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Reformulating Tax Shield Valuation: A Note

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ABSTRACT

Standard financial theory (in the absence of agency costs and personal taxes) implies that each dollar of debt contributes to the value of the firm in proportion to the firm's tax rate. To derive this result, incremental debt is assumed permanent. This paper shows that when the firm acts to maintain a constant market value leverage ratio, the marginal value of debt financing is much lower than the corporate tax rate. Since Hamada's [2] unlevering procedure for observed equity betas was derived under the assumption of permanent debt, we derive an unlevering procedure consistent with the assumption of a constant leverage ratio.

IMPLICIT IN THE MODIGLIANI-MILLER [4] hypothesis regarding the effect of leverage changes on firm value is that the firm maintains a constant level of debt. The purpose of this paper is to examine the implications for tax shield valuation of maintaining a constant market value leverage ratio instead of a constant debt level. Assuming a constant leverage ratio instead of a constant debt level does not imply any change in the normative implications of capital structure theory.

However, we show that when the firm's financing strategy is to maintain a constant market value leverage ratio, the marginal value of a change in debt level resulting from a change in this leverage ratio is much lower than the corporate tax rate. We also derive the relationship between the firm's equity beta and its unlevered beta under the assumption of a constant leverage ratio.

I. The Value of a Multiperiod Levered Stream

In this section, we develop an equation for the market value of a levered cash flow stream $(\tilde{X}_1 + \tilde{TS}_1), (\tilde{X}_2 + \tilde{TS}_2), \cdots, (\tilde{X}_T + \tilde{TS}_T)$ where $\tilde{X}_j$ are random unlevered cash flows and $\tilde{TS}_j$ are random debt-related tax shields in period j, respectively, for $j = 1, \cdots, T$.

At time $T - 1$, we can express the value of $(\tilde{X}_T + \tilde{TS}_T)$ by the CAPM as

$$V_{T-1} = \frac{E_{T-1}(\tilde{X}_T + \tilde{TS}_T) - \phi \operatorname{Cov}(\tilde{X}_T + \tilde{TS}_T, \tilde{r}_{mT})}{1 + r} \tag{1}$$

where

$E_i(\ ) =$ the time i expectation operator;
$\operatorname{Cov}(\ ) =$ the covariance operator;
$\phi =$ the market price of risk assumed constant across time;
$\tilde{r}_{mj} =$ the time j market rate of return; and
$r =$ the riskless rate of interest.

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Fama [1] and Myers and Turnbull [5] assume an expectation revision process specified in terms of

$$\tilde{e}_{ij} = \frac{\tilde{E}_i(\tilde{X}_j) - E_{i-1}(\tilde{X}_j)}{E_{i-1}(\tilde{X}_j)}. \tag{2}$$

This equation implies that

$$\tilde{X}_T = E_{T-1}(X_T)(1 + \tilde{e}_{TT}) \tag{3}$$

and, with the assumption that the market value leverage ratio is constant, $\tilde{TS}_T$ can be expressed as

$$\tilde{TS}_T = \tau rLV_{T-1}. \tag{4}$$

In this form, τ is the corporate tax rate and L is the leverage ratio.

At time $T - 1$, V_{T-1} is known with certainty. Hence, Equation (4) shows that $\tilde{TS}_T$, seen from time $T - 1$, is also known with certainty so that $\text{Cov}(TS_T, r_{mT}) = 0$. Substituting (3) and (4) into (1) and rearranging produces

$$V_{T-1} = \frac{E_{T-1}(\tilde{X}_T)\alpha_{TT} + \tau rLV_{T-1}}{1 + r} \tag{5}$$

where

$$\alpha_{ij} \equiv 1 - \phi \ \text{Cov}(\tilde{e}_{ij}, \tilde{r}_{mj}).$$

If we assume $\text{Cov}(\tilde{e}_{ij}, \tilde{r}_{mj})$ is constant for all i and j, then Equation (5) can be rewritten as

$$V_{T-1} = \frac{E_{T-1}(\tilde{X}_T)}{1 + \rho} + \frac{\tau rLV_{T-1}}{1 + r} \tag{6}$$

where

$$1 + \rho \equiv \frac{1 + r}{\alpha}. \tag{7}$$

At time $T - 2$,

$$V_{T-2} = \pi_{T-2}(\tilde{X}_{T-1}) + \pi_{T-2}(\tilde{TS}_{T-1}) + \pi_{T-2}(\tilde{V}_{T-1}) \tag{8}$$

