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# On Option Pricing Bounds

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ABSTRACT

The purpose of this article is to compare the Perrakis and Ryan bounds of option prices in a single-period model with option bounds derived using linear programming. It is shown that the upper bounds are identical but that the lower bounds are different. A comparison of these bounds, together with Merton's bounds and the Black-Scholes prices in a lognormal securities market, is presented.

THE BLACK-SCHOLES [1] OPTION pricing model is preference-free. Its simplicity derives from the continuous-trading assumption which enables a portfolio of stock and bonds to be constructed so as to replicate the payouts of the option. In an incomplete market with only finite opportunities to revise portfolios, no preference-free option price can be constructed [3, 6]. For this case, preference-dependent prices can be established. Rather than place restrictive assumptions on preferences, Perrakis and Ryan (PR) [5] have recently derived upper and lower bounds for a single- (and multiple-) period call option price. The development of these bounds is based on the single price law and arbitrage arguments and requires modest assumptions on preferences. The bounds exist as equilibrium values given a consensus on stock price distributions.

In this article, we shall develop bounds for option prices based on primitive prices in incomplete markets and compare these bounds to the single-period bounds derived by Perrakis and Ryan. Towards this goal, we first review the pricing approach for primitive securities and proceed to re-express the PR bounds as derived from their arbitrage portfolios. Then, using linear programming, bounds on option prices are derived. Linear programming approaches for obtaining relationships among option prices have been investigated. Garman [2], for example, established an algebra for evaluating hedge portfolios. Using this algebra, a one-period linear program can be established that provides a way to search for arbitrage opportunities over an infinite number of portfolio strategies. One of the features of the algebra is that only one linear program need be solved to determine if riskless arbitrage strategies can be established. By investigating the dual of the linear program, Garman obtained necessary and sufficient conditions for rational option pricing in a one-period setting. These bounds were shown to be consistent with the bounds determined by Merton [4].

Unlike Garman's approach, the linear programming approach presented here places constraints on time preferences, risk aversion, and state probabilities. Under specific sets of assumptions, linear programs are formulated where the objective function to be optimized is the call price. Minimization and maximi-

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zation of the objective function yield lower and upper bounds on the call. The simplicity of the results that are obtained derived from the fact that the special structure of the linear programs allow the optimal solutions to be obtained by inspection. Using this approach, in the first section, we place weak assumptions on the model and rediscover Merton's lower bound. With further assumptions on preferences and state probabilities, tighter bounds are then obtained. These tighter bounds are compared with the PR bounds. It is shown that, while the linear programming bound produces a different lower bound, the upper bounds are identical. A comparison of the Merton, Black-Scholes, PR, and linear programming bounds is made by varying the parameters for time, stock price, expected return, variance, and the riskless rate. The final section briefly discusses the estimation problems associated with practical implementation of these bounds.

## I. Notation

Current (end of period) stock price $= S_0(S_T)$
Current (end of period) option price $= C_0(C_T)$
Strike price $= X$
Time to expiration $= T$
Instantaneous riskless rate $= r$
Let $R = \exp(rT)$

For the single period problem, with no intermediate dividends, the law of one price implies that the value of the security can be written as

$$S_0 = \textstyle\sum_{j=1}^{n} s_j e_j \tag{1}$$

where $s_j$ is the value of the security in state $j$, and $e_j$ is the current price of a pure state $j$ security that pays out \$1 at time $T$, if state $j$ occurs [6]. We shall assume that the states are ordered in terms of aggregate wealth with state 1 representing the lowest aggregate wealth level and state $n$ representing the highest level.

Let $B_0$ be the current value of a portfolio containing all state securities. That is:

$$B_0 = \textstyle\sum_{j=1}^{n} e_j \tag{2}$$

Since this portfolio is riskless, we have:

$$B_0 = \textstyle\sum_{j=1}^{n} e_j = R^{-1} \tag{3}$$

Finally, let $c_j$ be the payout of an option in state $j$. Clearly,

$$c_j = \mathrm{Max}(0, s_j - X) \qquad j = 1, 2, \cdots, n$$

and

$$C_0 = \textstyle\sum_{j=1}^{n} c_j e_j \tag{4}$$

We shall assume that the payouts of the security increase as aggregate wealth increases. For simplicity, we shall assume $s_1 = 0$.

## II. The Perrakis-Ryan Bounds

Let $\pi_j$ be the ex ante probability that state $j$ occurs and define $d_j$ such that:

$$e_j = \pi_j \, d_j \tag{5}$$

Then, from (1), we have:

$$\begin{aligned} S_0 &= \textstyle\sum_{j=1}^{n} (s_j \, d_j)\pi_j \\ &= E(\tilde{S}_T \, \tilde{d}_T) \end{aligned}$$

Where $E(\quad)$ is the expectation operator, and a "$\tilde{}$" over a variable indicates that the variable is random.

