing and lending, but at different interest rates. In this case, the zero-beta rate is constrained to the interval between the respective rates. Such constraints may be accommodated by our procedures.

The portfolio efficiency problem is a special case of the more general problem of determining whether an efficient portfolio can be constructed from a given subset of a larger universe of assets. All of our results are presented in this general context which may be useful in the testing of multifactor equilibrium models. In the Appendix, Kandel's [8] method of obtaining the exact maximum likelihood estimator (MLE) of the zero-beta intercept is also generalized. Thus, the tests considered here are easily implemented and do not require the use of iterative computational procedures.

I. The Econometric Specification

Let R_t be an N-vector and R_{pt} a K-vector of security returns for period t. Our null hypothesis is that some portfolio of the K-security subset is efficient with respect to the total set of $N + K$ securities. For any scalar γ, consider the "excess return" multivariate linear regression

$$R_t - \gamma 1_N = \alpha(\gamma) + \beta[R_{pt} - \gamma 1_K] + \varepsilon_t \tag{1}$$

where α and β are $N \times 1$ and $N \times K$, respectively. Note that the vector α of regression intercepts is a function of γ while β and ε_t are invariant with respect to the value of γ. Under the null hypothesis described above, there exists a scalar γ, the zero-beta intercept, such that the vector $\alpha(\gamma)$ is equal to zero.[2] Our statistical analysis will focus on this parameter constraint.

Assuming the $N + K$ security returns are linearly independent and jointly normally distributed, the N-vector of disturbances ε_t is statistically independent of R_{pt} with a nonsingular covariance matrix which we call Σ. We also postulate that the returns are independent and identically distributed over T time periods.[3] For each γ, let $\hat{\alpha}(\gamma)$ and $\hat{\beta}$ be the usual unconstrained MLE's of $\alpha(\gamma)$ and β, respectively, from N separate OLS time series regressions. $\hat{\Sigma}$ is the unbiased estimator of Σ, computed from cross-products of OLS residuals. Like the corre-

[2] Conversely, if $\alpha(\gamma) = 0$ for some γ less than the expected return on the global minimum variance portfolio of the $N + K$ securities, then some portfolio of the K-security subset is efficient with respect to the larger set. This follows easily from the algebraic analysis of potential performance in Jobson and Korkie [6].

[3] Actually, it suffices that the ε_t be independent and identically normally distributed, conditional on R_p. R_p itself need not satisfy any particular distributional assumptions.

sponding parameters, $\hat{\alpha}$ depends on γ while $\hat{\beta}$ and $\hat{\Sigma}$ do not. Define

$$Q(\gamma) \equiv T\hat{\alpha}(\gamma)'\hat{\Sigma}^{-1}\hat{\alpha}(\gamma)/[1 + (\bar{R}_p - \gamma 1_K)'\Delta^{-1}(\bar{R}_p - \gamma 1_K)] \tag{2}$$

where Δ is the sample covariance matrix (MLE) and $\bar{R}_p$ the time series mean of the R_{pt}.

In any sample, the unique value of γ which minimizes $Q(\gamma)$ may be computed. Let $\hat{\gamma}$ denote this statistic. Using results from Kandel [8] and Shanken [11], we show in the Appendix that the LRT of the restriction $\alpha(\gamma) = 0$, for some γ, is an increasing transformation of $Q(\hat{\gamma})$. Furthermore, $\hat{\gamma}$ is the MLE of the zero-beta intercept under the assumption that this restriction is satisfied. For any scalar γ, Gibbons, Ross, and Shanken [5] show that the conditional distribution of $Q(\gamma)$, given R_p, is noncentral T^2 with degrees of freedom N and $T - K - 1$ and noncentrality parameter[4]

$$\lambda(\gamma) \equiv T\alpha(\gamma)'\Sigma^{-1}\alpha(\gamma)/[1 + (\bar{R}_p - \gamma 1_K)'\Delta^{-1}(\bar{R}_p - \gamma 1_K)]. \tag{3}$$

This conclusion does not depend on the validity of our null hypothesis. We use it to obtain a lower bound on the distribution of the LRT statistic.

