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Options on the Spot and Options on Futures

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ABSTRACT

This paper analyzes and compares the valuation of two types of options that relate to the same asset: options on the asset itself and options on the futures on the asset. The early exercise privilege plays a central role in explaining the differences between the values of the two options. It is shown that in the case of a cash instrument that does not make interim payments, such as gold, the value of a call option on the spot is smaller than the call option on the futures contract; the opposite is true for put options. The early exercise boundaries, which characterize when it pays to exercise, are also compared and analyzed.

THE FINANCIAL MARKETS HAVE recently witnessed a proliferation of various types of listed options on commodities and on financial assets. Commencing with call options on individual common stocks, recent innovations have included call and put options on Treasury bonds, metals, foreign currencies, and stock market indices. In some of these cases, markets have opened both in options on the underlying asset and options on futures contracts based on the asset.

The most recent examples of such twin instruments are the options on spot gold trading on the American Stock Exchange, and the options on gold futures, trading for two years now on the Commodities Exchange. Other assets that currently have futures, options on the spot, and options on futures are: Treasury bonds; the West German mark; the S&P 500 Stock Index; and the NYSE Composite Index.[1]

The introduction of such closely related assets naturally raises questions regarding the relative valuation of these two types of options. Aside from its theoretical appeal to academicians, this comparison is of considerable interest to investors and traders who wish to hedge their positions in the underlying assets.

Since options on futures are quite a recent phenomenon, few authors have dealt with the valuation of these options. Perhaps the first author to value options on forward and futures contracts is Black [2]. Black shows that, in the absence of interest rate uncertainty, a European option on a futures contract can be priced using a minor modification of the Black-Scholes [3] option pricing formula.

* Brenner is from The Hebrew University and New York University. Courtadon and Subrahmanyam are from New York University. A substantially similar version of this paper was presented at the American Stock Exchange Options Colloquium in March 1984. Comments from the participants of this conference improved the exposition of the paper. We would also like to thank Fischer Black and an anonymous referee for their helpful comments.

[1] It is interesting to note that all three financial instruments, the futures on the NYSE index, the options on the index, and the options on the futures are traded on the same floor, the New York Futures Exchange. This setup makes an important contribution to the efficiency of these markets.

Other authors such as Asay [1] and Ramaswamy and Sundaresan [11] attempt to value options on futures contracts by deriving rational pricing restrictions and valuation formulae for options on forward contracts. However, all of these studies neglect a crucial feature of these options: the early exercise privilege attached to American options. As shown in the remainder of this paper, the early exercise feature plays a central role in explaining the differences between the value of the option on the futures contract and that of the option on the cash instrument.

The only study to emphasize the importance of the early exercise feature in this context is Courtadon [6], who analyzed Treasury bond options on both the cash and futures instruments. However, that study took into account certain features specific to interest rate instruments, the most important among these being the process generating interest rates. In contrast, this paper analyzes the more general question of the relationship between option prices when the assets involved are not necessarily of the fixed income type.[2]

The following section presents rational pricing restrictions for European options on futures versus on cash instruments. In addition to providing a summary of the results for European options of the two types, this section provides a link with the previous literature in the area. Section II extends the comparison between the values of the two types of options to the case where these options are American, and hence can be prematurely exercised. Since early exercise is a crucial feature affecting the comparison of values, some interesting propositions regarding the optimal early exercise policies of the two types of options are derived. In particular, it is shown that, contrary to the case of an American call option on a cash[3] instrument, it may be profitable to exercise an American call option on the futures contract before the expiration date. The intuition behind this is that futures contracts lose value continuously, in relation to the spot, as would a stock that pays a continuous dividend.

I. European Options on Futures Contracts and Cash Instruments

We begin by making the usual assumptions of no taxes, transaction costs (including bid-ask spreads), or short sales restrictions. We also assume that the maturity dates for the futures contract and the two types of options coincide. Further, if interest rates are constant and no dividends are paid on the underlying instrument, then it follows that:

1. The futures (and forward) price, F, is the compounded value of the spot price, S. If t is the current calendar time and T is the maturity date, with an interest rate, r,

$$F = Se^{r(T-t)}. \quad (1)$$

This follows from first-order stochastic dominance arguments. In the ab-

[2] In general, the relative values of these options depend on the nature of the spot asset and the nature of the futures on that spot asset. Here, however, we concentrate on the early exercise feature that is common to all options and is of crucial importance in determining the relative values of the options.

[3] The term cash and spot are used interchangeably when referring to assets for immediate delivery.

sence of transaction costs, buying the cash instrument and holding it to maturity should yield the same as buying the futures contract and obtaining delivery at maturity.

2. Put-call parity holds for options on the spot instrument. If P_S and C_S, respectively, represent the price of the put and call written on the spot asset at an exercise price, E,

$$P_S + S - C_S = Ee^{-r(T-t)}. \tag{2}$$

The above result follows from arbitrage considerations. Consider a portfolio of buying the cash instrument and a put option on it, and selling a call option of the same maturity at the same exercise price. This portfolio will have a certain payoff at maturity of E, which has a current value of $Ee^{-r(T-t)}$.