Now, define a random variable, $\tilde{Y}_T$, such that:

$$\tilde{Y}_T = E(\tilde{Y}_T) \, \tilde{d}_T R$$

Then, we have:

$$S_0 = E(\tilde{S}_T \tilde{Y}_T)/RE(\tilde{Y}_T) \tag{6}$$

This equation, called Rubinstein's equation [6], states that any uncertain stream can be valued using a properly-adjusted expectation. Equation (6) can be rewritten as:

$$\begin{aligned} S_0 &= \underset{S_T}{E}(\underset{Y_T}{E}(\tilde{S}_T \tilde{Y}_T \mid S_T))/E(\tilde{Y}_T)R \\ &= \underset{S_T}{E}[\tilde{S}_T \bar{Z}(\tilde{S}_T)]/R \end{aligned} \tag{7}$$

where

$$\bar{Z}(S_T) = \underset{Y_T}{E}(\tilde{Y}_T \mid S_T)/E(\tilde{Y}_T)$$

is the expectation of $Y_T$, given $S_T$, relative to the unconditional expectation of $Y_T$.

Rubinstein shows that under suitable restrictions, primarily on preferences, $Y_T$ can be interpreted as the marginal utility of consumption of an average investor. That is:

$$Y_T = U'(W_T)$$

where $W_T$ is the consumption.

This result implies that the randomness of $Y_T$ is solely determined by per capita consumption. Also, since the utility function is concave, $Y_T$ must decrease as $W_T$ increases. This implies that securities tend to be more valuable if they have high payouts in states with low per capita consumption. Thus, we have:

$$S_0 = E(\tilde{S}_T \bar{Z}(\tilde{S}_T))/R \tag{8}$$

where

$$\bar{Z}(\tilde{S}_T) = E(U'(\tilde{W}_T) \mid S_T)/E(U'(\tilde{W}_T)) \tag{9}$$

Under the additional assumption that the normalized marginal utility of consumption given the stock price is nonincreasing in the stock price, Perrakis and Ryan were able to derive the following option bounds for the price of a call, $C_0$:

$$\text{Max}(0, J(S_0)) \leq C_0 \leq S_0 E(\tilde{C}_T)/E(\tilde{S}_T) \tag{10}$$

where

$$J(S_0) = (S_0 R + E(\tilde{C}_T) - E(\tilde{S}_T))/R \tag{11}$$

## III. Linear Programming Bounds

In an incomplete market, the prices of the state securities are not unique. Nevertheless, under suitable assumptions, bounds on prices can be obtained. We shall initially only require that investors always prefer more wealth to less. This implies that $e_j \geq 0$ for all $j$. The minimum price of a call can be established by solving the following linear program:

$$\begin{aligned} &\text{Minimize } C = \textstyle\sum_{j=1}^{n} c_j e_j \\ &\text{subject to} \quad \textstyle\sum_{j=1}^{n} s_j e_j = S_0 \\ &\text{and} \quad \textstyle\sum_{j=1}^{n} e_j = B_0 \\ &\qquad e_j \geq 0 \end{aligned} \tag{P1}$$

The upper bound is obtained by replacing the minimization objective by a maximization objective.

These optimization problems have a unique structure that can be readily exploited to obtain, by inspection, their optimal solutions. The unique structure derives from the fact that the piecewise linear function connecting adjacent $(s_j, c_j)$ coordinates is convex. This convexity property allows the optimal solutions to be simply obtained. To provide geometric insight of the results that will follow, consider the linear programming problem when $n = 3$. Figure 1 shows the graph of the objective function value plotted against the resource of the first constraint. Points $A$, $B$, and $C$ correspond to points $e_1 = B_0$, $e_2 = B_0$, $e_3 = B_0$, respectively. Notice that the piecewise linear function obtained by connecting $A$ to $B$ and $B$ to $C$ is convex. Points in the convex set $ABC$ satisfy the second constraint. However, only points along the line $DE$ satisfy the first constraint.

Clearly, the minimum objective value is attained at $D$ (linear coordinates of neighbors surrounding the resources $S_0$) while the maximum occurs at $E$ (linear combinations of the smallest and largest values).

This geometric insight motivates the next proposition, which provides the optimal solutions for the minimum and maximum option prices for problem (P1).

PROPOSITION 1

(*i*) *The optimal solution to the minimization problem, (P1), is given by*

$$\begin{aligned} e_j &= 0, \qquad j \neq j^*, j^* + 1 \\ e_{j^*} &= (s_{j^*+1} B_0 - S_0)/(s_{j^*+1} - s_{j^*}) \\ e_{j^*+1} &= (S_0 - s_{j^*} B_0)/(s_{j^*+1} - s_{j^*}) \end{aligned} \tag{12}$$