where $\pi_{T-2}(\ \ )$ is a generalized valuation operator. Equation (8) simply states that at time $T - 2$, the value of the levered stream is the sum of the time $T - 2$ values of the time $T - 1$ unlevered cash flow component, $\tilde{X}_{T-1}$, the time $T - 1$ tax shield component, $\tilde{TS}_{T-1}$, and the time $T - 1$ levered value, $\tilde{V}_{T-1}$. Generalizing from Equations (1) through (4), the following expressions for $\pi_{T-2}(\tilde{X}_{T-1})$ and $\pi_{T-2}(\tilde{TS}_{T-1})$ are easily obtained:

$$\pi_{T-2}(\tilde{X}_{T-1}) = \frac{E_{T-2}(X_{T-1})}{1 + \rho} \tag{9}$$

and

$$\pi_{T-2}(\tilde{TS}_{T-1}) = \frac{\tau rLV_{T-2}}{1 + r}. \tag{10}$$

To obtain an expression for $\pi_{T-2}(\tilde{V}_{T-1})$, we apply the single-period CAPM to $\pi_{T-2}(\tilde{V}_{T-1})$ producing

$$\pi_{T-2}(\tilde{V}_{T-1}) = \frac{E_{T-2}(\tilde{V}_{T-1}) - \phi \operatorname{Cov}(\tilde{V}_{T-1}, \tilde{r}_{M,T-1})}{1 + r}. \tag{11}$$

The time $T - 2$ uncertainty about $\tilde{V}_{T-1}$, $\operatorname{Cov}(\tilde{V}_{T-1}, \tilde{r}_{M,T-1})$, derives from the time $T - 2$ uncertainty about $\tilde{E}_{T-1}(\tilde{X}_T)$. Thus,

$$\tilde{V}_{T-1} = \frac{\tilde{E}_{T-1}(\tilde{X}_T)\alpha + \tau r L \tilde{V}_{T-1}}{1 + r}. \tag{12}$$

Substituting (12) into (11) and rearranging

$$\begin{aligned}\pi_{T-2}(\tilde{V}_{T-1}) = \Big\{&\{E_{T-2}[\tilde{E}_{T-1}(\tilde{X}_T)\alpha] - \phi \operatorname{Cov}[\tilde{E}_{T-1}(\tilde{X}_T)\alpha, \tilde{r}_{m,T-1}]\} \\ &+ \{E_{T-2}(\tau r L \tilde{V}_{T-1}) - \phi \operatorname{Cov}(\tau r L \tilde{V}_{T-1}, \tilde{r}_{m,T-1})\}\Big\} \Big/ (1 + r)^2\end{aligned} \tag{13}$$

is obtained. Equation (13) shows clearly that at time $T - 2$, the time T interest tax shield, $\tilde{TS}_T = \tau r L \tilde{V}_{T-1}$, is uncertain due to $\tilde{V}_{T-1}$ being a random variable. From Equation (2), we can write

$$\tilde{E}_{T-1}(\tilde{X}_T) = E_{T-2}(\tilde{X}_T)(1 + \tilde{e}_{T-1,T}). \tag{14}$$

Substituting (14) into (12) and solving the resulting equation for V_{T-1} results in

$$\tilde{V}_{T-1} = \frac{E_{T-2}(\tilde{X}_T)(1 + \tilde{e}_{T-1,T})\alpha}{(1 + r)Q} \tag{15}$$

where

$$Q \equiv 1 - \frac{\tau r L}{1 + r}. \tag{16}$$

The uncertainty about the time T tax shield is reflected in $\operatorname{Cov}(\tau r L \tilde{V}_{T-1}, \tilde{r}_{m,T-1})$ which, using Equation (15), we can specify as

$$\begin{aligned}\operatorname{Cov}(\tau r L \tilde{V}_{T-1}, \tilde{r}_{m,T-1}) &= \left[\tau r L \frac{E_{T-2}(\tilde{X}_T)\alpha}{(1 + r)Q}\right] \operatorname{Cov}(\tilde{e}_{T-1,T}, \tilde{r}_{mT}) \\ &= \tau r L\, [E_{T-2}(\tilde{V}_{T-1})] \operatorname{Cov}(\tilde{e}_{T-1,T}, \tilde{r}_{mT}).\end{aligned} \tag{17}$$

