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Derivation of the Capital Asset Pricing Model without Normality or Quadratic Preference: A Note

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ABSTRACT

Derivation of the capital asset pricing model requires various assumptions including normality or quadratic preference. The objective of this note is to show that the normality or quadratic preference assumption can be replaced by the fair game condition that assets' residual returns have zero mean conditional upon the return of the market portfolio.

SINCE SHARPE [9], LINTNER [4], AND Mossin [5] developed the capital asset pricing model (CAPM), there have been attempts to relax normality or quadratic preference assumption that underlies the model. For instance, Fama [3] derived a model quite similar to the CAPM for symmetric stable distribution, and Owen and Rabinovitch [6] for elliptical distribution. As Ross [7] showed, however, such restrictions on assets' entire return distributions are not necessary for the CAPM: certain distributional characteristics alone (called the two-fund separability conditions) are sufficient for the model.

In a recent article, Stapleton and Subrahmanyam [10] obtained the interesting result that the normality or quadratic preference assumption may be replaced by the two conditions:

(C1) Assets' residual returns have zero means conditional upon the return of the market portfolio.
(C2) There exists at least one investor who holds the market portfolio at equilibrium (more precisely, a portfolio that consists of risk-free asset and the market portfolio).

Chen and Ingersoll [1] also showed the result in the framework of a multifactor linear structure.

The objective of this note is to demonstrate that Condition (C1) implies Condition (C2). As a result, Condition (C1) alone is sufficient for the CAPM in the environment where other standard assumptions hold (i.e., perfect and competitive market, expected utility maximization of one period wealth, risk-averse preferences, risk-free asset, infinitely divisible assets, and short sales). The reason is roughly as follows. As Rothschild and Stiglitz [8] showed, risk-averse investors become better off if they can costlessly remove white noise from their portfolios. Condition (C1) means that assets' residual returns are white noise conditional

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upon the return of the market portfolio. Given (C1), therefore, risk-averse investors are seen to hold the market portfolio at equilibrium.

I. Statement of the Result

Throughout this note, we consider the environment where the standard assumptions hold as mentioned above.

Let $\tilde{y}_j$, represent the total return of asset j $(0 \le j \le n)$ where asset $j = 0$ is risk-free with $\tilde{y}_0 \equiv 1$. Regressing the returns, $\tilde{y}_j$, against the return of the market portfolio, $\tilde{y}_M = \sum_{j=1}^{n} \tilde{y}_j$, we can define assets' *residual returns*, ε_j, by the equation

$$\tilde{y}_j = \alpha_j + \beta_j \tilde{y}_M + \tilde{\varepsilon}_j \tag{1}$$

for all j, where

$$\alpha_j = E(\tilde{y}_j) - \frac{\operatorname{Cov}(\tilde{y}_j, \tilde{y}_M)}{\operatorname{Var}(\tilde{y}_M)} E(\tilde{y}_M) \quad \text{and} \quad \beta_j = \frac{\operatorname{Cov}(\tilde{y}_j, \tilde{y}_M)}{\operatorname{Var}(\tilde{y}_M)}. \tag{2}$$

Denote by p_j and $p_M = \sum_{j=1}^{n} p_j$ the respective prices of asset j and the market portfolio at equilibrium. The CAPM can then be written in the form

$$E(\tilde{y}_j) - \rho p_j = \beta_j [E(\tilde{y}_M) - \rho p_M] \tag{3}$$

for all j, where $\rho = 1/p_0$ (one plus risk-free rate of return).

THEOREM. *In addition to the standard assumptions, assume that assets' residual returns have zero mean conditional upon the return of the market portfolio*:

$$E(\tilde{\varepsilon}_j \mid \tilde{y}_M) = 0 \tag{4}$$

for all j. The CAPM must then hold at equilibrium.

Borrowing the term from Fama [2], we call condition (4) the "fair game" condition: if the return information about the market portfolio is publicly available, assets' equilibrium prices are determined in such a way that investors' market trading becomes a fair game. Note that the fair game condition does not include any parameters that are related to investors' preferences, nor is it explicitly associated with normal distribution. As a result, the normality or quadratic preference assumption is seen to be unnecessary for the CAPM.