3. In the case of options on the futures contract, put-call parity does not hold. Instead, put-call parity holds with respect to the implied spot price,

$$P_F - C_F = (E - F)e^{-r(T-t)} \tag{3a}$$

$$P_F + F - C_F \neq Ee^{-r(T-t)}. \tag{3b}$$

To see this, consider a portfolio created by buying a futures contract and a put option on the futures, and selling a call option of the same maturity with the identical exercise price on the same futures contract. At maturity, this portfolio will have a cash flow equal to the difference between the exercise price and the futures price contracted at time, t. This implies the first equation in (3) above. The second result follows from the first.

4. The value of a European option on the cash instrument will always be equal to the value of a European option written on the futures contract based on the instrument, provided that the two options have the same maturity date and exercise price. This result is obvious when it is pointed out that, under the conditions assumed, the stream of cash flows to be received at maturity on both options are identical, since neither option can be exercised before that date and the spot price equals the futures price on that date.

The propositions above deal with conclusions that can be drawn based on arbitrage or stochastic dominance arguments. Obviously, more specific assumptions regarding the stochastic process generating the return on the spot instrument are necessary in order to derive a closed-form expression for the value of the option. For example, if the spot instrument follows a geometric Wiener process, the Black-Scholes option pricing model would be applicable to the valuation of the call option on the spot. Under these conditions, $C_S(S, t)$, representing the value of the call option as a function of the spot value and time, will satisfy the following valuation equation:[4]

$$\frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C_S}{\partial S^2} + rS\,\frac{\partial C_S}{\partial S} - rC_S + \frac{\partial C_S}{\partial t} = 0 \tag{4}$$

where σ is the standard deviation of the instantaneous rate of return on the spot.

[4] The valuation equation for the put option (or any other contingent claim) would be the same, except that the boundary conditions would depend on the payoff formula.

Using the same framework, an argument similar to the one made by Black [2] would yield the valuation equation for the call option on the futures contract. If $C_F(F, t)$ is the value of the call option on the futures contract, the following equation would be satisfied

$$\frac{1}{2}\sigma^2 F^2 \frac{\partial^2 C_F}{\partial F^2} - rC_F + \frac{\partial C_F}{\partial t} = 0. \quad (5)$$

In order to obtain closed-form solutions for the values of the two options, the appropriate boundary conditions must be imposed on Equations (4) and (5).

The explicit valuation problem in the two cases is a straightforward application of the work of Black and Scholes [3] for options on the cash instrument and Black [2] for options on futures. An application of these two models to the pricing of European options on stock indices and stock index futures can be found in Asay [1].

II. American Options on Futures Contracts and Cash Instruments

The analysis of Section I demonstrates that as long as options on the spot and the equivalent options on the futures have the same maturity and are European, they will be valued identically. It follows, therefore, that a difference in value could result only if the options are American and, in particular, because of their early exercise privilege.[5]

It is obvious that early exercise may occur in the case of a call option due to the payment of dividends or interest. When a stock pays dividends, its price will fall on the ex-dividend day. If the call option is not payout-protected and if the premium over parity is relatively small, the drop in value may be more than the premium over parity and it may pay to exercise the option on the last cum-dividend day. In the case of options on commodities, like gold, where no interim payments are made, early exercise of call options on the spot will not be rational. As shown below, the early exercise of calls on futures is due to the fact that futures carry a premium and are marked to the market. The likelihood of exercise depends, in particular, on how large that premium is.

We will first derive valuation models for cash options and futures options assuming interim distributions. Options on assets with no interim payments are just a special case of these models. The possibility of early exercise will be characterized by an optimal early exercise boundary which gives the "trigger" price at which it is optimal to exercise at any point in time. We then translate these boundaries into values for the two options.

Equation (4) is a valid description of the valuation equation for options on stocks, between dividend payments, as argued by Schwartz [13] and Geske [9]. On the ex-dividend day, an adjustment must be made as suggested by Roll [12] and Geske [9]. By contrast, however, the derivation of the valuation equation for

[5] A divergence in value could also arise from institutional factors such as differences in maturity dates, margin requirements, exercise policies, tax treatments, or transactions costs. These issues are not explored in this paper.

futures options requires an assumption about the functional relationship between the futures and spot prices.

In the absence of interest rate uncertainty,[6] futures and forward prices are identical and equal to the compounded value until maturity of the present spot price. If there are any interim distributions, they must be taken into account. Thus, the futures price, F, at any time, t, can be written in terms of the contemporaneous spot price, S, and the maturity date, T:[7]

$$F(S, t, T) = Se^{r(T-t)} - \sum_{i=1}^{N} D_i e^{r(T-t_i)} \tag{6}$$

where N is the number of dividend payments between t and T, D_i is the amount of the ith dividend paid between t and T, and t_i is the corresponding ex-dividend date. For convenience, assume that the ex-dividend days and the dividend payment days are identical.