Now, by substituting (14) and (17) into (13) and rearranging, we obtain

$$\pi_{T-2}(\tilde{V}_{T-1}) = \frac{E_{T-2}(\tilde{X}_T)}{(1 + \rho)^2} + \frac{\tau r L[E_{T-2}(V_{T-1})]}{(1 + \rho)(1 + r)}. \tag{18}$$

From Equation (18), it is apparent that to obtain the time $T - 2$ value of the time T tax shield, $\tau r L V_{T-1}$, the appropriate discount rate for period T is the risk-free rate, r, and the unlevered cost of capital, ρ, is the appropriate discount rate for period $T - 1$. Thus, at time $T - 1$, the market knows the time T tax shield with certainty. However, viewed from time $T - 2$, the time $T - 1$ expectation about the time T tax shield is subject to the uncertain expectation revision reflected in Equation (17).

Using Equations (9), (10), and (18) in (8), we can express V_{T-2} as

$$V_{T-2} = \frac{E_{T-2}(\tilde{X}_{T-1})}{(1+\rho)} + \frac{E_{T-2}(\tilde{X}_T)}{(1+\rho)^2} + \frac{\tau r L V_{T-2}}{(1+r)} + \frac{\tau r L[E_{T-2}(\tilde{V}_{T-1})]}{(1+\rho)(1+r)}.$$

Continuing the backward iteration back to time 0, we obtain

$$V_0 = \sum_{j=1}^{T} \frac{E_0(\tilde{X}_j)}{(1+\rho)^j} + \sum_{j=1}^{T} \frac{E_0(\tilde{TS}_j)}{(1+\rho)^{j-1}(1+r)}. \tag{19}$$

where

$$E_0(\tilde{TS}_j) = \tau r L[E_0(\tilde{V}_{j-1})].$$

Equation (19) shows that when the market value leverage ratio is held constant, it is not generally correct to discount interest tax shields at the riskless rate. The tax shield expected at time j is discounted back one period at the riskless rate and the remaining $j-1$ periods at the unlevered cost of capital.

II. Implications for Tax Shields Valuation

The standard analysis of the present value of tax shields in a Modigliani-Miller (MM) tax world assumes that the expected unlevered cash income stream, $\tilde{X}_j$ is a level perpetuity and derives the following well-known expressions for levered market value:

$$\begin{aligned} V_0 &= \frac{E_0(\tilde{X})}{\rho} + \frac{\tau r D_0}{r} \\ &= \frac{E_0(\tilde{X})}{\rho} + \tau D_0 \end{aligned} \tag{20}$$

where D_0 is the level of current debt. Implicit in Equation (20) is that D_0 is permanent debt, and that all future tax shields are nonstochastic. Only in this case will each dollar of additional debt add τ dollars to total levered value. The result, however, is not consistent with Equation (19). To sharpen the contrast between Equation (19) and the typical approach regarding tax shield valuation, we will impose the level perpetuity assumption on (19). This assumption implies that $E_0(\tilde{X}_j) = \tilde{E}_0(\tilde{X})$ for $j > 0$ and, hence, $E_0(\tilde{V}_j) = V_0$ for $j > 0$ since future values are not "expected" to change. Therefore

$$\begin{aligned} E_0(\tilde{TS}_j) &= E_0(\tau r L \tilde{V}_{j-1}) \\ &= \tau r L V_0. \end{aligned}$$

Substituting this expression for the expected tax savings in Equation (19) and rearranging yields

$$V_0 = \frac{E_0(\tilde{X})}{\rho} + \left(\frac{1+\rho}{1+r}\right)\left(\frac{r}{\rho}\right)\tau L V_0. \tag{21}$$

Suppose the firm with the leverage ratio, L, were to select a higher leverage ratio which required that an additional dollar of debt be issued at time 0. Let L' represent this higher leverage ratio, V_0' be the new levered value associated with L', and D_0 be the amount of debt issued at time 0 to produce the higher leverage ratio, L'. Since the new leverage ratio will be maintained into perpetuity, expected future debt levels are also equal to D_0; i.e., $E_0(D_i) = D_0$ for $i = 1, \cdots, \infty$. Now, let dV_0/dD_0 represent the value created at time 0 by the higher debt level at time 0 as well as the higher expected debt levels at all future times. Modigliani and Miller imply $dV_0/dD_0 = \tau$, whereas Equation (21) implies that

$$\frac{dV_0}{dD_0} = \left[\frac{1+\rho}{1+r}\right]\left[\frac{r}{\rho}\right]\tau. \tag{22}$$