II. Proof of the Result

Consider the portfolio decision problem of investor i:

$$\max_{(\theta_{ij})_j} E\{u_i[\textstyle\sum_{j=0}^{n} \theta_{ij} \tilde{y}_j]\}$$

$$\text{subject to } \textstyle\sum_{j=0}^{n} \theta_{ij} p_j = \sum_{j=0}^{n} \bar{\theta}_{ij} p_j \tag{5}$$

where $u_i(\cdot)$ = the utility function of investor i, and $\bar{\theta}_{ij}$ = the initial ownership fraction of asset j by investor i. Assume that $(\theta_{ij}^*)_j$ solve (5) for all i: $\sum_i \theta_{ij}^* = 1$ for every j. As is well known, the equilibrium price of asset j, p_j, can then be expressed as

$$p_j = \frac{1}{\rho E[u_i'(\tilde{x}_i^*)]} E\{u_i'(\tilde{x}_i^*)y_j\} \tag{6}$$

for all i and j, where u_i' is the derivative of u_i, and $\tilde{x}_i^*$ is the return of the optimal portfolio of investor i:

$$\tilde{x}_i^* = \textstyle\sum_{j=0}^{n} \theta_{ij}^* \tilde{y}_j . \tag{7}$$

Two remarks are made for later reference. First, the return, $\tilde{x}_i^*$, can be written as the sum of two portfolio returns, where one depends on the return of the market portfolio and the other upon assets' residual returns only:

$$\tilde{x}_i^* = \tilde{z}_i + \tilde{\xi}_i \tag{8}$$

for all i, where

$$\begin{aligned} \tilde{z}_i &= (\textstyle\sum_{j=0}^{n} \theta_{ij}^* \alpha_j) + (\sum_{j=1}^{n} \theta_{ij}^* \beta_j)\tilde{y}_M \\ \tilde{\xi}_i &= \textstyle\sum_{j=1}^{n} \theta_{ij}^* \tilde{\varepsilon}_j \end{aligned} \tag{9}$$

[replace $\tilde{y}_j$ in (7) by expression (1) and note that $\beta_0 = \tilde{\varepsilon}_0 = 0$]. Secondly, given any portfolio, $\tilde{w} = \sum_{j=0}^{n} \theta_{ij}\tilde{y}_j$, its equilibrium price, $\pi(\tilde{w})$, can be computed by multiplying (6) by θ_{ij} and then summing over j, $0 \le j \le n$:

$$\pi(\tilde{w}) = \frac{1}{\rho E[u_i'(\tilde{x}_i^*)]} E\{u_i'(\tilde{x}_i^*)\tilde{w}\} \tag{10}$$

for all i. Note that the left side of (10) is independent of a particular investor, i. An important implication of this observation is that the choice of investor i in (10) is irrelevant for computation of the equilibrium price of a portfolio.

We are now ready to prove the theorem. For expository purposes, it will be convenient to establish first the two statements:

(S1) The equilibrium prices, $\pi(\tilde{\xi}_i)$, of portfolios $\tilde{\xi}_i$ in (9) are zero;
(S2) $\tilde{\xi}_i \equiv 0$ with probability one if investor i is strictly risk-averse.

Observe that the returns, $\tilde{\xi}_i$, depend upon assets' residual returns only [cf. (9)]. Statement (S1) thus indicates that investor i can *costlessly* remove noise, $\tilde{\xi}_i$, from his or her optimal portfolio, $\tilde{x}_i^*$. Statement (S2) implies that a strictly risk-averse investor i does diversify away the noise component $\tilde{\xi}_i$ at equilibrium.

