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Upper and Lower Bounds of Put and Call Option Value: Stochastic Dominance Approach

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ABSTRACT

Applying stochastic dominance rules with borrowing and lending at the risk-free interest rate, we derive upper and lower values for an option price for all unconstrained utility functions and alternatively for concave utility functions. The derivation of these bounds is quite general and fits any kind of stock price distribution as long as it is characterized by a "nonnegative beta."

Transaction costs and taxes can be easily incorporated in the model presented here since investors are not required to revise their portfolios continuously. The "price" that is paid for this generalization is that a range of values rather than a unique value is obtained.

IN THE LAST DECADE, there has been an increasing interest in the options market by both academicians and practitioners. The most well-known valuation models are the Black-Scholes [4] model, the jump stochastic process model (see Cox and Ross [6]), the binomial model (see Cox, Ross, and Rubinstein [7], Rendleman and Bartler [19], and Sharpe [22]), and Rubinstein's discrete model [21]. Recently, Perrakis and Ryan [18], using Rubinstein's approach, have derived upper and lower bounds for option prices.[1]

The relative pricing theory of contingent claims differs in the continuous time and discrete time models. In a continuous time framework, the instantaneous expected rate of return of the option and that of the underlying stock must follow a certain market equilibrium condition. In this case, the equilibrium condition is easily determined by a continuous hedging strategy. In the discrete time case, there is no hedging strategy to provide a simple relationship between the option and the stock price. However, for a given value distribution of the stock, the value distribution of the option is uniquely determined by the option exercise price, and this suggests some possible relationships between the stock and option values.

On the one hand, Brennan [5], assuming some specific stock value distributions and investor utility functions, derives a relative pricing relationship between the two financial assets. On the other hand, Merton [16], imposing no restrictions

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[1] It can be shown that Perrakis and Ryan [18] bounds are obtainable by applying the second degree stochastic dominance rule. However, their analysis does not cover all possible combinations of the risky asset with the riskless asset, hence their bounds are wider than the bounds obtained in this paper.

on the stock price behavior and the investors' characteristics, obtained upper and lower bounds on the option value relative to the stock value. Knowing these two extreme cases, can we deal with intermediate cases? That is, relaxing some of Brennan's assumptions while still assuming some knowledge of the investors' behavior (e.g., risk aversion) and stock price distribution, can we derive bounds of the option value? It is to this issue that our paper is addressed.

We apply stochastic dominance criteria to derive an upper and lower value for the call and put options. By applying first degree stochastic dominance with riskless assets (FSDR), we show that Merton's bounds [16] are in fact FSDR bounds. Thus, the paper suggests a procedure to compute Merton's type of bounds for other contingent claims. That is, instead of searching for a particular arbitrage condition (as in Merton), we suggest a procedure to determine the bounds for other claims, e.g., compound options, exchange options, and insurance contracts (since an insurance contract can be regarded as a put option). By applying a second degree stochastic dominance rule, we significantly strengthen Merton's bounds on the option value.

Section I presents the suggested model. In Section II, we derive the option bounds. Section III extends the analysis by allowing investors to hold other risky assets, e.g., the market portfolio, in addition to the option or its underlying stock. Section IV presents concluding remarks.

I. The Model

Throughout the paper, we assume a discrete time model.[2] The future stock price at the expiration date, T, is $\tilde{S}_T$, whose cumulative distribution is $F(\tilde{S}_T)$. We make the assumption of an objective stock price distribution, $F(\tilde{S}_T)$. However, the same analysis can be performed in the case of subjective beliefs. Let the current stock price be S and the investment in the stock be W_0. Hence, the investor's terminal wealth, W_s, is given by $W_s = (W_0/S)\tilde{S}_T$. Similarly, if the investor decides to invest W_0 dollars in the call whose current price is C, his terminal wealth, W_c, is given by

$$W_c = \max\left(0, \frac{W_0}{C}\,(\tilde{S}_T - X)\right) \tag{1}$$

where X stands for the exercise price. We denote the cumulative distributions of W_s and W_c by F_s and F_c, respectively.

The investor considers investing in the stock (which we call investment S) or in the call option (which we call investment C). Thus, we investigate below the relative merit of these two alternatives. The returns on the investment in an option (and hence the cumulative distribution F_c) is a function of the call market

[2] By comparing the cumulative distributions of returns on two alternative investments, we derive bounds on option value. However, the discrete time model given in this paper can be applied to many other situations, especially to cases when neither the contingent claim nor the underlying asset is traded, e.g., cases when costs are incurred only as a result of bankruptcy, when considering an option to buy a physical asset on expiration of the release contract, etc. For a list of cases when a discrete time model is appropriate, see Brennan [5]. Recently Kroll [11] employed stochastic dominance arguments to derive bounds for the premium charged on insurance contracts.

value, C. Obviously, other things being equal, the lower the value of C, the more attractive the investment in the option. In this paper, the upper bound of C is derived in such a way that F_s exactly does not dominate F_c, and the lower bound of the option is derived in such a way where F_c exactly does not dominate F_s. This analysis, however, should not be performed in isolation from other assets, since the investor can diversify S or C with other risky assets. However, in order to convey the main message of this paper, we start with the confining assumption that S and C are the only available risky alternatives, an assumption which will be relaxed in Section III.

Throughout the paper, we assume that borrowing and lending at the risk-free rate of interest is allowed. This assumption is crucial since it allows us to derive relatively narrow bounds. Note that it may be that neither F_s nor F_c dominates the other without the riskless asset. Yet, by allowing holdings of the riskless asset, we may find that *any* combination of one asset with the riskless asset is dominated by *some* combination of the other risky asset with the riskless asset. We further assume that the expected values of F_c and F_s are greater than the riskless interest rate, r. In the CAPM framework, this assumption implies a nonnegative beta.[3]

In deriving the bounds on the option value, we make two alternative assumptions on investors' preferences. We begin by assuming that $u \in U_1$, where U_1 is the set of all nondecreasing utility functions ($u' \geq 0$). The main results of this paper, however, are contained by assuming that $u \in U_2$, where U_2 is the set of all nondecreasing concave utility functions ($u' \geq 0$, $u'' \leq 0$).

II. Bounds on the Option Value

A. Bounds on Option Value for Nondecreasing Utility: First Degree Stochastic Dominance

Assuming that $C < S$, Figure 1 demonstrates the two cumulative distributions of W_c and W_s, which we denote by F_c and F_s, respectively. Note that F_s intersects F_c from below. Moreover, in general, there is a positive probability, p', that the investment in the call option ends up with zero terminal value, namely:

$$p' = \mathrm{pr}(\tilde{S}_T \leq X) \tag{2}$$

It is easy to verify that F_s and F_c intersect only once, and that the intersection point is given by $\tilde{S}_T = XS/(S - C)$, or in terms of the terminal wealth, W_T, by $W_T = W_0X/(S - C)$.[4] Thus, when the riskless asset is not allowed, and $C < S$, it

[3] It is interesting that the same restriction has been imposed by Perrakis and Ryan [18].