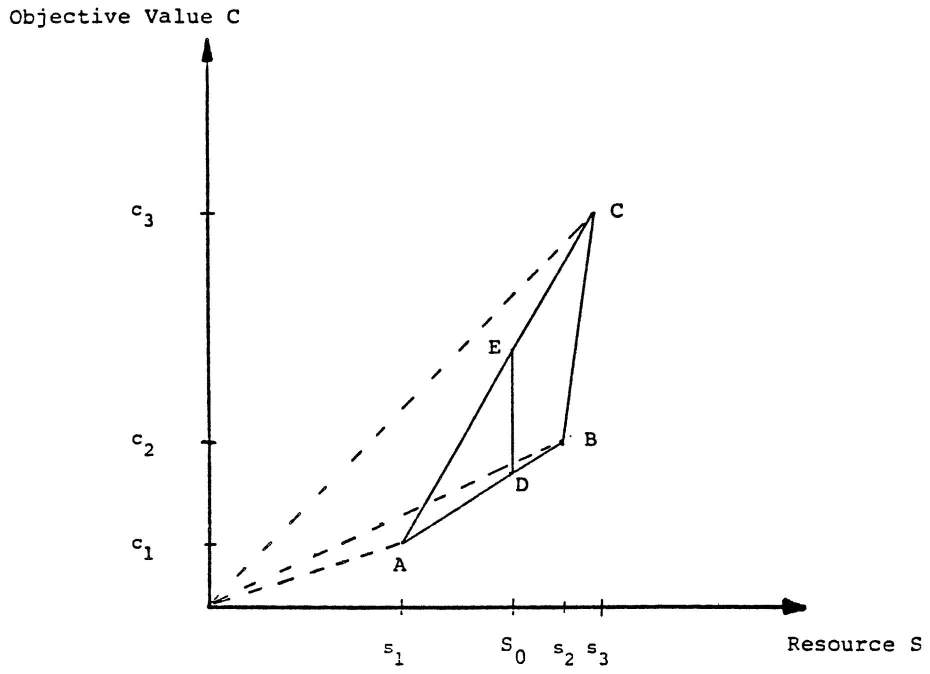


**Figure 1.** Objective Function Versus Resource

*where $j^*$ is chosen such that $s_{j^*}$ is the greatest value satisfying*

$$s_{j^*} \le S_0/B_0$$

$$s_{j^*+1} > S_0/B_0 \tag{13}$$

(*ii*) *The optimal solution to the maximization problem is given by*

$$e_1 = (s_n B_0 - S_0)/(s_n - s_1)$$

$$e_n = (S_0 - s_1 B_0)/(s_n - s_1)$$

$$e_j = 0 \qquad j = 2, 3, \cdots, n-1 \tag{14}$$

*Proof*: To prove that the solution given by (12) is the optimal solution to (P1), we shall show that it is feasible and that there exists a feasible solution to the dual problem that achieves the same objective value.

To prove feasibility, note from (13) that $s_{j^*}B_0 \le S_0$, and $s_{j^*+1}B_0 \ge S_0$. Since $s_{j^*+1} > s_{j^*}$, $e_{j^*} \ge 0$, and $e_{j^*+1} \ge 0$. Moreover, since

$$s_{j^*}e_{j^*} + s_{j^*+1}e_{j^*+1} = S_0 \tag{15}$$

and

$$e_{j^*c} + e_{j^*+1} = B_0, \tag{16}$$

the solution given by (12) is feasible.

The objective function value associated with this feasible solution is, after arrangement, given by:

$$C = [(c_{j^*}s_{j^*+1} - c_{j^*+1}s_{j^*})/(s_{j^*+1} - s_{j^*})]B_0 + [(c_{j^*+1} - c_{j^*})/(s_{j^*+1} - s_{j^*})]S_0 \quad (17)$$

To prove this solution is optimal, consider the dual of (P1). Let $u_1$ and $u_2$ be the dual variables. Then the dual is given by:

$$\underset{u_1,u_2}{\text{Maximize}}\ W = u_1B_0 + u_2S_0 \quad \text{(P2)}$$

subject to $u_1 + s_ku_2 \le c_k \qquad k = 1, 2, \cdots, n$

and $u_1$, $u_2$, unrestricted

Let

$$u_1 = (c_{j^*}s_{j^*+1} - c_{j^*+1}s_{j^*})/(s_{j^*+1} - s_{j^*}) \quad (18)$$

$$u_2 = (c_{j^*+1} - c_{j^*})/(s_{j^*+1} - s_{j^*}) \quad (19)$$

Then the dual objective function is given by (17). Hence, all that remains is to check whether $u_1$ and $u_2$, as defined, satisfy the dual constraints. Substituting $u_1$ and $u_2$ into (P2), and rearranging, we obtain the following necessary condition for feasibility:

$$s_{j^*+1}(c_{j^*} - c_k) + s_{j^*}(c_k - c_{j^*+1}) \le s_k(c_{j^*} - c_{j^*+1}) \quad \text{for} \quad k = 1, 2, \cdots, n \quad (20)$$