THEOREM 1. *Let $Q(\hat{\gamma})$ be the minimum value, over all γ, of $Q(\gamma)$ in* (2). *Similarly, let λ^* be the minimum value of $\lambda(\gamma)$ in* (3). *The conditional distribution function of $Q(\hat{\gamma})$, given R_p, is bounded below by the distribution function of a noncentral T^2-variate with degrees of freedom N and $T - K - 1$ and noncentrality parameter λ^*. Furthermore, $\lambda^* = 0$ if and only if $\alpha(\gamma) = 0$ for some γ.*

Proof: Let γ be any scalar. By definition, $Q(\hat{\gamma}) \leq Q(\gamma)$ in each sample. Thus, the distribution function of the statistic $Q(\hat{\gamma})$ must be bounded below by the distribution function of $Q(\gamma)$. As noted above, the latter distribution is that of a noncentral T^2 variate with noncentrality parameter $\lambda(\gamma)$. In particular, these statements hold for that value of γ which satisfies $\lambda(\gamma) = \lambda^*$ (the minimum will exist except in a degenerate case; see the Appendix). We state the theorem in terms of this value, since it provides the greatest lower bound within the class of distributions considered. This follows from the fact that the noncentral T^2-distribution function is decreasing in λ (see Johnson and Kotz [7, p. 193]). Q.E.D.

Theorem 1 implies that if the observed statistic, $Q(\hat{\gamma})$, exceeds a given critical level of the central T^2 distribution with degrees of freedom N and $T - K - 1$, then we may reject the null hypothesis that some portfolio of the securities in R_p is efficient. We need not resort to asymptotic approximations in such a case.[5] It is interesting to compare this result to previous results on testing portfolio efficiency. In the Appendix, we show that the CSRT in Shanken [11] is equal to $Q(\hat{\gamma}_C)$, where $\hat{\gamma}_C$ is the "generalized-least-squares" cross-sectional regression estimator of γ. Shanken suggests using the degrees of freedom $N - 1$ and $T - K - 1$ to obtain approximate inferences that, strictly speaking, are asymp-

[4] While γ is interpreted as a known riskless rate in that paper, the mathematics merely requires that γ be a given scalar.

[5] Of course, this result is conditional on our normality assumption. See footnote 3.

totically valid as $T \to \infty$. The T^2 distribution function based on these degrees of freedom is also shown to be an upper bound on the exact distribution function of the statistic $T\hat{\alpha}(\hat{\gamma}_C)'\hat{\Sigma}^{-1}\hat{\alpha}(\hat{\gamma}_C)$, the numerator of $Q(\hat{\gamma}_C)$. Roughly speaking, this bound is based on the idea that measurement error in $\hat{\beta}$ is a source of "noise" in the test statistic; ignoring this noise makes the statistic look more "significant" than it really is (p-value too low). In contrast, since $Q(\hat{\gamma})$ is computed in each sample by *minimizing* $Q(\cdot)$, inferences obtained by treating $\hat{\gamma}$ as if it were the true parameter value are biased toward *acceptance* of the null hypothesis (p-value too high).[6] Thus, Theorem 1 complements the earlier small-sample result.[7]

II. An Empirical Application

The bound in Theorem 1 is of particular interest in situations where the null hypothesis appears to be rejected. In this section, we apply the theorem to the data of Shanken [11], who rejects efficiency of the CRSP equal-weighted index using the approximate CSRT. Real returns on twenty portfolios stratified by "firm size" are employed in the tests which are conducted over three subperiods from 1953 through 1971 (see the earlier paper for additional details). The standard result, that a T^2-variate with n and m degrees of freedom is transformed into an F-variate with n and $m - n + 1$ degrees of freedom when multiplied by $(m - n + 1)/mn$, is utilized.

For each subperiod, Table I reports the MLE $\hat{\gamma}$ followed below by the corresponding statistic $Q(\hat{\gamma})$. $Q(\hat{\gamma})$ is transformed into two F-statistics with p-values in parentheses. The first statistic is based on the CSRT and yields an approximate p-value. The second F-statistic is based on Theorem 1. Therefore, the associated p-value is an upper bound on the true p-value for $Q(\hat{\gamma})$. The first set of F-statistics in Table I is similar to that reported by Shanken [11]. The slight differences are due to the fact that $Q(\cdot)$ is evaluated at the cross-sectional regression estimate of γ in that paper, whereas the results in Table I are based on the MLE. The rejection of the null hypothesis suggested by the first set is supported by the more "conservative" second set of F-statistics. The aggregate p-value for the three subperiods, 0.022, is an upper bound on the true p-value.[8] An approximate value is 0.009. With January returns deleted, the upper bound is 0.009 while an approximate p-value is 0.004.[9]

The relatively small differences between the approximate values and the upper bounds are encouraging and demonstrate the practical relevance of Theorem 1.