[4] By definition, $\mathrm{pr}[W_0/C(\tilde{S}_T - X) < W_0/C(Q(p) - X)] = p$ and

$$\mathrm{pr}\left[\frac{W_0}{S}\,\tilde{S}_T \leq \frac{W_0}{S}\,Q(p)\right] = p$$

where $Q(p)$ is the p^{th} quantile of the stock price distribution given by $\mathrm{pr}(\tilde{S}_T \leq Q(P)) = P$ (see footnote 5). Thus, the two cumulative distributions intersect at the point where $W_0/C(Q(p) - X) = W_0/S(Q(p))$. However, since $Q(p)$ is a quantile of the stock price distribution $\tilde{S}_T$, one can substitute $\tilde{S}_T$ for $Q(p)$ to obtain the intersection point $\tilde{S}_T = XS/(S - C)$. In terms of terminal wealth, $W_T = (W_0/S)\tilde{S}_T$, so the intersection point is at $W_T = W_0X/(S - C)$.

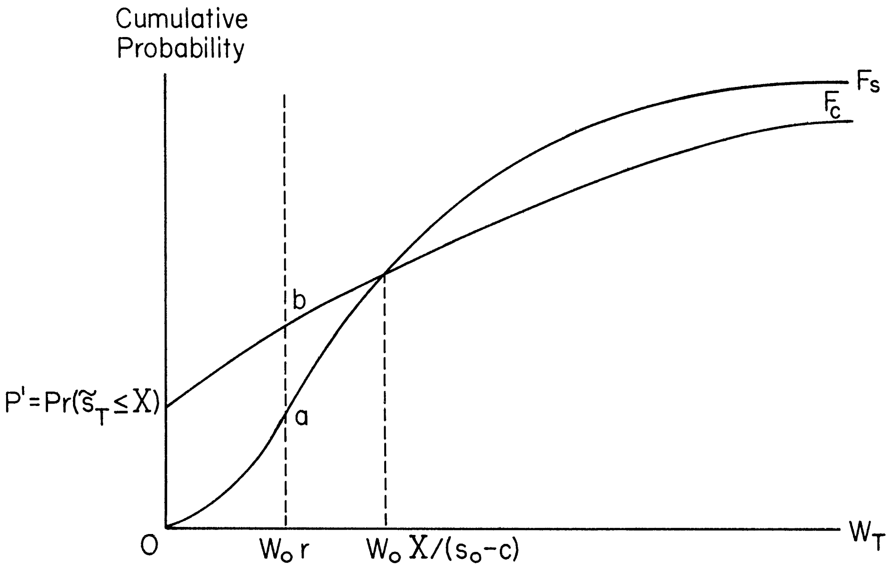

Figure 1. The Cumulative Distributions of the Return on the Option (F_c) and on the Stock (F_s)

is clear that neither F_c nor F_s dominates the other since the two distributions intersect. However, by adding the riskless asset, a dominance may occur. It is easier to show that such a dominance may take place and its economic implication by employing the distributions' quantiles rather than by analyzing directly F_s and F_c. Therefore, let us first analyze the relationship between the quantiles of the three distributions, F, F_c, and F_s, where F stands for the cumulative distribution of $\tilde{S}_T$, and F_c and F_s are as defined before. From the definition of the stock price cumulative distributions, we have:

$$F[Q(p)] = \text{pr}[\tilde{S}_T \leq Q(p)] = p \tag{3}$$

where $Q(p)$ is the p^{th} quantile of the stock price distribution.[5] Equation (3) can be rewritten as

$$p = \text{pr}\left[\frac{W_0}{S}\,\tilde{S}_T \leq \frac{W_0}{S}\,Q(p)\right] = \text{pr}[W_s < Q_s(p)] = F_s[Q_s(p)]$$

where $Q_s(p)$ is the p^{th} quantile of the distribution, F_s.

Recall that $(W_0/S)\tilde{S}_T = W_s$. Hence, the p^{th} quantile of the distribution of the return on investment, $Q(p)$, is related to the p^{th} quantile of the stock price distribution, $Q_s(p)$, as follows:

$$Q_s(p) = \frac{W_0}{S}\,Q(p) \tag{4}$$

[5] The quantile of order p of distribution $F(x)$ is defined as $\text{pr}(X < Q(P)) = P$, where X is a random variable whose cumulative distribution is $F(x)$. Note that $Q(\cdot)$ is the inverse function F^{-1}. Since $F(\cdot)$ is strictly monotone increasing, such an inverse exists.

By similar argument, the relationship of the p^{th} quantile of F_c to $Q(p)$ is given by

$$Q_c(p) = \max\left(0, \frac{W_0}{C}[Q(p) - X]\right) \tag{5}$$

Having these definitions and quantile relationships, we can turn to the stochastic dominance analysis. For simplicity, and without loss of generality, we assume that the initial investment either in the call or in the corresponding stock is $W_0 = \$1$. Thus, we hold a constant initial investment in the two alternatives.

Let F_s and F_c denote, as before, the cumulative distribution of investments S and C, respectively. Let us denote the cumulative distributions of the return on investments S_α and C_β by F_{s_α} and F_{c_β} where,

$$S_\alpha = \alpha S + (1 - \alpha)r,\ C_\beta = \beta C + (1 - \beta)r \text{ (for any } \alpha, \beta > 0)$$

Thus, C_β is a portfolio of C and the riskless asset, and S_α is a combination of portfolio S and the riskless asset; α, β are parameters which determine the mix of the risky and the riskless assets.[6]

Generally speaking, in order to achieve dominance of one distribution over the other, with only the assumption that $u' \geq 0$ ($u \in U_1$), the two distributions under consideration are not allowed to intersect, and the preferred investment is the one with the rightmost cumulative distribution. Such dominance is called first degree stochastic dominance (FSD).[7] When the riskless asset is introduced, the two distributions may intersect, but still, one option may dominate the other for all $u \in U_1$. Such dominance is called FSDR. Thus, we may have no dominance by FSD, but dominance by FSDR.

The FSDR dominance criterion is given in Theorem 1.

THEOREM 1. *Let F and G be any two cumulative distributions with quantiles $Q_F(p)$ and $Q_G(p)$, respectively. Then F dominates G by FSDR if and only if*

$$\text{Min}_{0 \leq p < F(r)} \frac{Q_G(p) - r}{Q_F(p) - r} \geq \text{Max}_{F(r) < p \leq 1} \frac{Q_G(p) - r}{Q_F(p) - r} \tag{6}$$

(see Levy and Kroll [15]).