The valuation equation of the option on the futures contract based on (6) above is:

$$\frac{1}{2}\,\sigma^2[F + \sum_{i=1}^{N} D_i e^{r(T-t_i)}]^2\,\frac{\partial^2 C_F}{\partial F^2} - rC_F + \frac{\partial C_F}{\partial t} = 0. \tag{7}$$

In the case of an American call option on the cash instrument, the boundary conditions to be imposed on Equation (4) are as follows:

$$C_S(S, T) = \text{Max}[0, S - E] \tag{8a}$$

$$C_S(S, t_{\bar{i}}) = \begin{cases} S - E & \text{if } C_S(S - D_i, t_i) \leq S - E \\ C_S(S - D_i, t_i) & \text{if } C_S(S - D_i, t_i) > S - E \end{cases} \tag{8b}$$

where $t_{\bar{i}}$ is the time just before the asset "goes ex-" the ith dividend. No other conditions need be imposed, since we know from Merton [10] that it would never be optimal to exercise such a call option between dividend payment dates. The valuation model of an American call option on a stock or a cash instrument [i.e., Equation (4) subject to boundary conditions (8a) and (8b)] does not have a closed-form solution and must be solved using numerical methods such as the one described by Schwartz [13].

In the case of the American call option on the futures contract, the valuation problem is somewhat different. On one hand, it is simpler since the futures price, unlike the spot price, does not decrease by a jump on the "ex" day, but decreases smoothly over time in anticipation of the future payments. Hence, a condition such as (8b) need not be imposed on the valuation model for this option. On the other hand, the problem is more complicated since the Merton [10] result about premature exercise just prior to the ex-dividend day does not hold in this case.

[6] In the presence of interest rate uncertainty, the relationship between the spot and futures prices is a complex one and depends on the specifics of the stochastic process governing interest rates and investor preferences as in Cox, Ingersoll, and Ross [7]. Since much of the literature in the area of futures and forward contracts (see, e.g., the most recent paper by Elton, Gruber, and Rentzler [8]) does not find much of a difference, in empirical terms, caused by interest rate uncertainty, we decided to ignore it and keep the analysis tractable.

[7] This approximation was used previously by Cornell and French [4] for the case of a continuous dividend payment.

Since the futures price converges smoothly toward the spot price at maturity depending on the force of interest, the call option on the futures contract is akin to one on a stock paying a continuous dividend, as in Merton [10]. This arises from the fact that, even if the spot price remains constant over a period of time, the futures price decreases as the time approaches the maturity date. This is evident from Equation (1). Therefore, as shown by Merton,[8] it may be optimal to exercise the American futures option at any time before the maturity date. Note that this effect is independent of the interim payments and would arise even if there were no such payments.

To obtain the value of the American call option on the futures contract, we must solve Equation (7) subject to:

$$C_F(F, T) = \text{Max}[0, F - E] = \text{Max}[0, S - E] \tag{9a}$$

$$C_F(F, s) = \text{Max}[C_F^*(F, s), F - E] \quad \text{for} \quad t \le s < T \tag{9b}$$

where $C_F^*(F, s)$ is the value of the call option if kept alive at time s. As in the previous case, the problem does not have a closed-form solution and must be solved numerically.

In the case of put options, the valuation problem is similar except for the boundary conditions. The value of the American put option on the spot is given by Equation (4), subject to the following boundary conditions:

$$P_S(S, T) = \text{Max}[0, E - S] \tag{10a}$$

$$P_S(S, s) = \text{Max}[P_S^*(S, s), E - S] \quad \text{for} \quad t \le s < T \tag{10b}$$

$$P_S(S, t_{\bar{i}}) = \begin{cases} P_S(S - D_i, t_i) & \text{if } P_S(S - D_i, t_i) > E - S \\ E - S & \text{if } P_S(S - D_i, t_i) \le E - S \end{cases} \tag{10c}$$

where $t_{\bar{i}}$ is the same as defined previously, and $P_S^*(S, s)$ is the value of the put if kept alive at time s.

For the case of the American put option on the futures contract, Equation (7) should be solved subject to:

$$P_F(F, T) = \text{Max}[0, E - F] = \text{Max}[0, E - S] \tag{11a}$$

$$P_F(F, s) = \text{Max}[P_F^*(F, s), E - F] \quad \text{for } t \le s < T \tag{11b}$$

to yield the value of the option, where $P_F^*(F, s)$ is the value of the option at time s if kept alive. As in the earlier cases, these two put valuation problems do not have a closed-form solution and can be solved numerically, using an approach similar to the one used in Schwartz [13].

III. A Comparative Analysis of Values and Optimal Exercise Boundaries for Options on Futures and Options on Cash

A. Options on Assets with No Interim Payments

As mentioned earlier, the value of an American option on the spot will be different from the value of an American option on the futures contract. This

[8] See Merton [10, footnote 61].

holds for puts and calls and whether or not there are interim payments. While the value of the American call option on the spot is equal to the value of a European call and is given by the Black-Scholes formula, the value of the American call option on the futures contract is obtained using a numerical solution.