To check if these constraints are satisfied, first note that since the piecewise linear function obtained by connecting adjacent $(s_j, c_j)$ coordinates is convex, we have

$$(c_{j^*}+1 - c_{j^*})/(c_{j^*+1} - s_{j^*}) \ge (c_{j^*} - c_k)/(s_{j^*} - s_k), \quad \text{for} \quad k < j^*$$

and

$$(c_{j^*+1} - c_{j^*})/(s_{j^*+1}\ s_{j^*}) < (c_k - c_{j^*})/(s_k - s_{j^*}), \quad \text{for} \quad k > j^*$$

Both equations imply:

$$(c_{j^*+1} - c_{j^*})(s_{j^*} - s_k) \ge (s_{j^*+1} - s_{j^*})(c_{j^*} - c_k) \quad k = 1, 2, \cdots, n$$

Rearranging this equation, we obtain:

$$s_{j^*+1}(c_{j^*} - c_k) + s_{j^*}(c_k - c_{j^*+1}) + s_k(c_{j^*+1} - c_{j^*}) \le 0 \quad \text{for} \quad k = 1, 2, \cdots, n$$

But this equation is Equation (20). Hence, feasibility is confirmed.
The proof of the maximization problem follows, using similar arguments.

PROPOSITION 2

(*i*) *The linear programming upper bound for the single period option price is given by:*

$$c_{\max} = S_0 - XS_0/s_n \quad (21)$$

(*ii*) *The linear programming lower bound for the single period option price is given by*:

$$c_{\min} = [(c_{j^*}s_{j^*+1} - c_{j^*+1}s_{j^*})B_0 + (c_{j^*+1} - c_{j^*})S_0]/(s_{j^*+1} - s_{j^*}) \tag{22}$$

*where* $j^*$ *is chosen such that*

$$s_{j^*} < S_0/B_0$$

$$s_{j^*+1} > S_0/B_0$$

(*iii*) *Passing to the limit of continuous states, the upper bound remains unchanged, but the lower bound converges to*:

$$C_{\min} = \mathrm{Max}(0,\ S_0 - XB_0) \tag{23}$$

*Proof*: The proofs of (i) and (ii) follow directly from Proposition 1. Notice that for the upper bound, it is assumed that the lowest stock price, $s_1$, is zero, and hence $c_1 = 0$. Moreover, the strike price, $X$, is assumed to be less than the greatest possible stock price, $s_n$. Hence $c_n = s_n - X$. In this case, from Equation (14), we have:

$$\begin{aligned} c_{\max} = c_n e_n &= (s_n - X)S_0/s_n \\ &= S_0 - XS_0/s_n \end{aligned} \tag{24}$$

Notice that the upper bound depends only on the values, $s_1$ and $s_n$. The lower bound, however, depends on $j^*$ and $j^* + 1$. As the number of states between $s_1$ and $s_n$ increase, the solution to the minimization problem decreases. Since the sequence of optimal solutions form a nonincreasing sequence which is bounded below, a limiting value must exist. This value is obtained by choosing $j^*$ such that:

$$s_{j^*} = S_0/B_0$$

In this case

$$e_{j^*} = B_0$$

$$c_{\min} = c_{j^*}e_{j^*}$$

and

$$\begin{aligned} &= \mathrm{Max}(s_{j^*} - X,\ 0)B_0 \\ &= \mathrm{Max}(S_0 - XB_0,\ 0) \end{aligned} \tag{25}$$

Proposition 2 provides us with bounds similar to those derived by Merton [4]. Indeed, the lower bound provided by Equation (23) is precisely Merton's lower bound. The upper bound, however, depends on the largest price that the stock can take. From Equation (22), we see that this price can never exceed Merton's upper bound, namely the price of the stock itself.

From Equation (24), we can argue that the current value of a call must equal at most the present value of the benefits from exercise (i.e., $S_0$), minus the lowest possible present value of the costs of exercise. This cost equals the strike price discounted at the highest possible rate. This rate is, in turn, the highest possible stock price, divided by the current stock price.

## IV. Option Bounds in a Risk-Averse Incomplete Market

To obtain tighter bounds on the price of the option requires additional assumptions. So far, the only preference constraint imposed was that investors prefer more wealth to less. We shall now assume that investors are risk averse. In addition, we assume that there exists a set of ex ante consensus probabilities for the states. Specifically, we have

$$e_j = \pi_j d_j \qquad j = 1, 2, 3, \cdots, n$$

where $d_j$ is the state discount factor that is nonincreasing with respect to aggregate wealth.

That is:

$$d_1 \geq d_2 \geq d_3 \cdots \geq d_n \geq 0$$

With these additional assumptions, (P1) is now replaced by the following optimization problem:

$$\begin{aligned} &\underset{\{dj\}}{\text{Minimize}} \ \textstyle\sum_{j=1}^{n} (c_j\pi_j)\, d_j \\ &\text{subject to } \textstyle\sum_{j=1}^{n} (s_j\pi_j)\, d_j = S_0 \\ &\qquad \textstyle\sum_{j=1}^{n} \pi_j\, d_j = B_0 \\ &\qquad d_1 \geq d_2 \geq d_3 \geq d_4 \cdots \geq d_n \geq 0 \end{aligned} \tag{P3}$$

This problem is a linear program, with $n + 1$ constraints. However, by making the following transformation, the problem can be converted into a linear program with two constraints.