Applying the condition given in Equation (6) to our specific distributions, F_s and F_c, we obtain the following theorem:

THEOREM 2. *Let F_c and F_s be the cumulative distributions of the return on investments C and S, respectively. Then,*

[6] Note that the constraint α, $\beta > 0$ can be easily relaxed. However, if the expected rate of return on the stock is greater than the risk-free interest rate, in our framework no investor would consider short-selling. Hence, we will obtain that α, $\beta > 0$.

[7] Let F and G be the cumulative distributions of two risky investments. We say that F dominates G by FSD if and only if $F(x) \leq G(x)$ for all x (with at least one strict inequality). Namely, $F(x) \leq G(x)$ for all x if and only if $E_F u(x) \geq E_G u(x)$, for all $u \in U_1$. For more details on FSD, see [9] and [10]. Note that in the expected utility framework, we do not focus explicitly on the distribution moments (e.g., preference for mean, variance, and skewness) but look at the entire distribution which implicitly incorporates the information about all the distribution moments.

(*a*) *A necessary and sufficient condition for dominance of C over S by FSDR is* $F_c(r) < F_s(r)$.

(*b*) *A necessary and sufficient condition for dominance of S over C by FSDR is* $C > S$.

The proof of Theorem 2 is rather long. Thus, we only sketch its steps.[8] First, note that for any two distributions, F and G, a necessary condition for dominance of F over G by FSDR is that $F(r) < G(r)$ (see Levy and Kroll [15]). Thus, what is left to be proved is that $F_c(r) < F_s(r)$ is also a sufficient condition for dominance. Applying the condition of Equation (6) to the distributions, F_c and F_s, we obtain

$$\mathrm{Min}_{0 \le p < F_c(r)} \frac{Q_s(p) - r}{Q_c(p) - r} \ge \mathrm{Max}_{F_c(r) < p \le 1} \frac{Q_s(p) - r}{Q_c(p) - r} \tag{7}$$

We first substitute the quantiles of $Q_s(P)$ and $Q_c(p)$ from Equations (4) and (5) into Equation (7). Then it can be shown that $F_c(r) < F_s(r)$ implies that Equation (7) holds. Thus, the necessary condition for dominance by FSDR becomes a necessary and sufficient one in the specific case when one compares the distribution of the return on a stock with the distribution of the return on its corresponding option.

COROLLARY. *The bound,* $C \ge S - X/r$, *proposed in Merton* [16] *is the FSDR lower bound.*

Proof: We have to show that the condition, $F_c(r) < F_s(r)$, is identical to Merton's lower bound, $C > S - X/r$ [16], To see this, recall that

$$F_s(r) = \mathrm{pr}\left(\frac{\tilde{S}_T}{S} \le r\right) = \mathrm{pr}(\tilde{S}_T \le rS) = F(rS),$$

and

$$F_c(r) = \mathrm{pr}\left(\frac{\tilde{S}_T - X}{C} \le r\right) = \mathrm{pr}(\tilde{S}_T \le rC + X) = F(rC + X)$$

where $\tilde{S}_T$ is the stock price with cumulative distribution F. Thus, $F_c(r) < F_s(r)$ implies that $rC + X < rS$ or $C < S - X/r$. Now if $F_c(r) < F_s(r)$ or $C < S - X/r$, then by Theorem 2, investment C dominates investment S for all nondecreasing utility functions, $u \in U_1$. To avoid such dominance and to prevent arbitrage (selling S and buying C), we must have $F_c(r) \ge F_s(r)$ which implies that $C \ge S - X/r$, which is exactly Merton's lower bound for the option value.

This result is not surprising, since Merton makes in his derivation the same assumptions which underlie the FSDR, namely that $u' \ge 0$ and that the riskless asset is available to all investors. However, note that Merton creates *a specific combination* of the call and the riskless asset, and compares it to an investment in stock. In contrast, the condition of Theorem 2 allows the investor to create any portfolio of the form $C_\alpha = \alpha C + (1 - \alpha)r$ (and not one specific portfolio as

[8] A proof of this claim is in Levy [14]. For the sake of brevity, we do not provide it in this study. The proof is available on request from the author.

Merton suggests), and also any portfolio of the form $S_\alpha = \alpha S + (1 - \alpha)r$. We have found that if $F_c(r) < F_s(r)$, then for any arbitrary portfolio S_{α_0} that one chooses, one can find some mix of C and the riskless asset, say, C_{α_1}, such that C_{α_1} dominates S_{α_0} by FSD. Namely, $F_{c_{\alpha_1}}(W_T) \leq F_{s_{\alpha_0}}(W_T)$ for all terminal wealth levels of W_T, or for all states of nature, exactly as in Merton's framework.

To sum up, instead of searching for a particular arbitrage condition (as in Merton), we suggest a procedure to determine the bounds, a procedure which has potential use for the valuation of other financial claims.

B. Bounds on Option Value for Concave Utility: Second Degree Stochastic Dominance

Let us first state the risk aversion dominance criteria:

THEOREM 3. *Let X and Y be two random variables whose cumulative distributions are F and G, respectively. Then F dominates G by second degree stochastic dominance (SSD) if and only if*

$$\int_{-\infty}^{z} [G(t) - F(t)]\, dt \geq 0,$$

for all z, with a strict inequality for at least one value, z_0. (For proof, see Fishburn [8], *Hadar and Russell* [9], *Hanoch and Levy* [10], *Rothschild and Stiglitz* [20] *and others).*

When riskless borrowing and lending is allowed, one can create an infinite set of portfolios of the form $X_\alpha = \alpha X + (1 - \alpha)r$, which we denote by $\{F_\alpha\}$, and a second set, $Y_\alpha = \alpha Y + (1 - \alpha)r$, which we denote by $\{G_\alpha\}$. Levy and Kroll [15] proved that if there is one value α such that F_α (the cumulative distribution of X_α) dominates G (the cumulative distribution of Y) by SSD, then the entire set $\{F_\alpha\}$ dominates the entire set $\{G_\alpha\}$ by SSD. The interpretation of this dominance is that for any mix of the random variable with the riskless asset, say $Y_\beta = \beta Y + (1 - \beta)r$, there is a mix in $\{F_\alpha\}$, say $X_\gamma = \gamma X + (1 - \gamma)r$, which dominates Y_β by SSD. Thus, every risk averter will be better off by holding a portfolio of F and the riskless asset rather than by holding a portfolio of G and the riskless asset. If such dominance of $\{F_\alpha\}$ over $\{G_\alpha\}$ exists, we say that F dominates G by second degree stochastic dominance with riskless asset (SSDR). We employ these results in the proof of Theorems 4 and 5.

B.1. The Lower Bound of the Option Value, C_L

THEOREM 4. *Let F_c and F_s be the cumulative distributions of the return on investment C and investment S, respectively. F_c dominates F_s by SSDR if and only if $p_c \leq p_s$, where p_c and p_s are the values which solve*

$$\int_0^{p_c} [Q_c(t) - r]\, dt = 0 \quad \text{and} \quad \int_0^{p_s} [Q_s(t) - r]\, dt = 0, \tag{8}$$

respectively.

