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Source: *The Journal of Finance*, Vol. 40, No. 4 (Sep., 1985), pp. 1245-1250
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328406
Accessed: 26/11/2009 08:46

THE JOURNAL OF FINANCE • VOL. XL, NO. 4 • SEPTEMBER 1985

Easy Proofs of Unanimity and Optimality without Spanning: A Pedagogical Note

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WE OFFER EASY INTUITIVE proofs of shareholder unanimity and constrained Pareto optimality for competitive stock markets.

The features that are noteworthy are:

(i) there is no assumption of spanning required for unanimity or optimality;
(ii) agents are assumed to have consistent competitive conjectures, but not necessarily correct competitive conjectures;
(iii) competitive conjectures are characterized in terms of the invariance of individuals' shadow prices for income, as is standard in the finance literature; and
(iv) there are no short sales of shares permitted.

Regarding these points, (i) follows the line introduced in Hart [6, 7] and further developed in Makowski [8, 9]. And (ii) follows Pepall's weakening of "correct" to "consistent" competitive conjectures, which as demonstrated in [10] allows for a more general optimality result. It also allows, as we show, for a more general unanimity result; and it allows one to drop the assumption that any firm's production decision does not effect other firms' production decisions (cf. (2) in [8]). Next, (iii) follows the line adopted in the finance literature (cf. the no-surplus approach adopted in [8, 9]). Finally, (iv) follows Makowski [8, 9], which show by way of simple examples that unanimity without spanning is not true as a general result, without a short-sales restriction.

These proofs should fill in a gap between [8] and [9]. The former paper is very general but hard, and therefore not likely to receive many explorers. The latter is easy but quite informal, and therefore not likely to make many "converts" to the idea that spanning has been given a false role in the finance literature. The true import of "spanning" is to provide *information* about pricing new securities (in the language below, information about people's shadow prices), not to ensure unanimity or optimality given such information. This theme is developed and explored in Pepall [10].

Benninga and Muller [2] also contain some results on weakening the spanning assumption. They show that sometimes current spanning is sufficient to ensure approval by at least a majority of a firm's initial shareholders. However, as pointed out in Winter [11] and acknowledged in Benninga and Muller [3], the results in [2] are ostensibly relevant only when a firm's choice of outputs across

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states is restricted to a one-dimensional manifold. Thus, the framework in [2] is much less general than the Arrow-Debreu framework we shall be working with below.

I. A Standard Financial Model

The economy consists of I individuals indexed by i, and F firms indexed by f. There are two periods, $t \in [0, 1]$, and S possible states in period $t = 1$, indexed by s. Also, there is only one commodity available for consuming or investing in each period.

A production decision by firm f is a $y^f \in Y^f \subset R^{S+1}$, where $y^f \equiv (y_0^f, y_1^f) \in R_- \times R_+^S$. Similarly, a consumption decision by individual i is an $x^i \equiv (x_0^i, x_1^i) \in R_+^{S+1}$. Each individual has a concave differentiable utility function u^i from R_+^{S+1} to the reals, with always positive marginal utility from initial consumption, i.e., $u_0^i(x^i) > 0$ for all x^i. He or she also has an initial commodity endowment, $\underline{x}^i \in R_+^{S+1}$ and initial shareholdings, $\underline{s}^{if} \geqq 0$ such that $\sum_i \underline{s}^{if} = 1$ for all f.

An *exchange equilibrium given* y is defined as an $(x, s, y, p) \in R_+^{I(S+1)} \times R_+^{IF} \times_f Y^f \times R^F$, such that for all i, (x^i, s^i) maximizes $u^i(\tilde{x}^i)$

subject to

$$\tilde{x}_0^i + \textstyle\sum_f \tilde{s}^{if} p^f \leqq \underline{x}_0^i + \sum_f \underline{s}^{if}(p^f + y_0^f)$$

$$\tilde{x}_1^i = \underline{x}_1^i + \textstyle\sum_f \tilde{s}^{if} y_1^f$$

$$\tilde{x}^i \geqq 0, \quad \tilde{s}^i \geqq 0 \tag{1}$$

and

$$\textstyle\sum_i s^i = \sum_i \underline{s}^i = (1, 1, \cdots, 1) \in R^F \tag{2}$$