Let $x_1, x_2, \cdots, x_n$ be nonnegative variables such that

$$\textstyle\sum_{i=j}^{n} x_i = d_j \qquad j = 1, 2, 3, \cdots, n$$

Then the optimization problem can be written as:

$$\begin{aligned} &\underset{\{x_j\}}{\text{Minimize}} \ c = \textstyle\sum_{j=1}^{n} (\sum_{i=1}^{j} c_i\pi_i)x_j \\ &\text{subject to} \quad \textstyle\sum_{j=1}^{n} (\sum_{i=1}^{j} \pi_i)x_j = B_0 \\ &\qquad \textstyle\sum_{j=1}^{n} (\sum_{i=1}^{j} s_i\pi_i)x_j = S_0 \\ &\qquad x_j \geq 0 \qquad j = 1, 2, \cdots, n \end{aligned} \tag{P4}$$

Now let $y_j = (\sum_{i=1}^{j} \pi_i)x_j \qquad j = 1, 2, \cdots, n$

Then (P4) is given by:

$$\begin{aligned} &\text{Minimize } c = \textstyle\sum_{j=1}^{n} \bar{c}_j y_j \\ &\text{subject to} \quad \textstyle\sum_{j=1}^{n} y_j = B_0 \\ &\qquad\qquad\quad \textstyle\sum_{j=1}^{n} \bar{s}_j y_j = S_0 \\ &\qquad\qquad\qquad\quad y_j \geq 0 \qquad j = 1, 2, \cdots, n \end{aligned} \tag{P5}$$

where

$$\bar{s}_j = \textstyle\sum_{i=1}^{j} s_i \pi_i / \sum_{i=1}^{j} \pi_i$$

and

$$\bar{c}_j = \textstyle\sum_{i=1}^{j} c_i \pi_i / \sum_{i=1}^{j} \pi_i$$

are, respectively, the expected stock price and option price, conditional on the stock price being less than $s_j$ at the expiration date.

If the piecewise linear function connecting adjacent $(\bar{s}_j, \bar{c}_j)$ coordinates can be shown to be convex function, then, using Proposition 1, bounds on the option prices can be obtained.

PROPOSITION 3

(*i*) *The optimal solution to the maximization problem given by* (*P*5) *is*:

$$c_{\max} = E(\tilde{C}_T) S_0 / E(\tilde{S}_T) \tag{26}$$

(*ii*) *The optimal solution to the minimization problem given by* (*P*5) *is*:

$$c_{\min} = [(\bar{c}_{j^*}\bar{s}_{j^*+1} - \bar{c}_{j^*+1}\bar{s}_{j^*})/(\bar{s}_{j^*+1} - \bar{s}_{j^*})]B_0 + [(\bar{c}_{j^*+1} - \bar{c}_{j^*})/(\bar{s}_{j^*+1} - \bar{s}_{j^*})]S_0 \tag{27}$$

*where $j^*$ is chosen such that* $\bar{s}_{j^*} \leq S_0/B_0$, and $\bar{s}_{j^*+1} > S_0/B_0$

(*iii*) *For the continuous state problem, the lower bound converges to:*

$$c_{\min} = E(\tilde{C}_T \mid \tilde{S}_T < s_{j^*}) B_0 \tag{28}$$

*where $s_{j^*}$ is chosen such that*

$$E(\tilde{S}_T \mid \tilde{S}_T < s_{j^*}) = S_0/B_0 \tag{29}$$

*Proof*: Since the piecewise linear function obtained by connecting adjacent $(s_j, c_j)$ coordinates is convex, and since

$$\bar{c}_j = \textstyle\sum_{i=1}^{j} c_k k_i, \quad \bar{s}_j = \sum_{i=1}^{j} s_i k_i$$

where $k_i = \pi_i / \sum_{i=1}^{j} \pi_i$, it follows that the piecewise linear function obtained by connecting adjacent $(\bar{s}_j, \bar{c}_j)$ coordinates is convex. Hence, Proposition 1 can be applied.