In order to avoid such dominance, the call option price must be greater than some value, C_L, given by

$$C > C_L \equiv S - X/r + \frac{1}{rp_c}\int_0^X (X - \tilde{S}_T)f(\tilde{S}_T)\, d\tilde{S}_T \tag{9}$$

The proof that $p_c \leq p_s$ is a necessary and sufficient condition for dominance is given in the Appendix. We prove below that Equation (9) follows from this dominance condition.

Suppose that $p_c < p_s$ and hence, that F_c dominates F_s. Every risk averter will sell the stock and buy the call, the stock price will go down, and the call price will rise. This price change causes a shift to the right in the distribution, F_s and to the left in the distribution, F_c. Hence, p_c will go up and p_s will go down (see Figure A1). As long as $p_c \leq p_s$, no one will hold the stock.[9] Thus, to avoid such dominance, in equilibrium, we must have $p_c > p_s$. One can solve for the value, C, which equates p_s and p_c and can safely conclude that the market call price must be greater than the value which equates p_s and p_c. Recalling the definition of p_c and p_s (see Equation (8)), and the formulas for the quantiles (see Equations (4) and (5)), we have (note that $p' = \mathrm{pr}(S_T \leq X)$; see Equation (2)):

$$rp_c = \int_{p'}^{p_c} \frac{1}{C}[Q(t) - X]\, dt \quad \text{and} \quad rp_s = \int_0^{p_s} \frac{1}{S} Q(t)\, dt \tag{8$'$}$$

Change C until $p_s = \mathrm{p}_c$, namely, until we obtain

$$\frac{1}{C}\int_{p'}^{p_c} [Q(t) - X]\, dt = \frac{1}{S}\int_0^{p_c} Q(t)\, dt$$

Hence, the value C which solves this equation is given by

$$C = \frac{S\int_{p'}^{p_c} [Q(t) - X]\, dt}{\int_0^{p_c} Q(t)\, dt} = \frac{S\int_0^{p_c} [Q(t) - X]\, dt}{\int_0^{p_c} Q(t)\, dt} - \frac{S\int_0^{p'} [Q(t) - X]\, dt}{\int_0^{p_c} Q(t)\, dt}$$

which can be rewritten as

$$C = \frac{S\int_0^{p_c} Q(t)\, dt - SXp_c}{\int_0^{p_c} Q(t)\, dt} - \frac{S\int_0^{p'} [Q(t) - X]\, dt}{\int_0^{p_c} Q(t)\, dt} = S\left[1 - \frac{X}{\frac{1}{p_c}\int_0^{p_c} Q(t)\, dt}\right]$$

$$- \frac{S\int_0^{p'} [Q(t) - X]\, dt}{\int_0^{p_c} Q(t)\, dt}$$

[9] It is interesting to note that $p_c \leq p_s$ implies that $E_{F_c}(W_T) > E_{F_s}(W_T)$. This follows from the fact that if F_c dominates F_s by SSDR, there must be $0 < \alpha < 1$ such that $\alpha E_{F_c}(W_T) + (1 - \alpha)r > E_{F_s}(W_T)$ and this can hold only if $E_{F_c}(W_T) > E_{F_s}(W_T)$. Recall that if $\alpha > 1$, the left tail of F_c will always lie above the left tail of F_s, and investment C cannot dominate investment S. Thus, we have $\alpha < 1$ if dominance is indeed observed.

However, since we assume $p_c = p_s$, then by Equation (8′), we have $1/S \int_0^{p_c} Q(t)\, dt = rp_c$ or $1/p_c \int_0^{p_c} Q(t)\, dt = rS$. Substituting this term, we obtain for the lower bound value, C_L

$$C_L = S\left[1 - \frac{X}{rS}\right] - \frac{\int_0^{p'} [Q(t) - X]\, dt}{\frac{1}{S}\int_0^{p_c} Q(t)\, dt} = \left(S - \frac{X}{r}\right) - \frac{\int_0^{p'} [Q(t) - X]\, dt}{rp_c}$$

Since for all $p < p'$, $Q(t) < X$, the integral on the right-hand side is negative, hence the lower SSDR bound is greater than Merton's lower bound.

If C is smaller than the above value, then C dominates S by SSDR. In order to guarantee that such dominance does *not* exist, C must be greater than this critical value and we obtain a lower bound for the option value,

$$C > C_L = S - \frac{X}{r} + \frac{1}{rp_c}\int_0^{p'} [X - Q(t)]\, dt \tag{10}$$

It is easy to verify[10] that

$$\int_0^{p'} [X - Q(t)]\, dt = \int_0^{X} F(\tilde{S}_T)\, d\tilde{S}_T = \int_0^{X} (X - \tilde{S}_T) f(\tilde{S}_T)\, d\tilde{S}_T$$

Hence, Equation (10) can be rewritten as in Equation (9). Since the call price can never be negative, we can safely assert that $C > \text{Max}[0, C_L]$.

B.2. The Upper Bound on the Option Value, C_U

We turn now to derive the upper bound of the call value (which we denote by C_U) in the face of risk aversion.

THEOREM 5. Denote, as before the cumulative distribution of investments C and S by F_c and F_s, respectively. A necessary and sufficient condition for dominance of S over C by SSDR is that $E_s(W_T) \geq E_c(W_T)$, namely the expected value of investment S is not smaller than the expected value of investment C. In order to avoid dominance of S over C, the call option price must be lower than some upper value, C_U, given by

$$C_U = \frac{S}{E(\tilde{S}_T)}\int_X^{\infty} (\tilde{S}_T - X) f(\tilde{S}_T)\, d\tilde{S}_T \tag{11}$$

[10] For the stock price distribution $F(\tilde{S}_T)$ with the quantile $Q(t)$, we have $\int_0^{p'} [X - Q(t)]\, dt = \int_0^X F(\tilde{S}_T)\, d\tilde{S}_T$ since by definition $p' = \text{pr}(\tilde{S}_T < X)$ and the left-hand side is nothing but the area under $F(S_t)$ from zero up to point X. However, the right-hand side can be developed as

$$\int_0^X 1\cdot F(\tilde{S}_T)\, d\tilde{S}_T = \tilde{S}_T F(\tilde{S}_T)\Big|_0^X - \int_0^X \tilde{S}_T\, dF(\tilde{S}_T) = XF(X) - \int_0^X \tilde{S}_T\, dF(\tilde{S}_T)$$
$$= \int_0^X (X - \tilde{S}_T)\, dF(\tilde{S}_T) = \int_0^X (X - \tilde{S}_T) f(\tilde{S}_T)\, d\tilde{S}_T$$

Proof:

Sufficiency: To see this, note that[11]

$$E_s(W_T) - E_c(W_T) = \int_0^\infty [F_c(W_T) - F_s(W_T)]\, dW_T$$

Since $E_s(W_T) - E_c(W_T) \geq 0$ by assumption, also

$$\int_0^\infty [F_c(W_T) - F_s(W_T)]\, dW_T \geq 0$$

However, in our specific case, F_s intersects F_c from below, and there is only one intersection (see Figure 1). Hence, by decreasing the upper limit of the integral, we only eliminate some negative area, hence for any value, $x < \infty$, we have

$$\int_0^x [F_c(W_T) - F_s(W_T)]\, dW_T \geq 0$$

which implies second degree dominance of F_s over F_c.