Using the numerical methods described in Schwartz [13] or Courtadon [5], Tables I and II present the differences in value between the futures option and

Table I

Difference in Value Between the American Futures-Call Option and the Spot-Call Option* (No Payout Case)

		$r = 0.15$			$\sigma = 0.25$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		0.0000	0.0000	0.0000	0.0002	0.0007
		—	0.00%	0.00%	1.20%	2.02%
0.9000		0.0000	0.0000	0.0001	0.0008	0.0025
		0.00%	0.00%	0.47%	1.58%	3.15%
1.0000		0.0001	0.0003	0.0007	0.0027	0.0065
		0.29%	0.56%	1.02%	2.48%	4.51%
1.1000		0.0007	0.0015	0.0025	0.0071	0.0142
		0.61%	1.15%	1.72%	3.79%	6.34%
1.2000		0.0026	0.0048	0.0069	0.0153	0.0273
		1.22%	2.13%	2.89%	5.52%	8.70%
		$r = 0.15$			$\sigma = 0.20$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		0.0000	0.0000	0.0000	0.0001	0.0004
		—	0.00%	0.00%	1.12%	1.78%
0.9000		0.0000	0.0000	0.0001	0.0006	0.0020
		0.00%	0.00%	0.74%	1.57%	3.13%
1.0000		0.0001	0.0003	0.0006	0.0026	0.0065
		0.34%	0.66%	1.01%	2.68%	4.98%
1.1000		0.0009	0.0018	0.0030	0.0082	0.0164
		0.80%	1.42%	2.13%	4.58%	7.64%
1.2000		0.0026	0.0055	0.0084	0.0188	0.0326
		1.22%	2.45%	3.55%	6.88%	10.58%
		$r = 0.10$			$\sigma = 0.20$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		0.0000	0.0000	0.0000	0.0000	0.0002
		—	0.00%	0.00%	0.00%	1.27%
0.9000		0.0000	0.0000	0.0000	0.0003	0.0008
		0.00%	0.00%	0.00%	1.00%	1.62%
1.0000		0.0001	0.0002	0.0003	0.0012	0.0027
		0.37%	0.49%	0.57%	1.46%	2.51%
1.1000		0.0005	0.0010	0.0015	0.0038	0.0072
		0.46%	0.84%	1.15%	2.38%	3.87%
1.2000		0.0017	0.0034	0.0049	0.0096	0.0157
		0.82%	1.57%	2.18%	3.82%	5.67%

* Entries in percent represent differences in value as a percent of the spot-option value.

Table II

Difference in Value Between the American Futures-Put Option and the Spot-Put Option* (No Payout Case)

		$r = 0.15$			$\sigma = 0.25$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		−0.0099	−0.0200	−0.0292	−0.0497	−0.0645
		−4.95%	−10.00%	−14.60%	−24.85%	−32.25%
0.9000		−0.0087	−0.0121	−0.0149	−0.0224	−0.0291
		−8.70%	−12.10%	−14.80%	−21.66%	−27.56%
1.0000		−0.0010	−0.0022	−0.0035	−0.0073	−0.0110
		−4.22%	−7.03%	−9.64%	−16.04%	−21.61%
1.1000		0.0000	−0.0003	−0.0007	−0.0022	−0.0041
		0.00%	−4.69%	−6.93%	−12.09%	−17.45%
1.2000		0.0000	0.0000	−0.0001	−0.0006	−0.0015
		0.00%	0.00%	−4.55%	−9.09%	−14.42%
		$r = 0.15$			$\sigma = 0.20$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		−0.0099	−0.0200	−0.0301	−0.0568	−0.0767
		−4.95%	−10.00%	−15.05%	−28.40%	−38.35%
0.9000		−0.0105	−0.0172	−0.0218	−0.0318	−0.0392
		−10.50%	−17.20%	−21.80%	−31.80%	−39.20%
1.0000		−0.0011	−0.0023	−0.0036	−0.0073	−0.0107
		−6.04%	−9.75%	−13.28%	−21.99%	−29.23%
1.1000		0.0000	−0.0001	−0.0004	−0.0015	−0.0028
		0.00%	−3.70%	−8.51%	−15.96%	−22.40%
1.2000		0.0000	0.0000	0.0000	−0.0003	−0.0007
		—	0.00%	0.00%	−13.04%	−17.95%
		$r = 0.10$			$\sigma = 0.20$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		−0.0066	−0.0133	−0.0200	−0.0373	−0.0504
		−3.30%	−6.65%	−10.00%	−18.65%	−25.20%
0.9000		−0.0068	−0.0104	−0.0127	−0.0181	−0.0231
		−6.80%	−10.40%	−12.70%	−17.89%	−22.47%
1.0000		−0.0006	−0.0014	−0.0023	−0.0048	−0.0073
		−3.06%	−5.36%	−7.54%	−12.31%	−16.52%
1.1000		0.0000	−0.0001	−0.0003	−0.0011	−0.0022
		0.00%	−3.03%	−5.00%	−8.87%	−12.79%
1.2000		0.0000	0.0000	0.0000	−0.0002	−0.0006
		—	0.00%	0.00%	−6.06%	−9.84%

* Entries in percent represent differences in value as a percent of the spot-option value.

the spot option. From Table I, which deals with call options, we can make the following inferences:

(a) Call options on futures are worth more than call options on the spot.
(b) The difference in value is a nondecreasing function of time to maturity.
(c) The difference in value is a nondecreasing function of the rate of interest.
(d) The difference in value is a nondecreasing function of volatility for

relatively low values of S/E and a nonincreasing function of volatility for relatively high values of S/E.