Two testable predictions can be inferred from Equation (22). First, dV_0/dD_0 can be much less that τ even in the absence of personal tax biases. Second, dV_0/dD_0 is a decreasing function of the firm's business risk (as measured by either ρ or β_u, the unlevered beta coefficient). If the firm has no business risk, then $\rho = r$, and Equation (22) implies $dV_0/dD_0 = \tau$ which is the MM result. However, if $\rho > r$, then $dV_0/dD_0 < \tau$. These implications are depicted in Figure 1 which shows the relationships between dV_0/dD_0 and the unlevered beta coefficient for both the standard MM analysis and the analysis of this section.

III. Unlevering Beta

In the previous section, we demonstrated that when the market value leverage ratio is held constant, leverage contributes less to market value than the standard

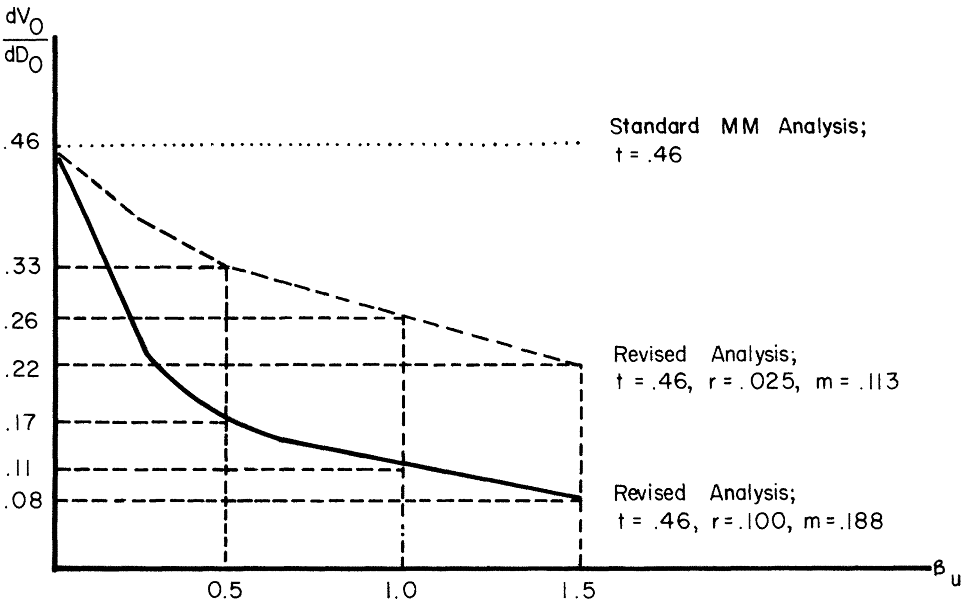

Figure 1. The Relationship Between dV_0/dD_0 and the Unlevered Beta Coefficient where τ, r, and r_m are the Corporate Tax Rate, Risk-free Rate, and Expected Return on the Market, Respectively.

analysis of MM valuation theory implies. In this section, we show that leverage increases the equity beta more than implied by the standard analysis.

Hamada [2] assumed that Equation (20) was correct and developed the following familiar equation for unlevering the equity beta:

$$\beta_u^* = \left(\frac{1 - L}{1 - \tau L}\right)\beta_E \tag{23}$$

where β_u^* and β_E are, respectively, the unlevered beta and the equity beta. The asterisk indicates that Hamada's unlevering procedure is used. To derive (23), it must be assumed that future interest tax write-offs are certain and this holds only if the firm's future debt levels are certain. However, if the firm maintains a constant leverage ratio, then the correct unlevering procedure is quite different from the one specified above since, as shown previously, tax shields occurring beyond time 1 are uncertain.