$$\textstyle\sum_i x^i = \sum_i \underline{x}^i + \sum_f y^f. \tag{3}$$

Note that $\tilde{s}^i \geqq 0$ in (1) is a no-short-sales constraint. Also note that in any exchange equilibrium given y, the Kuhn-Tucker theorem implies

$$\text{for all } f \text{ and all } i\text{: } p_1^i y_1^f \leqq p^f, \text{ with } < \text{ implying } s^{if} = 0 \tag{4a}$$

where $p_1^i \equiv u_1^i(x^i)/u_0^i(x^i)$ is the vector of i's shadow prices for state contingent income in period $t = 1$, measured in terms of initial period income. Since $\sum s^{if} = 1 > 0$ by the market clearing condition (2), we also know:

$$\text{for all } f \text{ and some } i\text{: } p_1^i y_1^f = p^f. \tag{4b}$$

Combining (4a) and (4b) shows that:

$$\text{for all } f,\ p^f = \operatorname*{Max}_i \{p_1^i y_1^f\} \equiv v(y^f). \tag{5}$$

Define $p^i \equiv (1, p_1^i) \in R^{S+1}$ and let $\pi^f(y^f) \equiv v(y^f) + y_0^f$ represent f's net market value in the exchange equilibrium (x, s, y, p). The vector of i's shadow prices, p^i, has a lovely geometric interpretation (see Figure 1): it describes the slope of a hyperplane, namely $\{\tilde{x}^i\colon p^i\tilde{x}^i = p^i\underline{x}^i + \sum_f \underline{s}^{if}\pi^f(y^f)\}$, that goes through x^i and is tangent to $\{\tilde{x}^i\colon u^i(\tilde{x}^i) \geqq u^i(x^i)\}$. To see this, observe that in any exchange

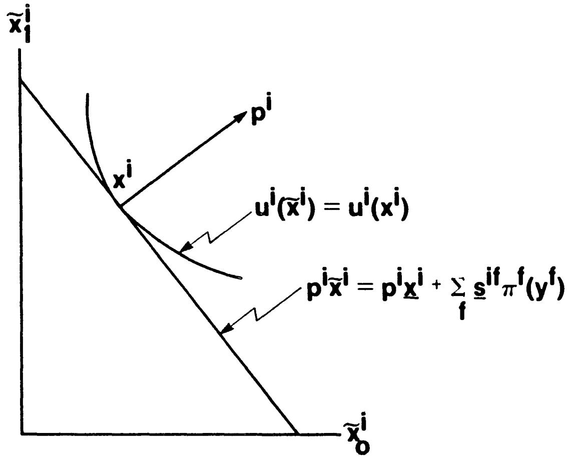

Figure 1. The Geometry of any Exchange Equilibrium (x, s, y, p)

equilibrium, (x, s, y, p), $x_1^i = \underline{x}_1^i + \sum_f s^{if} y_1^f$. So, multiplying by p_1^i and using the fact that $p_1^i y_1^f = v(y^f)$ if $s^{if} > 0$:

$$p_1^i x_1^i = p_1^i \underline{x}_1^i + \textstyle\sum_f s^{if} v(y^f). \tag{6}$$

But also in any exchange equilibrium, the budget constraint of i is just met, so

$$x_0^i + \textstyle\sum_f s^{if} v(y^f) = \underline{x}_0^i + \sum_f \underline{s}^{if}(p^f + y_0^f). \tag{7}$$

Summing (6) and (7) shows that, for all i:

$$p^i x^i = p^i \underline{x}^i + \textstyle\sum_f \underline{s}^{if} \pi^f(y^f).$$

That is, the above hyperplane does indeed go through x^i. Furthermore, by the concavity of u^i, for any $\tilde{x}^i$, $u^i(\tilde{x}^i) \leqq u^i(x^i) + [u_0^i(x^i), u_1^i(x^i)](\tilde{x}^i - x^i)$ (geometrically, this expresses the familiar fact that on the graph of u^i, $[\tilde{x}^i, u^i(\tilde{x}^i)]$ lies below the tangent line through $[x^i, u^i(x^i)]$); so $u^i(\tilde{x}^i) \geqq u^i(x^i)$ implies $p^i(\tilde{x}^i - x^i) \geqq 0$.