First, consider (S1): $\pi(\tilde{\xi}_i) = 0$ for all i. Note that these prices sum to zero:

$$\textstyle\sum_i \pi(\tilde{\xi}_i) = \pi(\sum_i \tilde{\xi}_i) = \pi[\sum_i(\sum_j \theta_{ij}^* \tilde{\varepsilon}_j)] = \pi(\sum_j \tilde{\varepsilon}_j) = 0$$

[cf. (1), (2), (10); $\sum_i \theta_{ij}^* = 1$ for all j and $\sum_j \tilde{\varepsilon}_j = 0$]. Consequently, it suffices to show that $\pi(\tilde{\xi}_i) \le 0$ for all i (if the sum of nonpositive numbers is zero, each of these numbers must be zero). To the contrary, assume that $\pi(\tilde{\xi}_i) > 0$ for *some* i. Since $\tilde{x}_i^* = \tilde{z}_i + \tilde{\xi}_i$ by (8), it then follows that $\pi(\tilde{x}_i^*) = \pi(\tilde{z}_i) + \pi(\tilde{\xi}_i) > \pi(\tilde{z}_i)$: portfolio $\tilde{z}_i$ is less costly than $\tilde{x}_i^*$. On the other hand,

$$\begin{aligned} Eu_i(\tilde{x}_i^*) &= E\{Eu_i(\tilde{z}_i + \tilde{\xi}_i) \mid \tilde{y}_M\} \\ &\le E\{u_i[E(\tilde{z}_i \mid \tilde{y}_M) + E(\tilde{\xi}_i \mid \tilde{y}_M)]\} \end{aligned}$$

by Jensen's inequality. Since $E(\tilde{z}_i \mid \tilde{y}_M) = \tilde{z}_i$ by definition and $E(\tilde{\xi}_i \mid \tilde{y}_M) = 0$ by Condition (4) [cf. (9)], the above inequality is reduced to

$$Eu_i(\tilde{x}_i^*) \le Eu_i(\tilde{z}_i) \tag{11}$$

which means that portfolio $\tilde{z}_i$ is *not* dominated by $\tilde{x}_i^*$ for investor i. Since $\tilde{z}_i$ is less costly, this contradicts the optimality of $\tilde{x}_i^*$. By the principles of contrapositive proof, statement (S1) must then hold for every i.

Next, consider (S2): $\tilde{\xi}_i \equiv 0$ with probability one if $u_i'' < 0$. For this result, however, it suffices to note that inequality (11) is strict whenever $u_i'' < 0$ and $\tilde{\xi}_i \not\equiv 0$ (cf. Rothschild and Stiglitz [8]).

Finally, consider the main result: Condition (4) implies the CAPM. Since the CAPM trivially holds if *all* investors are risk-neutral, we may assume that $u_i'' < 0$ for *some* i. As we have shown in (S2), though, this investor i holds at equilibrium a portfolio that consists of risk-free asset and the market portfolio:

$$\tilde{x}_i^* = \tilde{z}_i = \left(\textstyle\sum_{j=0}^{n} \theta_{ij}^* \alpha_j\right) + \left(\textstyle\sum_{j=1}^{n} \theta_{ij}^* \beta_j\right)\tilde{y}_M$$

[cf. (9)]. The analysis in Chen and Ingersoll [1] and Stapleton and Subrahmanyam [10] then indicates that the CAPM holds at equilibrium.

III. Conclusions

The CAPM does not include any parameters that are related to investor preferences. Also, assets' mean returns and covariances that appear in the model are not unique to normal distributions. These two observations seem to imply that the CAPM should hold even in the absence of normality or quadratic preference. In fact, Stapleton and Subrahmanyam [10] have recently shown that the normality or quadratic preference assumption can be replaced by the two conditions. (C1) Assets' residual returns have zero mean conditional upon the return of the market portfolio. (C2) Some investor holds the market portfolio at equilibrium. As the analysis in this note shows, however, Condition (C2) can further be relaxed: in the usual environment (perfect and competitive market, risk-averse expected utility maximization, risk-free asset, infinite divisibility, and short sale), Condition (C2) follows from (C1) and, thus, Condition (C1) alone is sufficient for the CAPM.

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