Thus, if $E_s(W_T) \geq E_c(W_T)$, S dominates C by SSD rule. Since dominance by SSD implies dominance by SSDR, the sufficiency side of the proof is complete.[12]

Necessity: Suppose that $E_s(W_T) < E_c(W_T)$. Recall that a necessary condition for dominance of S over C by SSDR is that there is at least one value, $\alpha > 0$ such that the mix S_α dominates C, where the mix S_α is given by $S_\alpha = \alpha S + (1 - \alpha)r$ (with $\alpha > 0$). We will show that one cannot find such a value, α, if indeed $E_s(W_T) < E_c(W_T)$.

(a) Suppose that $0 < \alpha \leq 1$. In this case

$$E_{s_\alpha}(W_T) = \alpha E_s(W_T) + (1 - \alpha)r < E_c(W_T)$$

since by assumption $r < E_s(W_T)$ and $r < E_c(W_T)$. However, $E_{s_\alpha}(W_T) \geq E_c(W_T)$ constitutes a necessary condition for dominance, hence S_α does not dominate C by SSD.

(b) Let us turn to the second possible case, $\alpha > 1$. In this case, due to the leverage, F_s pivots clockwise around the point a (see Figure 1). Note that for $\alpha > 1$, if the return on the stock is equal to zero, we get the following return on the mix, S_α: $\alpha \cdot 0 + (1 - \alpha)r = 0 + (1 - \alpha)r < 0$. Thus, the left tail of distribution F_{s_α} is above the left tail of distribution $F_c(W_t)$. There-

[11] For any two distributions, F and G, the expected value difference is given by

$$E_F(z) - E_G(z) = \int_{-\infty}^{\infty} z[f(z) - g(z)]\, dz.$$

Integration by parts reveals that the difference of the means is equal to $\int_{-\infty}^{\infty} [G(z) - F(z)]\, dz$. If the distributions are defined only on the positive range, the integration is from zero to infinity.

[12] Recall that to guarantee dominance by SSDR, it is sufficient to find one combination, S_α, which dominates C by SSD. Choose $\alpha = 1$ to prove that dominance by SSD implies dominance by SSDR.

fore, for point $z = 0$, we have,

$$\int_{\infty}^{0} [F_c(W_T) - F_{s_\alpha}(W_T)]\, dW_T < 0$$

and hence, also in the case $\alpha > 1$, F_{s_α} does not dominate F_c by SSD. Since, neither for $0 < \alpha < 1$ nor for $\alpha > 1$, does F_{s_α} dominate F_c by SSD, we can safely conclude that F_s does not dominate F_c by SSDR as long as $E_s(W_T) < E_c(W_T)$. Once again, recall that if one cannot find α such that S_α dominates C by SSD, S does not dominate C by SSDR.

We turn now to the derivation of the upper bound, C_U (Equation (11)). First, recall that

$$E_s(W_T) = \frac{1}{S}\int_0^\infty \tilde{S}_T f(\tilde{S}_T)\, d\tilde{S}_T = \frac{1}{S} E(\tilde{S}_T) \tag{12}$$

The expected value of investment C is,

$$E_c(W_T) = 0\cdot \mathrm{pr}(\tilde{S}_T < X) + \frac{1}{C}\int_X^\infty (\tilde{S}_T - X) f(\tilde{S}_T)\, d\tilde{S}_T$$

Thus, S dominates C if

$$\frac{1}{S} E(\tilde{S}_T) \geq \frac{1}{C}\int_X^\infty (\tilde{S}_T - X) f(\tilde{S}_T)\, d\tilde{S}_T$$

or if

$$C \geq \frac{S}{E(\tilde{S}_T)}\int_X^\infty (\tilde{S}_T - X) f(\tilde{S}_T)\, d\tilde{S}_T = C_U$$

Obviously, the higher the value of C, the higher the chance that S dominates C. Thus, the smallest value of C such that S still dominates C is given by Equation (11), and in equilibrium, when S does not dominate C, we must have $C < C_U$.

Equation (11) has a very simple interpretation: the upper bound on the call value is equal to the expected value of the option cash flow discounted at the stock's required rate of return, $r_s = E(\tilde{S}_T)/S$. We have proved that $C \leq C_U$; thus, we have really shown that the call option is riskier than the corresponding stock since the appropriate discount rate for the call option, r_c, must satisfy $r_c > r_s$ (see Equation (11)).

Note also that the value of the call option at expiration is $\mathrm{Max}\{0, \tilde{S}_T - X\}$. This value is stochastically dominated by $\tilde{S}_T$. Therefore $E(C) < E(\tilde{S}_T)$, and hence

$$C_U = \frac{E(C)}{r_s} < \frac{E(\tilde{S}_T)}{r_s} = S \tag{13}$$

This shows that the SSDR upper bound is less than S, as one might expect.

While it is not immediate, it also can be shown that the bounds increase with an increase in S and r, and with a decrease in X.

B.3. Put Option Bounds

Having upper and lower bounds, one can use the put-call parity relationship to derive bounds on put options. Let C, P, and S be the call option, put option, and stock values, respectively. Let r_c and r_s equal one plus the appropriate discount rate of the call option and stock, respectively. $\tilde{C}_T$, $\tilde{P}_T$, and $\tilde{S}_T$ are the stochastic values of these three variables at the expiration date, respectively. From Equations (9) and (11), we obtain:

$$S - \frac{X}{r} + \frac{E(\tilde{P}_T)}{rp_c} < C < \frac{E(\tilde{C}_T)}{r_s} \tag{14}$$

where $E(P_T) = \int_0^X (X - \tilde{S}_T) f(\tilde{S}_T)\, d\tilde{S}_T$ is the expected return on the put option. The well-known put-call parity equation is

$$S - \frac{X}{r} + P = C \tag{15}$$

Substitute (15) into (14) to obtain the bounds for the put option.[13]

$$\frac{E(\tilde{P}_T)}{rp_c} < P < \frac{E(\tilde{C})}{r_s} + \frac{X}{r} - S \tag{16}$$

One can apply arguments similar to those given in this paper and in the Appendix to derive directly upper and lower bounds to the put option, not using put-call parity. We demonstrate below the direct derivation of the lower bound and compare it to the lower bound given in (16).