This difference in value can be significant. For example, if the annual rate of interest is 15%, the standard deviation is 20%, time to maturity is 270 days and the option is at-the-money, the difference in value is about 5% of the spot call value. If the option is 20% in-the-money, the difference in value amounts to about 10.6%.

Similar inferences can be made from Table II, which deals with put options:

(a) Put options on futures are worth less than put options on the spot.
(b) The difference in value is a nonincreasing function of time to maturity.
(c) The difference in value is a nonincreasing function of the rate of interest.
(d) The difference in value is a nondecreasing function of volatility for relatively low values of S/E and a nonincreasing function of volatility for relatively high values of S/E.

In the case of put options, the difference is more significant. Using the same values of the parameters as for call options above, the difference in value for the at-the-money put is about 29%, and for the 20% in-the-money put is about 38%. These differences in value are a reflection of the early exercise possibilities, which can be characterized by optimal early exercise boundaries. Examples of these boundaries are provided in Figure 1. The analysis is simple for the case of a call option on the spot: as proven by Merton [10] and as corroborated by the numerical algorithm, it will never be optimal to exercise this option before maturity. For the American call option on the futures, there will always exist a value of the futures price above which it will be optimal to exercise the option early. For example, if r is 15%, σ is 25%, and 120 days are left to maturity, it is optimal to exercise when the value of the spot is 22% above the exercise price (or, when the value of the future is 28% above the exercise price). When plotted as a function of time, these values constitute an optimal exercise boundary. This boundary is a nondecreasing function of the volatility parameter, a nonincreasing function of the rate of interest, and a nondecreasing function of the time to maturity.

In the case of American put options, the early exercise boundary will be defined by the time series of spot prices or futures prices below which it will be optimal to exercise the option early. The comparative statics analyses are similar for both put options: the boundaries are nonincreasing functions of the volatility parameter and of time to maturity, and nondecreasing functions of the rate of interest. It is worth noting that the futures-put option is less likely to be exercised than the spot-put option.

B. Options on Instruments with Interim Payments

In this subsection, we analyze the difference in value between the two options when the underlying instrument provides interim payments. We will use the example of a quarterly payout. For simplicity, we assume a single known payment