[6] A similar theorem holds when testing expected return relations where the "factors" are not portfolio returns. In this case, however, the degrees of freedom are $N - K - 1$ and $T - K - 1$ in the approximate T^2-distribution, but still N and $T - K - 1$ in the lower bound.

[7] Theorem 1 also provides information about the power of the LRT; specifically, an upper bound on the probability of rejecting the null hypothesis when it is false.

[8] For each subperiod statistic, the standard normal z corresponding to the given p-value is determined. The subperiod z's are then added and divided by $\sqrt{3}$ to obtain an aggregate $N(0,1)$ statistic from which the aggregate p-value is determined. This procedure was suggested in Shanken [11].

[9] Similar results are obtained using nominal returns or returns in excess of the one month T-bill rate.

Table I

Tests of Efficiency of the CRSP Equal-weighted Index[a]

Statistic	Subperiod 2/53–3/59	4/59–5/65	6/65–7/71	Aggregate p-Value
γ^b	0.743	1.094	0.022	
$Q(\gamma)$	49.2	36.6	37.3	
$F(19, 54)$	1.94	1.44	1.47	
	(0.03)	(0.15)	(0.13)	(0.009)
$F(20, 53)$	1.81	1.35	1.37	
	(0.04)	(0.19)	(0.18)	(0.022)
γ^b	0.425	1.254	−0.002	
$Q(\gamma)$	57.2	37.4	44.1	
$F(19, 48)$	2.19	1.43	1.69	(0.004)
	(0.01)	(0.16)	(0.07)	
$F(20, 47)$	2.04	1.33	1.57	
	(0.02)	(0.21)	(0.10)	(0.009)

[a] Real monthly returns on twenty portfolios stratified by firm size are employed. The first F-statistic provides approximately valid inferences. The second F-statistic yields an upper bound on the true p-value. January returns have been deleted from tests summarized in the second panel. P-values are in parentheses.

[b] Multiplied by 100.

Examining the individual subperiod results, the differences tend to increase with the level of the approximate p-value. This suggests that the bound may not be very useful in cases where there is only "mild" evidence against the null hypothesis.

If we interpret γ as a real riskless rate, the estimates in the second subperiod seem quite high—in excess of 12% per annum. The F-statistics do not permit rejection of the null hypothesis for this subperiod. It appears, however, that efficiency might be rejected with the real riskless rate constrained to a "reasonable" set of values. Theorem 1 is easily extended to accommodate such considerations. We simply minimize $Q(\gamma)$ and $\lambda(\gamma)$ over the given constraint set and redefine $\hat{\gamma}$ and λ^* accordingly. The modified $Q(\hat{\gamma})$ statistic is subject to the bound given in Theorem 1, by essentially the same argument. In general, the modified statistic rises as we shrink the constraint set. In the limiting case, where the zero-beta intercept is contrained to a single number, the bound coincides with the exact distribution. This is the Gibbons, Ross, and Shanken [5] theorem cited earlier.

Results from an application of this analysis are presented in Table II. The real riskless rate is constrained to the interval between zero and twice the average real T-bill return over the given subperiod. As expected, the (conservative) p-values drop considerably in the second subperiod; from 0.19 to 0.11 with January returns included and from 0.21 to 0.08 without these returns. This is reflected in the aggregate p-values which are reduced from 0.022 to 0.009 and from 0.009 to 0.003, respectively. Of course, these are all upper bounds on the corresponding true p-values.

Table II

Tests of Efficiency of the CRSP Equal-weighted Index with the Real Riskless Rate Constrained to the Interval Between Zero and Twice the Average Real T-bill Return Over the Given Subperiod[a]

Statistic	Subperiod 2/53–3/59	4/59–5/65	6/65–7/71	Aggregate p-Value
T-bill Return[b]	0.047	0.129	0.074	
γ^c	0.094	0.258	0.022	
$Q(\gamma)$	52.7	41.9	37.3	
$F(20, 53)$	1.94	1.54	1.37	
	(0.03)	(0.11)	(0.18)	(0.009)
γ^c	0.094	0.258	0.000	
$Q(\gamma)$	58.1	46.6	44.1	
$F(20, 47)$	2.07	1.66	1.57	
	(0.02)	(0.08)	(0.10)	(0.003)

[a] Real monthly returns on twenty portfolios stratified by firm size are employed. January returns have been deleted from tests summarized in the second panel. P-values in parentheses are upper bounds in the true values.