Suppose that investors consider buying either the stock or the put. As before, $Q(p)$ denotes the p^{th} quantile of the stock price distribution. By definition,

$$\text{pr}(\tilde{S}_T > Q(1 - p)) = p$$

or

$$\text{pr}(-S_T \le -Q(1 - p)) = p$$

which also implies

$$\text{pr}\left(\frac{1}{P}(X - \tilde{S}_T) < \frac{1}{P}(X - Q(1 - P))\right) = p$$

Since $(1/P(X - S_T))$ is the return on the investment of \$1 in the put option (where $X > S_T$), we obtain that the quantile of the return on the investment in the put options is as follows:

$$Q_P(p) = \begin{cases} 0 & \text{if } X < \tilde{S}_T \\ \dfrac{1}{P}(X - Q(1 - p)) & \text{if } X \ge \tilde{S}_T \end{cases}$$

The formula for $Q_P(p)$ reveals that the cumulative distributions of the investment in the put option, F_P, is very similar to the one of the call option which is shown in Figure 1, namely, there is only one intersection between F_s and F_P, and

[13] I would like to thank the referee for this idea.

F_s intersects F_P from below. Using the same technique as in Theorem 5, it can easily be shown that a necessary and sufficient conditions for SSDR dominance of S over P is that $E(W_s) \geq E(W_p)$, where W_s and W_P are the terminal wealth of an investment of \$1 in the stock and in the put option, respectively. Thus, to avoid such dominance, we must have,

$$\frac{E(\tilde{S}_T)}{S} < \frac{1}{P} \int_0^X (X - \tilde{S}_T) f(\tilde{S}_T) \, d\tilde{S}_T$$

Hence,

$$P < \frac{S}{E(\tilde{S}_T)} \int_0^X (X - \tilde{S}_T) f(\tilde{S}_T) \, d\tilde{S}_T = P_U \tag{17}$$

where P_U denotes the upper bound on the put option. Add and subtract from Equation (17) the term,

$$\frac{S}{E(\tilde{S}_T)} \int_X^\infty (X - \tilde{S}_T) f(\tilde{S}_T) \, d\tilde{S}_T$$

to obtain

$$P_U = \frac{S}{E(\tilde{S}_T)} (X - E(\tilde{S}_T)) - \frac{S}{E(\tilde{S}_T)} \int_X^\infty (X - \tilde{S}_T) f(\tilde{S}_T) \, d\tilde{S}_T$$

or

$$P_U = \frac{X}{r_s} - S + \frac{1}{r_s} \int_X^\infty (\tilde{S}_T - X) f(\tilde{S}_T) \, d\tilde{S}_T$$

which can finally be rewritten as

$$P_U = \frac{E(\tilde{C})}{r_s} + \frac{X}{r_s} - S \tag{18}$$

A comparison of Equations (18) and (16) reveals that P_U is smaller than the upper value given in Equation (16). To see this, recall that, by assumption, the expected rate of return on the stock is greater than the riskless interest rate, hence $r_s > r$. For stocks with $E(\tilde{S}_T)$ which is close to the risk-free interest rate, r_s approaches r and the two bounds coincide.

An explanation for the fact that Equation (18) provides a smaller upper bound than Equation (16) stems from the implicit assumption we have made in deriving these two equations. In Equation (16), we implicitly assume that investors buy either the stock or the "synthetic call" which is a portfolio composed of one put, one share of stock and borrowings of E/r. Thus, we are able to find bounds for the "synthetic call," which is a portfolio, $P + S - E/r$, from which we derive the put value bounds (see Equation (16)). Thus, the proportions of the components in the "synthetic call" are fixed and hence we lose flexibility in the dominance comparisons. On the other hand, in Equation (18), we compare directly the distribution of the rate of return on the put with the distribution of the rate of return on the stock. No constraint is imposed on the put (i.e., to be included in some fixed proportions with other assets), hence tighter bounds are obtained.

Before we turn to portfolio consideration, let us illustrate our bounds by mean of a numerical example. Assume that $F(\tilde{S}_T)$ is a lognormal distribution with $\sigma = 0.10$ and mean $\mu = 5\%$. Also assume that $S = \$100$, $X = \$90$, and the riskless interest rate is 3% and that the time to expiration is $T = 0.25$. Employing the Black-Scholes formula reveals that for these parameters, $C = \$13.04$. Using our Equations (9) and (11) show that $C_L = \$12.81$ and $C_U = \$15.03$. Merton's lower and upper bounds are \$12.62 and \$100, respectively. Thus, by adding the risk-aversion assumption, we have succeeded in significantly tightening up Merton's bound. It is true that this result is a function of the chosen parameters. However, the results are quite robust, and we get a similar decrease in Merton's bounds for a large set of parameters.

III. Portfolio Consideration

In deriving the upper and lower bounds on the call option value in Section II, we assumed that investors hold either the call or the stock, i.e., holding a single-risky-asset portfolio. In this section, we extend the result to the case where investors are allowed to hold a third risky asset which can be the market portfolio.

Denote by x_c, x_s, and x_m the returns on the call option, the underlying stock, and a third risky asset, e.g., the market portfolio.[14] The following theorem extends our results to the general case where investors are allowed to diversify in a third risky asset, assuming arbitrary dependence between the returns on the various risky assets.

THEOREM 6. *Let $f(x_c, x_m)$ and $f(x_s, x_m)$ be the joint density distribution functions of x_c and x_m, and of x_s and x_m, respectively. Denote by $f(x_c \mid x_m)$ and $f(x_s \mid x_m)$ the appropriate conditional density functions. Then:*

(*a*) *If $f(x_c \mid x_m)$ dominates $f(x_s \mid x_m)$ by SSD for all possible values of x_m, then any portfolio of x_s and x_m is dominated (by SSD) by at least one portfolio constructed from x_c and x_m.*

(*b*) *If $f(x_s \mid x_m)$ dominates $f(x_c \mid x_m)$ by SSD for all values of x_m, then any portfolio of x_c and x_m is dominated (by SSD) by at least one portfolio constructed from x_s and x_m.*

We first prove this theorem and then discuss its implication to the call options bounds.

Proof: By definition, the expected utility of a portfolio comprising x_c in proportion α and x_m in the proportion $(1 - \alpha)$ is given by

$$Eu(\alpha x_c + (1-\alpha)x_m) = \int_{x_c}\int_{x_m} u(\cdot) f(x_c, x_m)\, dx_c\, dx_m$$

Substituting $f(x_c \mid x_m) f(x_m)$ for the joint density $f(x_c, x_m)$ yields

$$Eu(\alpha x_c + (1-\alpha)x_m) = \int_{x_c}\int_{x_m} u(\cdot) f(x_c \mid x_m) f(x_m)\, dx_c\, dx_m \tag{19}$$

[14] Note that in order to simplify the formulas, we use here different notation, compared to the previous section.