We begin by writing levered value from Equation (19) as

$$\begin{aligned} V_0 &= \Sigma_{j=1}^{T} \frac{E_0(X_j)}{(1 + \rho)^j} + \frac{\tau r L V_0}{(1 + r)} + \Sigma_{j=2}^{T} \frac{E_0(TS_j)}{(1 + \rho)^{j-1}(1 + r)} \\ &= \frac{E_0(\tilde{X}_1 + \tilde{V}_1^u)}{(1 + \rho)} + \frac{\tau r L V_0}{(1 + r)} + \frac{E_0(\tilde{V}_1^\tau)}{(1 + \rho)} \end{aligned} \tag{24}$$

where $\tilde{V}_1^u$ is the time 1 value of the unlevered cash flows beyond time 1 and $\tilde{V}_1^\tau$ is the time 1 value of all interest tax write-offs beyond time 1. In Equation (24), the time 0 levered value is partitioned into three components. The first component is simply unlevered value, the second is the present value of the time 1 tax shield, and the third is the value of all interest tax write-offs beyond time 1. The levered beta coefficient, β_L, can be specified as a weighted average of the betas of these three components.

The beta coefficient of the first value component of Equation (24) is β_u, the unlevered beta coefficient. Since the third component has the same expected rate of return as the first component, its beta coefficient is also β_u. Since the rate of return on the second value component is riskless, its beta coefficient is zero. The market value of this component is $\frac{\tau r L}{1 + r} V_0$ implying that $\frac{\tau r L}{1 + r}$ is the fraction of total value with a zero beta. Hence, the remaining fraction, $1 - \frac{\tau r L}{1 + r}$ has a beta of β_u. Therefore,

$$\beta_L = \left[1 - \frac{\tau r L}{1 + r}\right]\beta_u. \tag{25}$$

Since we also know that

$$\beta_L = L\beta_D + (1 - L)\beta_E \tag{26}$$

and following the Hamada assumption that β_D, the debt beta, is zero, we can obtain from Equations (25) and (26) the following relationship between β_E and

Table I

A Comparison of Unlevering Procedures

	$L = 0.2$		$L = 0.4$		$L = 0.6$	
β_E	β_u^*	β_u	β_u^*	β_u	β_u^*	β_u
0.5	0.44	0.40	0.37	0.31	0.28	0.21
1.0	0.88	0.81	0.74	0.61	0.55	0.41
1.5	1.32	1.21	1.11	0.92	0.83	0.62

Risk-free Rate = 10%

β_u:

$$\beta_u = \beta_E \left[\frac{1 - L}{1 - \dfrac{\tau r L}{1 + r}}\right]. \tag{27}$$

To compare this unlevering procedure with Hamada's, suppose an equity beta of β_E is observed. Letting β_u represent the unlevered beta specified in Equation (27), the percentage error in using the Hamada approach is

$$\frac{\beta_u^*}{\beta_u} - 1 = \frac{1 - \dfrac{\tau r L}{1 + r}}{1 - \tau L} - 1$$

$$= \frac{\tau L}{(1 + r)(1 - \tau L)}.$$

Of course, this is the correct error measurement only if firms maintain a constant leverage ratio. Note that for $\tau > 0$ and $L > 0$, $\beta_u < \beta_u^*$ and $\beta_u - \beta_u^*$ is an increasing function of the market value leverage ratio. Table I shows the estimated unlevered betas using both the Hamada procedure and the procedure developed in this section for selected combinations of the equity beta and the leverage ratio. Using β_u^* rather than β_u as a measure of asset risk will "severely" overestimate the cost of capital. This error would be compounded in an adjusted present value framework by adding the present value of tax shields discounted at the riskless rate.

IV. Conclusions

Figure 1 provides a convenient summary of our analysis. For the average firm (i.e., a firm with an unlevered beta of 0.75), the marginal value of debt financing is between \$0.15 and \$0.30 where the actual value will depend on the ratio of the risk-free rate to the unlevered cost of capital as shown in our Equation (22). Our estimate for the marginal value of debt in an MM tax world is substantially less than the \$0.46 put forth by Modigliani and Miller. Masulis [3], in a study of debt-equity swaps, concludes that the tax effect of debt financing is positive but smaller than the \$0.46 predicted by MM so that Masulis' observations are consistent with our analysis. (Masulis' results are also consistent with a personal tax bias in favor of equity income.) We use our valuation model to derive a relationship between the firm's equity beta and its unlevered beta which is significantly different from the relationship derived by Hamada. We have shown specifically that the Hamada relationship may cause a sizable overestimation of unlevered beta.

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