To summarize we have shown in any exchange equilibrium, (x, s, y, p), x^i maximizes $u^i(\tilde{x}^i)$

subject to

$$\begin{aligned} p^i\tilde{x}^i &\leqq p^i\underline{x}^i + \textstyle\sum_f \underline{s}^{if}\pi^f(y^f) \\ \tilde{x}^i &\geqq 0. \end{aligned} \tag{8}$$

II. Competitive Conjectures

Firm f is said to have *competitive conjectures* if it believes that it cannot effect any individual's shadow prices, p^i. This is the usual way of characterizing competitive conjectures in the finance literature (e.g., see [1]).

Note that if any firm f changes production from y^f to any $\bar{y}^f \in Y^f$, some new exchange equilibrium $(\bar{x}, \bar{s}, \bar{y}, \bar{p})$ will result. And, analogous to (5), in the new exchange equilibrium

$$\bar{p}^f = \operatorname*{Max}_i \{\bar{p}_1^i y_1^f\} \tag{9}$$

where $\bar{p}_1^i \equiv u_1^i(\bar{x}^i)/u_0^i(\bar{x}^i)$. To be "consistent" with its competitive conjectures, f must believe $\bar{p}_1^i = p_1^i$ for all i; so, f must believe that the market value of its shares will change to

$$v(\bar{y}^f) \equiv \operatorname*{Max}_i \{p_1^i \bar{y}_1^f\}.$$

This motivates the following definition. Firm f has *consistent competitive conjectures* if it believes the function v from R^{S+1} to the reals characterizes the value of its shares that will result from any production decision, $\bar{y}^f \in Y^f$.

Define a *competitive stock market equilibrium* as any four-tuple (x, s, y, p) that is an exchange equilibrium given y and satisfies, for all f,

$$y^f \operatorname*{Maximizes}_{\tilde{y}^f \in Y_f} \pi^f(\tilde{y}^f) \equiv v(\tilde{y}^f) + \tilde{y}_0^f. \tag{10}$$

Note that (10) says that each firm maximizes its conjectured net market value under its consistent competitive conjectures.

III. Unanimity

Suppose all shareholders i in firm f also have consistent competitive conjectures. So, if f changes from y^f to any $\bar{y}^f \in Y^f$, then i believes a new exchange equilibrium will result, say $(\bar{x}, \bar{s}, \bar{y}, \bar{p})$, such that $\bar{p}^i = p^i$ for all i, and $\bar{p}^f = v(\bar{y}^f)$ for all f. We show that all shareholders in firm f will want f to maximize net market value.

THEOREM 1 (unanimity). *Suppose (x, s, y, p) is a competitive stock market equilibrium and $(\bar{x}, \bar{s}, \bar{y}, \bar{p})$ is any other exchange equilibrium given $\bar{y}$ such that $\bar{p}^i = p^i$ for all i and $\bar{p}^f = v(y^f)$ for all f. Then, $u^i(\bar{x}^i) \leqq u^i(x^i)$ for all i.*

Proof: Using (8), we know that

$$p^i x^i = p^i \underline{x}^i + \textstyle\sum_f \underline{s}^{if} \pi^f(y^f).$$

And, for any i, $u^i(\bar{x}^i) > u^i(x^i)$ implies $p^i\bar{x}^i > p^i x^i$. But in any exchange equilibrium $(\bar{x}, \bar{s}, \bar{y}, \bar{p})$ we know, again using (8), that

$$\bar{p}^i \bar{x}^i = \bar{p}^i \underline{x}^i + \textstyle\sum_f \underline{s}^{if}(\bar{p}^f + \bar{y}_0^f).$$

Or, given the assumptions that $\bar{p}^i = p^i$ and $\bar{p}^f = v(\bar{y}^f)$,

$$p^i \bar{x}^i = p^i \underline{x}^i + \textstyle\sum_f \underline{s}^{if} \pi^f(\bar{y}^f) \leqq p^i \underline{x}^i + \textstyle\sum_f \underline{s}^{if} \pi^f(y^f) = p^i x^i.$$

Hence, a contradiction of $u^i(\bar{x}^i) > u^i(x^i)$ for any i. Q.E.D.