Equation (19) can be rewritten as

$$E_c u(\cdot) = \int_{x_m} f(x_m)\left(\int_{x_s} u(\cdot) f(x_c \mid x_m)\, dx_c\right) dx_m \tag{20}$$

when the subscript c indicates that the expected utility is calculated for a portfolio which comprises the call option, x_c, and a third risky asset, x_m. Similarly, the expected utility of a portfolio which comprises x_s and x_m can be rewritten as

$$Eu(\alpha x_s + (1 - \alpha)x_m) = \int_{x_m} f(x_m)\left(\int_{x_s} u(\cdot) f(x_s \mid x_m)\, dx_s\right) dx_m \tag{21}$$

or

$$E_s u(\cdot) = \int_{x_m} f(x_m)\left(\int_{x_s} u(\cdot) f(x_s \mid x_m)\, dx_s\right) dx_m \tag{22}$$

when the subscript s denotes a portfolio which combines the stock, x_s, and the risky asset, x_m.

Assume that $f(x_c \mid x_m)$ dominates $f(x_s \mid x_m)$ by SSD for all x_m. Then, for all nondecreasing concave utility functions, $u(\cdot)$ ($u' \geq 0$ and $u'' \leq 0$), we have

$$\int_{x_c} u(\cdot) f(x_c \mid x_m)\, dx_c > \int_{x_s} u(\cdot) f(x_s \mid x_m)\, dx_s \quad \text{for all } x_m \tag{23}$$

Since $f(x_m)$ is common to Equations (20) and (22), inequality (23) implies that $E_c u(\cdot) \geq E_s u(\cdot)$ for all nondecreasing concave $u(\cdot)$. Thus, we have found one portfolio, $\alpha x_c + (1 - \alpha)x_m$, which dominates the portfolio, $\alpha x_s + (1 - \alpha)x_m$. By the same token, we can show that if $x_s \mid x_m$ dominates $x_c \mid x_m$ for all x_m, then the expected utility of Equation (22) is greater than the expected utility of Equation (20), which completes the proof.

Note that in order to check whether one conditional distribution dominates the other, it is sufficient to know one conditional distribution since the other can be derived from it. Indeed, Equations (4) and (5) reveal that the return on the stock and the call option are related to each other. Hence, $f(x_c \mid x_m)$ and $f(x_s \mid x_m)$ are also related. Nevertheless, knowing, for example, that $f(x_c \mid x_{m_1})$ dominates $f(x_s \mid x_{m_1})$, one cannot employ the relationships of Equation (4) or (5) in order to make a safe statement regarding the dominance of, say, $f(x_c \mid x_{m_2})$ $f(x_s \mid x_{m_2})$ (when x_{m_1} and x_{m_2} are two values that x_m can take).

Let us turn to investigate the implications of the conditions of Theorem 6 to the option's bounds. Suppose that $x_c \mid x_m$ dominates $x_s \mid x_m$ for every x_m. Then, Theorem 6 implies that every portfolio comprising the stock x_s and the risky asset x_m is dominated by some portfolio composed of the option x_c and the same risky asset x_m. We claim that in equilibrium such dominance is impossible. If such dominance exists, the market price of the call option C will rise until the dominance of the portfolio consisting of x_c and x_m over the portfolio of x_s and x_m is wiped out. To avoid such dominance, it is necessary that the conditional distribution $f(x_c \mid x_m)$ would not dominate the conditional distribution $f(x_s \mid x_m)$ for at least one value of x_m (see Equations (20) and (22)). Namely, if no dominance is observed for one pair of conditional distributions, we can no longer guarantee dominance of the mix of x_c and x_m over the mix of x_s and x_m.

Suppose that x_m can take the values $x_{m_1}, x_{m_2}, \cdots, x_{m_k}$. We first find the condition that $x_c \mid x_{m_1}$ does *not* dominate $x_s \mid x_{m_1}$ (while for the other values of x_{m_i} $(i \neq 1)$, dominance may or may not exist). As in Section II, from the nondominance condition, one can derive that $C > C_L^1$ where C_L^1 stands for the lower bound on the option value given that x_{m_1} occurs. Note that having switched to conditional distributions $f(x_c \mid x_{m_1})$ and $f(x_s \mid x_{m_1})$, we are in fact dealing with distributions like those shown in Figure 1, and the analysis of Section II may be repeated verbatim to derive the bounds on the option's value.

Another possible reason for nondominance is that $f(x_c \mid x_{m_2})$ does not dominate $f(x_s \mid x_{m_2})$ (while with respect to other values of x_{m_i} $(i \neq 2)$, dominance may or may not exist). From the corresponding nondominance condition, one can establish that $C > C_L^2$, where C_L^2 is the lower bound derived when x_{m_2} occurs. Thus, for each x_{m_i}, we obtain a unique lower bound on the call option value, $C > C_L^i$.

If the option market price, C, is greater than all these C_L^i $(i = 1, 2, \cdots, k)$, there is certainly no dominance of $x_c \mid x_{m_i}$ over $x_s \mid x_{m_i}$ for any of the possible values of x_{m_i}. However, dominance of the mix of x_c and x_m over the mix of x_s and x_m is not guaranteed as soon as $x_c \mid x_{m_i}$ does not dominate $x_s \mid x_{m_i}$ for only *one* value of x_{m_i}. Therefore, the no-dominance condition implies the following lower bound on the call option value:

$$C > \mathrm{Min}(C_L^1, C_L^2, \cdots, C_L^k)$$

If the call price is greater than the minimum lower bound (C_L^j, say) but smaller than all the other lower bounds $C_L^i (i \neq j)$, $x_c \mid x_{m_i}$ dominates $x_s \mid x_{m_i}$ for all x_{m_i} apart from the one value x_{m_j} for which the minimum lower bound $C_L^j = \mathrm{Min}(C_L^1, \cdots, C_L^k)$ is obtained. But if $x_c \mid x_{mj}$ does not dominate $x_s \mid x_{m_j}$, we cannot safely say that the mix of x_c and x_m dominates the mix of x_s and x_m, and it is impossible to assert that in equilibrium, the call price, C, must rise.

Hence, as long as $C > \mathrm{Min}(C_L^1, C_L^2, \cdots, C_L^k)$, we cannot assert with certainty that all risk averters will switch from the portfolio (x_s, x_m) to the portfolio (x_c, x_m) and so any call price greater than $\mathrm{Min}_i(C_L^i)$ may represent an equilibrium price.