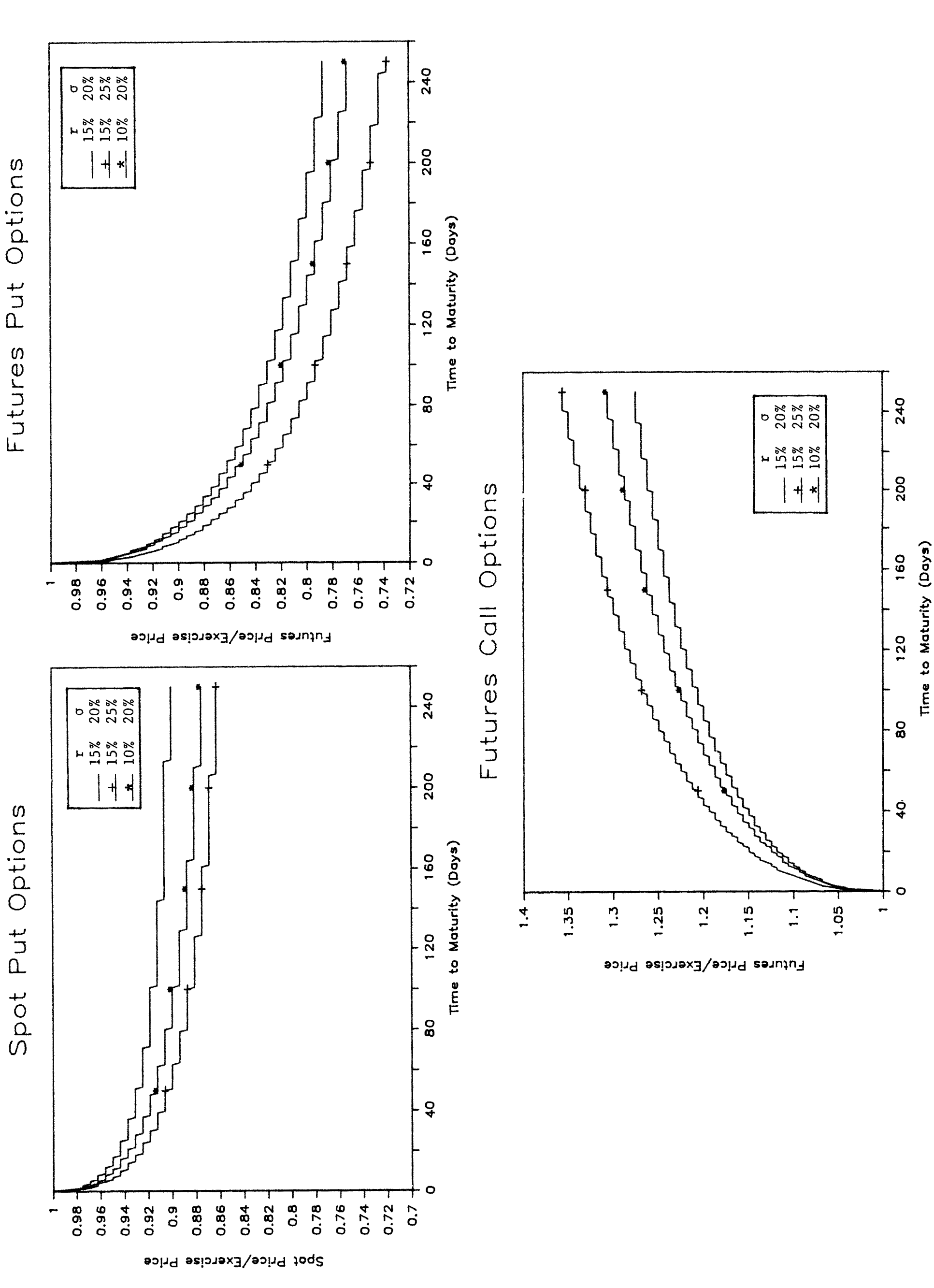

Figure 1. Optimal Exercise Boundaries (No Payout Case)

each quarter. Tables III and IV correspond to Tables I and II, respectively, with the addition that the spot pays a quarterly dividend of 0.01 (the number is already scaled by the exercise price of the option), and the last payout is 30 days before the options' expiration date.

The comparative statistics results that we derived in the previous section with respect to the volatility parameter and the rate of interest still hold. However,

Table III

Difference in Value Between the American Futures-Call Option and the Spot-Call Option* (Quarterly Payout of 0.01; Last Payout Made 30 Days Before Maturity)

		$r = 0.15$			$\sigma = 0.25$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		0.0000	0.0000	0.0000	0.0001	0.0005
		—	0.00%	0.00%	0.76%	1.85%
0.90000		0.0000	0.0000	0.0001	0.0006	0.0018
		0.00%	0.00%	0.54%	1.40%	2.73%
1.0000		0.0001	0.0003	0.0006	0.0022	0.0051
		0.34%	0.63%	0.95%	2.27%	4.07%
1.1000		0.0006	0.0013	0.0022	0.0060	0.0116
		0.57%	1.07%	1.60%	3.50%	5.76%
1.2000		0.0024	0.0043	0.0063	0.0133	0.0228
		1.19%	2.00%	2.75%	5.12%	7.91%

		$r = 0.15$			$\sigma = 0.20$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		0.0000	0.0000	0.0000	0.0000	0.0002
		—	0.00%	0.00%	0.00%	1.23%
0.9000		0.0000	0.0000	0.0000	0.0004	0.0014
		0.00%	0.00%	0.00%	1.29%	2.76%
1.0000		0.0000	0.0002	0.0005	0.0021	0.0048
		0.00%	0.50%	0.93%	2.50%	4.34%
1.1000		0.0008	0.0016	0.0026	0.0067	0.0129
		0.77%	1.36%	1.97%	4.12%	6.78%
1.2000		0.0025	0.0051	0.0079	0.0166	0.0278
		1.24%	2.38%	3.48%	6.51%	9.88%

		$r = 0.10$			$\sigma = 0.20$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		0.0000	0.0000	0.0000	0.0000	0.0000
		—	0.00%	0.00%	0.00%	0.00%
0.9000		0.0000	0.0000	0.0000	0.0000	0.0003
		0.00%	0.00%	0.00%	0.00%	0.78%
1.0000		0.0000	0.0000	0.0000	0.0005	0.0014
		0.00%	0.00%	0.00%	0.71%	1.55%
1.1000		−0.0001	0.0000	0.0004	0.0022	0.0046
		−0.10%	0.00%	0.33%	1.53%	2.81%
1.2000		−0.0002	0.0015	0.0029	0.0067	0.0113
		−0.10%	0.72%	1.34%	2.86%	4.49%

* Entries in percent represents difference in value as a percent of the spot-option value.