(*i*) The maximum value is obtained by linear combinations of $y_1$ and $y_n$. That is:

$$y_1 + y_n = B_0 \tag{30}$$

$$\bar{s}_1 y_1 + \bar{s}_n y_n = S_0 \tag{31}$$

Now, since $\bar{s}_1 = 0$, and $\bar{s}_n = E(\tilde{S}_T)$, we have from (31), $y_n = S_0/E(\tilde{S}_T)$.
Hence, $c_{\max} = \bar{c}_n y_n$
$= E(\tilde{C}_T)S_0/E(\tilde{S}_T)$

To prove (ii), we have, analogous to Equations (15) and (16), the following conditions.

$$\bar{s}_{j^*}y_{j^*} + \bar{s}_{j^*+1}y_{j^*+1} = S_0 \tag{32}$$

$$y_{j^*} + y_{j^*+1} = B_0 \tag{33}$$

where $j^*$ is chosen such that $\bar{s}_{j^*} \leq S_0/B_0$ and $\bar{s}_{j^*+1} > S_0/B_0$

Solving (32) and (33) for $y_{j^*}$ and $y_{j^*+1}$, and substituting into the objective function, we obtain the result.

The proof of (iii) follows in the same way as Proposition 2. Namely, there exists a $j^*$ such that

$$\bar{s}_{j^*}y_{j^*} = S_0$$

and

$$y_{j^*} = B_0$$

Hence,

$$c_{\min} = \bar{c}_j B_0$$

and

$$\bar{s}_j = S_0/B_0$$

Hence,

$$c_{\min} = E(\tilde{C}_T \mid \tilde{S}_T < s_{j^*})B_0,$$

where $j^*$ is chosen such that

$$E(\tilde{S}_T \mid \tilde{S}_T < s_{j^*}) = S_0/B_0$$

Equation (26) says that the maximum price of a call should be set equal to that value, $c_{\max}$, that equates the expected return on the call to the expected return on the stock. This makes good economic sense, especially if one views the call as a leveraged position in the stock. Notice that this equation is precisely the same bound as the PR bound (Equation (10)).

The lower bound is more difficult to interpret. For continuous distributions, Equation (29) requires that a critical price, $s_{j^*}$, be determined such that, if the future stock price was known to be less than this value, then the expected rate of return on the stock would be the riskless rate. Given this critical value, the lower bound on the call price, given by Equation (28), is:

$$c_{\min} = 0 \quad \text{for} \quad s_{j^*} \leq X \tag{34}$$

$$c_{\min} = E(\tilde{C}_T \mid \tilde{S}_T < s_{j^*}) \quad \text{for} \quad s_{j^*} > X \tag{35}$$

By substituting $\tilde{C}_T = \text{Max}(0, \tilde{S}_T - X)$ into Equation (35), using Equation (29), and rearranging, we eventually obtain:

$$c_{\min} = (S_0 - XB_0) + E(\tilde{P}_T | \tilde{S}_T < s_{j^*})B_0 \tag{36}$$

where $\tilde{P}_T = \text{Max}(0, X - \tilde{S}_T)$ is the price of a put option with strike $X$ at the expiration date.

Notice that the first term in Equation (36) is Merton's lower bound. However, $c_{\min}$ exceeds Merton's lower bound by the nonnegative second term. This term represents the discounted value of the expected price of a put option at expiration, given that the stock price is lower than the critical value.

## V. Comparison of Option Bounds in an Incomplete Lognormal Securities Market

In order to compare the PR bounds with the linear programming bounds, Merton's bounds, and the Black-Scholes equation, consider the valuation of options in a lognormal securities market. Table I presents the fundamental data used for the illustrations.

In all the tables that follow, the first column denotes the variable upon which a sensitivity analysis is to be performed. The second column gives the Black-Scholes option price. The third column provides Merton's lower bound, $(S_0 - X\exp(-rT))$. The column identified by PR provides the values of $J(S_0)$ for the Perrakis-Ryan lower bound (see Equation (11)), while the last two columns provide the linear programming bounds. $C_{\max}$ also corresponds to the Perrakis-Ryan upper bound.

Table II indicates how the bounds change as stock prices change. For deep-in-the-money contracts, option bounds converge. For at- and out-the-money options, the linear programming bound, $C_{\min}$, is usually higher than the Perrakis-Ryan bound.

Table III presents option bounds as the instantaneous expected return, $\mu$ changes. The results are for an at-the-money call. Notice that the Black-Scholes and Merton bound prices do not depend on expectations. The gap between lower and upper bounds widens as $\mu$ increases.

Table IV shows how instantaneous variances affect the option prices.

Table V illustrates the sensitivity of option bounds to interest rates. As interest rates increase, the option bounds tend to converge.

**Table I**

Original Data

| Original Values |
|---|
| Stock Price = 50 |
| Strike Price = 50 |
| Risk-Free Rate = 0.1 |
| The Instantaneous Expected Return on Asset = 0.2 |
| The Instantaneous Standard Deviation of Asset = 0.2 |
| Length of Option in Years = 1 |