Note the proof, as in DeAngelo [4], is based on a simple revealed preference argument: any increased net market value of a firm is conjectured by its shareholders to just move their budget constraints out in a parallel direction (see Figure 2). The proof shows that DeAngelo's revealed preference argument works even in the absence of spanning. In [4], DeAngelo seems to assert that it requires spanning; this assertion is critiqued in Makowski [9].

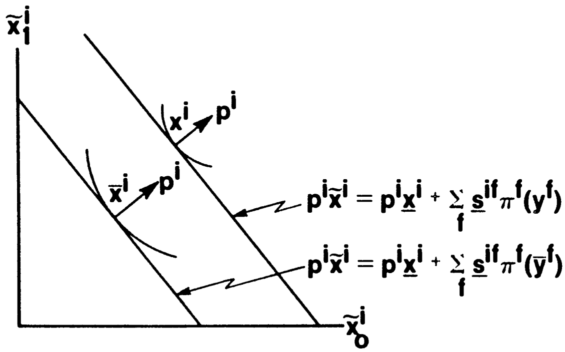

Figure 2. The Idea Behind Unanimity

IV. Optimality

Following Diamond [5], we say (x, y) is constrained Pareto optimal if there exists no allocation $(\bar{x}, \bar{y})$ that is: (a) feasible (i.e., $\sum_i \bar{x}^i = \sum_i \underline{x}^i + \sum_f \bar{y}^f$ with each $\bar{x}^i \geqq 0$ and each $\bar{y}^f \in Y^f$); (b) Pareto dominates (x, y) (i.e., $u^i(\bar{x}^i) \geqq u^i(x^i)$ for all i, with $>$ for some i); and (c) satisfies, for all i, $\bar{x}_1^i = \underline{x}_1^i + \sum_f \bar{s}^{if}\bar{y}_1^f$, for some shares $\bar{s}^{if} \geqq 0$ such that $\sum_i \bar{s}^{if} = 1$ for all f.

THEOREM 2. *If (x, s, y, p) is a competitive stock market equilibrium, then (x, y) is constrained Pareto optimal.*

Proof: If the contrary were true, then (8) implies $p^i\bar{x}^i \geqq p^ix^i$ for all i and $p^i\bar{x}^i > p^ix^i$ for some i. So,

$$\textstyle\sum_i p^i\bar{x}^i > \sum_i p^ix^i = \sum_i p^i\underline{x}^i + \sum_f \pi^f(y^f).$$

We show a contradiction. Since, by assumption, for all i:

$$\bar{x}_1^i = \underline{x}_1^i + \textstyle\sum_f \bar{s}^{if}\bar{y}_1^f,$$

multiplying by p_1^i and summing over the i implies

$$\textstyle\sum_i p_1^i\bar{x}_1^i = \sum_i p_1^i\underline{x}_1^i + \sum_i \sum_f \bar{s}^{if}p_1^i\bar{y}_1^f.$$

Or, since by feasibility, $\sum_i \bar{x}_0^i = \sum_i \underline{x}_0^i + \sum_f \bar{y}_0^f$,

$$\textstyle\sum_i p^i\bar{x}^i = \sum_i p^i\underline{x}^i + \sum_i \sum_f \bar{s}^{if}p_1^i\bar{y}_1^f + \sum_f \bar{y}_0^f.$$

But we know $p_1^i\bar{y}_1^f \leqq v(\bar{y}^f)$, for all i, so

$$\textstyle\sum_i p^i\bar{x}^i \leqq \sum_i p^i\underline{x}^i + \sum_i \sum_f \bar{s}^{if}v(\bar{y}^f) + \sum_f \bar{y}_0^f = \sum_i p^i\underline{x}^i + \sum_f \pi^f(\bar{y}^f).$$

And since $\pi^f(\bar{y}^f) \leqq \pi^f(y^f)$ for all f,

$$\textstyle\sum_i p^i\bar{x}^i \leqq \sum_i p^i\underline{x}^i + \sum_f \pi^f(y^f) = \sum_i p^ix^i,$$

contradicting $u^i(\bar{x}^i) > u^i(x^i)$ for any i. Q.E.D.

The analogy between this proof and the usual revealed preference proof of optimality for competitive models should be obvious. Note that consistent competitive conjectures will lead to a constrained Pareto optimum whether or not there is spanning.

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