Table IV

Difference in Value Between the American Futures-Put Option and the Spot-Put Option* (Quarterly Payout of 0.01; Last Payout Made 30 Days Before Maturity)

		$r = 0.15$			$\sigma = 0.25$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		−0.0098	−0.0098	−0.0195	−0.0327	−0.0420
		−4.67%	−4.90%	−9.75%	−16.35%	−21.00%
0.9000		−0.0095	−0.0069	−0.0088	−0.0138	−0.0184
		−8.64%	−6.71%	−8.64%	−12.93%	−16.64%
1.0000		−0.0013	−0.0023	−0.0026	−0.0052	−0.0078
		−4.56%	−6.48%	−6.60%	−10.30%	−13.59%
1.1000		−0.0001	−0.0004	−0.0007	−0.0018	−0.0031
		−3.33%	−5.19%	−6.03%	−8.45%	−11.07%
1.2000		0.0000	0.0000	−0.0001	−0.0006	−0.0012
		0.00%	0.00%	−3.85%	−7.41%	−9.23%
		$r = 0.15$			$\sigma = 0.20$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		−0.0098	−0.0098	−0.0200	−0.0385	−0.0525
		−4.67%	−4.90%	−10.00%	−19.25%	−26.25%
0.9000		−0.0108	−0.0085	−0.0141	−0.0196	−0.0248
		−9.82%	−8.50%	−14.10%	−19.54%	−24.48%
1.0000		−0.0015	−0.0024	−0.0027	−0.0054	−0.0080
		−6.52%	−8.66%	−9.03%	−14.32%	−18.87%
1.1000		0.0000	−0.0002	−0.0004	−0.0013	−0.0023
		0.00%	−5.56%	−6.90%	−11.11%	−14.47%
1.2000		0.0000	0.0000	0.0000	−0.0003	−0.0007
		—	0.00%	0.00%	−10.00%	−12.96%
		$r = 0.10$			$\sigma = 0.20$	
S/E	Days to Maturity:	30	60	90	180	270
0.8000		−0.0065	−0.0049	−0.0099	−0.0186	−0.0246
		−3.10%	−2.43%	−4.95%	−9.30%	−12.30%
0.9000		−0.0071	−0.0054	−0.0055	−0.0085	−0.0111
		−6.45%	−5.21%	−5.45%	−8.07%	−10.17%
1.0000		−0.0009	−0.0016	−0.0018	−0.0030	−0.0043
		−3.67%	−5.23%	−5.25%	−6.67%	−8.25%
1.1000		0.0000	−0.0001	−0.0003	−0.0009	−0.0015
		0.00%	−2.33%	−4.11%	−5.77%	−6.82%
1.2000		0.0000	0.0000	0.0000	−0.0002	−0.0005
		—	0.00%	0.00%	−4.44%	−6.02%

* Entries in percent represent differences in value as a percent of the spot-option value.

the other results concerning the signs of the deviation and the effects of changes in the time to maturity, and the effects of changes in the underlying spot value or futures value are no longer valid. One interesting result is that an increase in the payout will make the difference in value between the futures-call option and the spot-call option less positive and the difference in value between the futures-put option and the spot-put option less negative. Indeed, for a sufficiently high value of the interim payment given a value of the rate of interest, the sign of the value difference can be the opposite of what it was in the no-dividend case.

Other interesting results are obtained by analyzing the effect of an increase in payout on the optimal early exercise boundaries. The early exercise boundary for the spot-call option is only a series of discrete points in time, since it is only optimal to exercise this option just before the spot goes ex-dividend. The sensitivity of this boundary with respect to the rate of interest is the opposite of that of the early exercise boundary of the futures-call option. The spot-call option early exercise boundary is a nondecreasing function of the rate of interest. With respect to the volatility parameter the analysis shows that, for the futures-call option, the early exercise boundary is a nondecreasing function of volatility. The effect of volatility on the spot-call option, however, is ambiguous.[9]

Finally, a larger payout on the spot will mean a higher probability of early exercise for the spot-call option and a lower probability of early exercise for the futures-call option. This corroborates intuition, since an increase in payout lowers the futures price given a constant spot price while the futures option early exercise boundary remains practically the same, in terms of the futures price to exercise price ratio, as shown by Figure 2.

Concerning put options, it is unambiguous from Figures 1 and 2 that an increase in payout increases the probability of early exercise of the futures-put option. This result follows from the fact that the early exercise boundary remains practically unchanged while the futures price decreases given a constant spot price. The results concerning the effect of an increase in payout on the probability of early exercise of the spot-put option are more ambiguous. It is apparent from Figure 2 that an increase in payout implies a downward shift of the spot-put option early exercise boundary. Whether the increase in payout also implies an increase in the probability of early exercise of the spot-put option is unclear, however, since the increase in payout does imply an increase in the probability that the stock price will reach low values before the maturity of the option.

IV. Summary and Conclusions

This paper deals with the comparative valuation of options on the spot and options on futures. The main reason for differences in values is the early exercise privilege attached to any American option. As discussed earlier, the effect of the force of interest on the early exercise of a futures option is similar to the effect of a continuous payout on the early exercise of a spot option. In fact, if the spot had a continuous and proportional payout exactly equal to the rate of interest, the two options would have the same value. In reality, however, the spot asset either has a zero payout or a positive payout at discrete points in time while the futures price converges to the spot. Thus, differences in value must arise.