**Table II**

Sensitivity of Option Prices to Changes in Stock Prices

The Variable to Be Analyzed Is $S$

| Stock Price | Black-Scholes | Merton | PR | $C_{MIN}$ | $C_{MAX}$ |
|---|---|---|---|---|---|
| 20.000 | 0.000 | 0.000 | −2.550 | 0.000 | 0.000 |
| 25.000 | 0.003 | 0.000 | −3.166 | 0.000 | 0.019 |
| 30.000 | 0.054 | 0.000 | −3.571 | 0.000 | 0.225 |
| 35.000 | 0.374 | 0.000 | −3.224 | 0.000 | 1.098 |
| 40.000 | 1.395 | 0.000 | −1.566 | 0.272 | 3.135 |
| 45.000 | 3.481 | 0.000 | 1.441 | 2.278 | 6.367 |
| 50.000 | 6.635 | 4.758 | 5.449 | 5.791 | 10.487 |
| 55.000 | 10.624 | 9.758 | 10.017 | 10.143 | 15.104 |
| 60.000 | 15.129 | 14.758 | 14.849 | 14.888 | 19.954 |
| 65.000 | 19.908 | 19.758 | 19.788 | 19.797 | 24.901 |
| 70.000 | 24.816 | 24.758 | 24.768 | 24.767 | 29.883 |
| 75.000 | 29.779 | 29.758 | 29.761 | 29.760 | 34.877 |
| 80.000 | 34.766 | 34.758 | 34.759 | 34.754 | 39.875 |

**Table III**

Sensitivity of Option Prices to Changes in Instantaneous Expected Return

The Variable to Be Analyzed Is $\mu$

| Expected Return | Black-Scholes | Merton | PR | $C_{MIN}$ | $C_{MAX}$ |
|---|---|---|---|---|---|
| 0.100 | 6.635 | 4.758 | 6.374 | 6.431 | 7.238 |
| 0.150 | 6.635 | 4.758 | 5.838 | 6.061 | 8.824 |
| 0.200 | 6.635 | 4.758 | 5.449 | 5.789 | 10.487 |
| 0.250 | 6.635 | 4.758 | 5.180 | 5.582 | 12.187 |
| 0.300 | 6.635 | 4.758 | 5.004 | 5.436 | 13.890 |
| 0.350 | 6.635 | 4.758 | 4.894 | 5.309 | 15.567 |
| 0.400 | 6.635 | 4.758 | 4.830 | 5.213 | 17.200 |
| 0.450 | 6.635 | 4.758 | 4.794 | 5.133 | 18.775 |

Table VI illustrates how time-to-expiration affects the bounds.

The results indicate that the linear programming bounds are usually tighter than the PR bounds. It is interesting to note that the Black-Scholes price always falls in the bounds but is usually quite close to the lower boundary.

## VI. Implicit Estimation of Parameters

The practical value of the linear programming bounds may, at first glance, appear limited because of the necessity of having to estimate the instantaneous mean return. This parameter is not required by the Black-Scholes model. By substituting actual call prices, however, into Equations (26) and (28), implicit bounds on the space of possible mean and volatility parameters can be obtained. To

**Table IV**

Sensitivity of Prices to Change in Instantaneous Volatility

The Variable to Be Analyzed is $\sigma$

| Standard Deviation | Black-Scholes | Merton | PR | $C_{MIN}$ | $C_{MAX}$ |
|---|---|---|---|---|---|
| 0.100 | 5.154 | 4.758 | 4.795 | 4.863 | 9.301 |
| 0.150 | 5.834 | 4.758 | 5.029 | 5.240 | 9.764 |
| 0.200 | 6.635 | 4.758 | 5.449 | 5.789 | 10.487 |
| 0.250 | 7.488 | 4.758 | 5.970 | 6.436 | 11.386 |
| 0.300 | 8.367 | 4.758 | 6.537 | 7.101 | 12.403 |
| 0.350 | 9.259 | 4.758 | 7.118 | 7.776 | 13.504 |
| 0.400 | 10.158 | 4.758 | 7.699 | 8.457 | 14.667 |
| 0.450 | 11.068 | 4.758 | 8.268 | 9.125 | 15.875 |
| 0.500 | 11.970 | 4.758 | 8.822 | 9.778 | 17.119 |
| 0.550 | 12.870 | 4.758 | 9.357 | 10.424 | 18.387 |

**Table V**

Sensitivity of Prices to Change in Instantaneous Risk-Free Rate

The Variable to Be Analyzed Is $H$

| Risk-Free Return | Black-Scholes | Merton | PR | $C_{MIN}$ | $C_{MAX}$ |
|---|---|---|---|---|---|
| 0.020 | 4.465 | 0.990 | 1.738 | 2.768 | 10.487 |
| 0.040 | 4.962 | 1.961 | 2.694 | 3.501 | 10.487 |
| 0.060 | 5.494 | 2.912 | 3.631 | 4.249 | 10.487 |
| 0.080 | 6.053 | 3.844 | 4.549 | 5.020 | 10.487 |
| 0.100 | 6.635 | 4.758 | 5.449 | 5.797 | 10.487 |
| 0.120 | 7.238 | 5.654 | 6.331 | 6.581 | 10.487 |
| 0.140 | 7.860 | 6.532 | 7.196 | 7.367 | 10.487 |
| 0.160 | 8.499 | 7.393 | 8.043 | 8.147 | 10.487 |
| 0.180 | 9.151 | 8.236 | 8.874 | 8.935 | 10.487 |
| 0.200 | 9.815 | 9.063 | 9.688 | 9.717 | 10.487 |
| 0.220 | 10.487 | 9.874 | 10.487 | 10.483 | 10.487 |