[b] Multiplied by 100. Taken from Stambaugh [12].

[c] Multiplied by 100.

It is interesting to note the relatively minor movement of $Q(\hat{\gamma})$ in the first subperiod, with January deleted, as the MLE $\hat{\gamma}$ changes from 0.425% to the constrained value of 0.094%. Since this represents a compound difference of over 4% per annum, one might have anticipated a greater impact on the likelihood function. The formal analysis belies this naive intuition, however, and reflects the flatness of the likelihood function.

Appendix

In this section, we consider computation of the MLE $\hat{\gamma}$ and the test statistic $Q(\hat{\gamma})$. In the case $K = 1$, using a different parameterization, Kandel [8] observes that $\hat{\gamma}$ minimizes the LRT statistic viewed as a function of γ. This follows from the fact that γ is not a parameter in the unconstrained likelihood function, so that the LRT statistic is inversely related to the value of the constrained likelihood function. The same observation may be made in the general case considered here. Shanken [11] notes, in Appendix A, that the LRT statistic is an increasing function of $Q(\hat{\gamma})$ [the equality of $Q(\cdot)$ and Shanken's CSRT statistic Q_A follows from (A1) below]. Therefore, the constrained MLE $\hat{\gamma}$ may be obtained by minimizing $Q(\gamma)$ over γ.

While Theorem 1 is conveniently presented in terms of the excess return multivariate linear regression, we now consider the raw return system of regression equations:

$$R_t = a + \beta R_{pt} + \varepsilon_t.$$

Note that β and ε_t equal their counterparts in (1) and that the same estimator, $\hat{\beta}$, is obtained whether raw or excess returns are involved. Letting $\hat{a}$ equal the OLS estimator of the intercept vector, a, it is easily demonstrated that for any γ,

$$\hat{\alpha}(\gamma) = \hat{a} - \gamma(1_N - \hat{\beta}1_K) \tag{A1}$$

and likewise for the corresponding parameters. The "generalized least-squares" cross-sectional regression estimator $\hat{\gamma}_C$ is obtained by minimizing the quadratic form $\hat{\alpha}(\gamma)'\hat{\Sigma}^{-1}\hat{\alpha}(\gamma)$ over γ.

By (A1), the numerator of $Q(\gamma)$ in (2) is

$$(Z'\hat{\Sigma}^{-1}Z)\gamma^2 - (2Z'\hat{\Sigma}^{-1}\hat{a})\gamma + (\hat{a}'\hat{\Sigma}^{-1}\hat{a}) \tag{A2}$$

where $Z \equiv 1_N - \hat{\beta}1_K$. As γ approaches plus or minus infinity, $Q(\gamma)$ converges to the asymptotic value

$$q_\infty \equiv T(Z'\hat{\Sigma}^{-1}Z)/(1_K'\Delta^{-1}1_K).$$

It may be verified, using (2) and (A2), that $Q'(\gamma)$ has the same sign, for all γ, as the quadratic

$$H(\gamma) = A\gamma^2 + B\gamma + C,$$

where

$$A \equiv (1_K'\Delta^{-1}1_K)(Z'\hat{\Sigma}^{-1}\hat{a}) - (1_K'\Delta^{-1}\bar{R}_p)(Z'\hat{\Sigma}^{-1}Z)$$

$$B \equiv (1 + \bar{R}_p'\Delta^{-1}\bar{R}_p)(Z'\hat{\Sigma}^{-1}Z) - (1_K'\Delta^{-1}1_K)(\hat{a}'\hat{\Sigma}^{-1}\hat{a})$$

and

$$C \equiv -(1 + \bar{R}_p'\Delta^{-1}\bar{R}_p)(Z'\hat{\Sigma}^{-1}\hat{a}) + (1_K'\Delta^{-1}\bar{R}_p)(\hat{a}'\hat{\Sigma}^{-1}\hat{a}).$$

It can also be shown that, with probability one, A is nonzero and H has two real roots. Thus, $Q'(\gamma)$ changes sign exactly twice.

It follows that if A is greater (less) than zero, $Q(\gamma)$ increases (decreases) from the value q_∞, attains a global maximum (minimum) at the smaller root of H, decreases (increases) to a global minimum (maximum) at the larger root of H and then increases (decreases) back toward q_∞. This simple characterization of the behavior of Q makes it easy to obtain the value of γ which minimizes Q over the real line or any specific subset. In the latter context, it is more precise to speak of the infimum which could equal q_∞.

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