For assets that make no interim payments, like gold or silver, call options on futures should have higher values than options on the spot while put options on

[9] It can be shown that for low values of the rate of interest in relation to the annual payout rate, the boundary is a nondecreasing function of volatility, while for sufficiently large values of the interest rate, Merton's sufficient condition for no exercise [10, Equation (12)] will obtain.

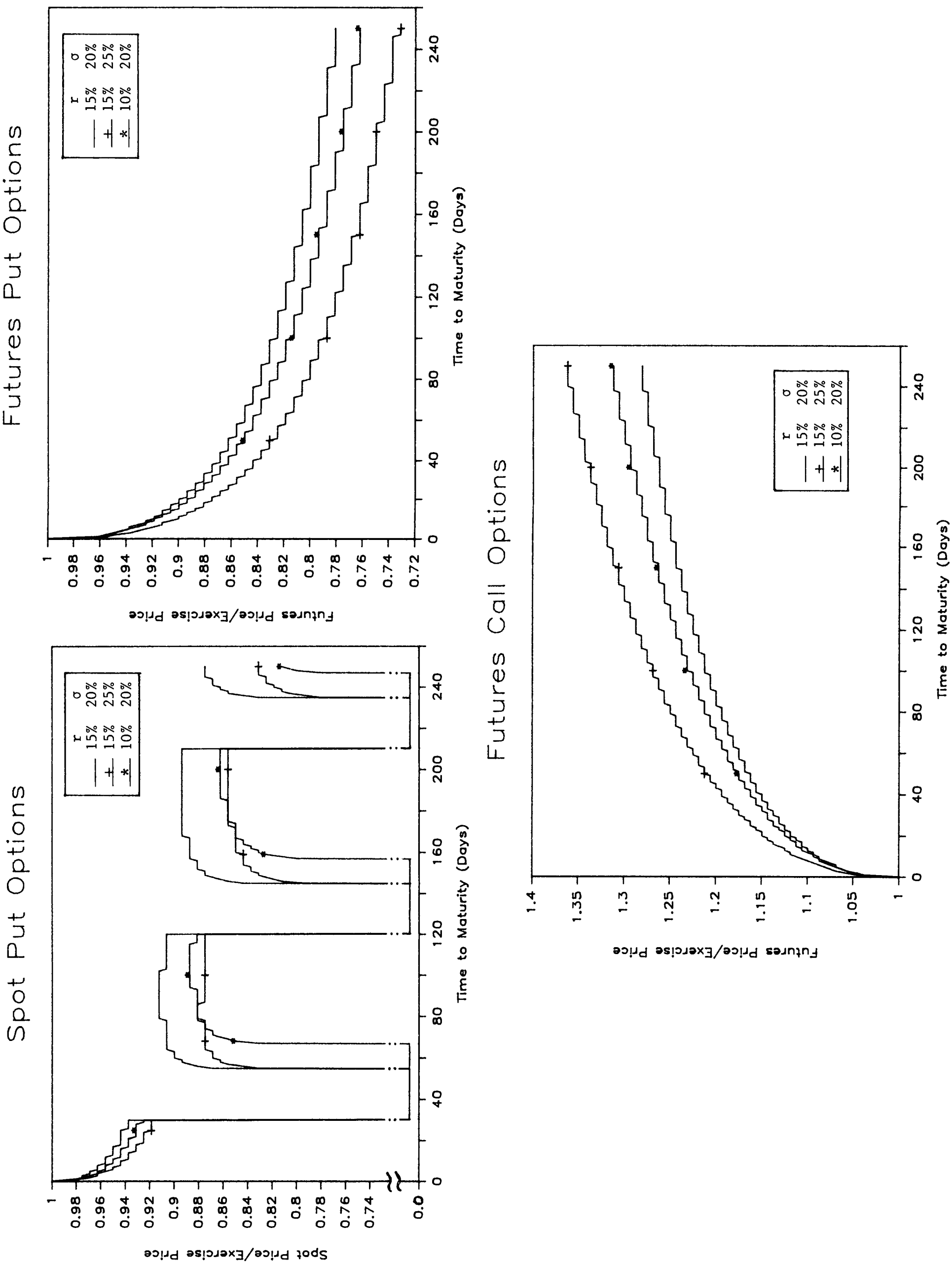

Figure 2. Optimal Exercise Boundaries (Quarterly Payout of 0.01)

futures should have lower values than options on the spot. The difference in value is more pronounced for put options than for call options. Regarding optimal exercise policies, the early exercise boundary for call options is a nondecreasing function of volatility and time to maturity and a nonincreasing function of the interest rate. The opposite is true concerning early exercise of put options.

For assets that do make interim payments, the difference in value for call options and put options will decrease as the payments increase. As for the optimal exercise boundaries, it is shown that an increase in payments will decrease the probability of early exercise of call options on futures and increase the probability of early exercise of put options on futures.

Finally, it should be mentioned again that the relative values of these options may be affected by various institutional arrangements such as, delivery and exercise policies, margin requirements, and transactions costs. For example, call options on stock indices, when exercised, are settled on a cash basis while futures-call options provide the holder with a futures contract. This difference in exercise policy may affect early exercise differentially and thus affect relative values.

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