**Table VI**

Sensitivity of Prices to Changes in Time to Expiration

The Variable to Be Analyzed Is $T$

| Time | Black-Scholes | Merton | PR | $C_{MIN}$ | $C_{MAX}$ |
|---|---|---|---|---|---|
| 0.100 | 1.519 | 0.498 | 1.285 | 1.360 | 1.866 |
| 0.100 | 2.305 | 0.990 | 1.887 | 2.024 | 3.028 |
| 0.100 | 2.970 | 1.478 | 2.400 | 2.594 | 4.083 |
| 0.100 | 3.574 | 1.961 | 2.873 | 3.099 | 5.092 |
| 0.100 | 4.139 | 2.439 | 3.325 | 3.583 | 6.043 |
| 0.100 | 4.674 | 2.912 | 3.764 | 4.051 | 6.976 |
| 0.100 | 5.188 | 3.380 | 4.194 | 4.502 | 7.884 |
| 0.100 | 5.684 | 3.844 | 4.617 | 4.946 | 8.771 |
| 0.100 | 6.166 | 4.303 | 5.035 | 5.371 | 9.638 |
| 0.100 | 6.635 | 4.758 | 5.449 | 5.789 | 10.487 |

illustrate the idea, assume stock prices follow a lognormal distribution and that volatility estimates are obtained using historical data. In this case, the upper bound on the call price, given by Equation (26), reduces to

$$C_0 \leq \hat{C}(\mu + \sigma^2/2) \tag{37}$$

where $\hat{C}(\mu + \sigma^2/2)$ is the Black-Scholes equation with $\mu + \sigma^2/2$ replacing the risk-free rate. With the observed call price inserted into the left-hand side, Equation (37) imposes restrictions on feasible $\mu$ and $\sigma^2$ values. Given an estimate of volatility, Equation (37) imposes a lower bound constraint on $\mu$.

Similarly, Equations (28) and (29) impose restrictions on $\mu$ and $\sigma^2$. For lognormal distributions, Equation (28) requires that the critical value, $s^* = s^*(\mu, \sigma^2)$ satisfy the following condition:

$$\frac{\Phi(a(s^*) - \sigma\sqrt{t})}{\Phi(a(s^*))} = \exp((r - \mu - \sigma^2/2)t) \tag{38}$$

where

$$a(s^*) = [-\log(S_0/s^*) - \mu t - \sigma^2 t]/\sigma\sqrt{t}$$

and $\Phi(\ \ )$ is the cumulative normal distribution function.

Equation (29) then reduces to the following feasibility requirement for $\mu$ and $\sigma$:

$$\Phi(a(s^*)) \geq k(\mu, \sigma^2) \tag{39}$$

where

$$k(\mu, \sigma^2) = \frac{X\Phi(a(X)) - S_0\exp(mt)\Phi(a(X) - \sigma\sqrt{t})}{\exp(rt)(C_0 - S_0 + X\exp(-rt))} \tag{40}$$

From Equation (38), we see that since the left-hand side is less than one, $\mu$ must be strictly greater than the quantity, $(r - \sigma^2/2)$. For a given $\sigma^2$, Equations (37), (38), and (39) can be used to generate the feasible range of $\mu$ values.

Clearly, the approach outlined here is suggestive. It remains for future research to establish whether or not these option bounds can be used for obtaining joint implicit range estimates of the market's consensus opinion of future expectations (and variances). Finally, note from Table III that the lower bound on the option price is not as sensitive to small changes in $\mu$ as the upper bound. This suggests that estimation risk for the lower bound may not be too severe a problem.

## VII. Conclusions

Establishing bounds on option prices in an incomplete market is important for two primary reasons. First, the bounds do not necessitate severe restrictions on preferences. Second, the framework allows dividends and/or transaction costs to be incorporated into the analysis.

Under the assumption that investors prefer more wealth to less, the linear programming bounds yielded Merton's lower bound. A value for the upper bound, lower than Merton's upper bound of the stock price, could be derived if a maximum possible stock price at expiration could be assessed. Under additional assumptions on risk aversion and state probabilities, tighter bounds were developed. Although the linear programming upper bounds were found to be identical to the PR bounds, different lower bounds were obtained. A comparison of these bounds was conducted for lognormal stock prices. Practical limitations of the linear programming bounds are caused by the fact that, in addition to the volatility, the instantaneous mean return must be estimated. This latter parameter is not required by the Black-Scholes model. Implicit interval bounds on this parameter may be obtained using observed option prices. The effectiveness of this approach remains a topic for future research. Another possible research topic would be to extend the single-period model to a multi-period model.

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