By exactly the same argument as in Section II, for each value of $x_{m_i}(i = 1, \cdots, k)$, we find an upper bound C_U^i such that $f(x_s \mid x_{m_i})$ does not dominate $f(x_c \mid x_{m_i})$ for $C < C_U^i$. Recall that it is enough that one conditional distribution, $f(x_s \mid x_m)$, does not dominate $f(x_c \mid x_m)$ in order to rule out the situation where the mix of x_s and x_m with certainty dominates the mix of x_c and x_m, for all possible probability distributions of x_m. For example, take the extreme case where $f(x_s \mid x_{m_j})$ does not dominate $f(x_c \mid x_{m_j})$ and that $\mathrm{pr}(x_m = x_{m_j}) = 1$. In this case, we can safely assert that the mix of x_s and x_m does not dominate the mix of x_c and x_m as long as $f(x_s \mid x_{m_j})$ does not dominate $f(x_c \mid x_{m_j})$ (see Equations (20) and (22)).

Since any upper bound value derived from one nondominance condition on the conditional distributions is sufficient to rule out sure dominance of the mix (x_s, x_m) over the mix (x_c, x_m), we can safely claim that the equilibrium call value must be smaller than the maximum of all $C_U^i (i = 1, 2, \cdots, k)$.

If C is smaller than the maximum of C_U^i, we have the situation that $f(x_s \mid x_m)$

does not dominate $f(x_c \mid x_m)$ for one value of x_m. If C is smaller than, say, two values, C_U^i and C_U^j, then $f(x_s \mid x_m)$ does not dominate $f(x_c \mid x_m)$ for two values of x_m. Since a no-dominance situation for one value of x_m is sufficient to rule out the sure dominance of the mix (x_s, x_m) over the mix (x_c, x_m), we can safely assert that as long as $C > \text{Max}_i(C_U^i)$ we have dominance of the mix (x_s, x_m) over the mix (x_c, x_m). To avoid such dominance, we simply require that C be smaller than the greatest of all the upper bounds. Thus, the upper bound condition becomes

$$C < \text{Max}_i(C_U^i)$$

Combining the upper and the lower bounds, we obtain

$$\text{Min}_i(C_L^i) < C < \text{Max}_i(C_U^i) \tag{24}$$

IV. Concluding Remarks

Interest by both academicians and practitioners in option valuation models has increased tremendously in recent years. The existing models which employ the idea that investors create a perfectly hedged portfolio yield single-valued equilibrium formulas for option values, which are very useful and easy for practitioners to implement. These strong results are achieved by making some assumptions regarding the distribution process and by assuming no taxes, no transaction costs and that the proceeds from short sales are received by short sellers.

In this paper, we suggest a stochastic dominance framework for option valuation. Since the proposed model is discrete, taxes, transaction costs, etc., are easily incorporated in the analysis. Also, the model is distribution-free and can be used by applying either subjective or objective probabilities.

The bounds on the call option value can be derived either by a partial analysis when investors are assumed to hold either the option alone or the underlying stock alone, but do not hold other risky assets, or in a portfolio context, when investors may hold as many risky assets as desired. In the portfolio context, the bounds are derived by examining the conditional distributions of the call and stock returns $f(x_c \mid x_m)$ and $f(x_s \mid x_m)$, given the market portfolio return, x_m, rather than the marginal return distributions, $f(x_c)$ and $f(x_s)$. In practice, it is reasonable to assume two alternative states for the market portfolio returns, x_m, e.g., a bull market or a bear market, and proceed to derive the upper and lower bounds for each of these two states. Then the highest of all upper bounds and the lowest of all lower bounds may be used as the relevant bounds on the call option value in the portfolio context.

The disadvantage of the suggested framework in comparison to other valuation models is that we obtain a range, an upper and a lower bound, rather than one value. This property is, of course, inferior with respect to the existing models, which provide one equilibrium price for the option.

Appendix

In this Appendix, we probe Theorem 2 of the text. Let us repeat the Theorem's claim:

Let F_c and F_s be the cumulative distributions of the return on investments C and S, respectively. F_c dominates F_s by SSDR if and only if $p_c \leq p_s$ where p_c and p_s are the values which solve,

$$\int_0^{p_c} (Q_c(t) - r)\, dt = 0 \quad \text{and} \quad \int_0^{p_s} (Q_s(t) - r)\, dt = 0 \quad \text{(see Figure A1)}$$

Proof: Levy and Kroll [15] proved that for any two distributions, C and S (not necessarily distributions of the return on the call and stock), a necessary condition for dominance of C over S by SSDR is $p_c \leq p_s$, hence this condition is clearly a necessity in our specific case. So, what is left to prove is that in our specific case, this condition is also a sufficient one. We need the following two definitions, which will be used in the proof:

(a)
$$\delta_c(p) \equiv \frac{Q_s(p) - r}{Q_c(p) - r}, \quad \gamma_c(p) \equiv \frac{\int_0^p [Q_s(t) - r]\, dt}{\int_0^p [Q_c(t) - r]\, dt} \tag{A1}$$

where the investment in the call is denoted by C and the investment in the corresponding stock is denoted by S, with quantiles $Q_c(p)$ and $Q_s(p)$, respectively, and where r equals one plus the risk-free interest rate.

(b) Let X and Y be two random variables (returns) on two distinct risky options whose cumulative distributions are F and G, respectively. Define

$$X_\alpha = \alpha X + (1 - \alpha)r, \quad \alpha \geq 0$$

$$Y_\beta = \beta Y + (1 - \beta)r, \quad \beta \geq 0$$

The set of all combinations, X_α and Y_β, are denoted by $\{F_\alpha\}$ and $\{G_\beta\}$, respectively. We assert that the set $\{F_\alpha\}$ dominates the set $\{G_\beta\}$ if, for any linear

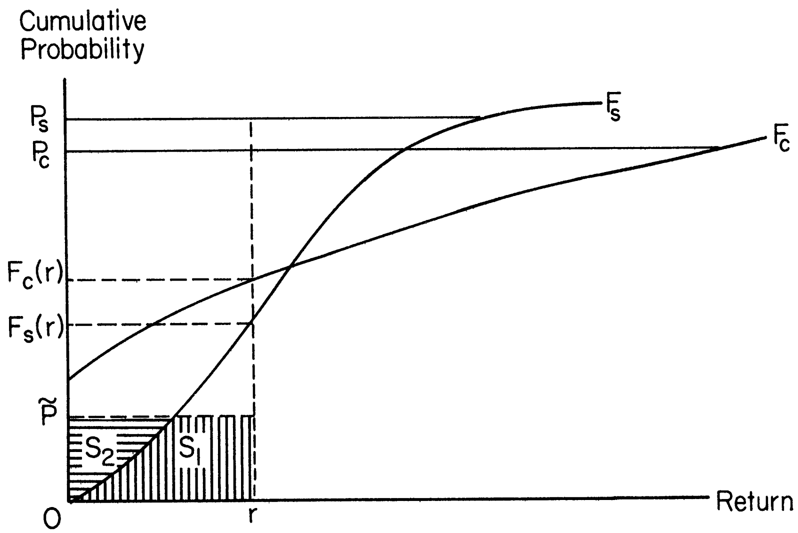

Figure A1. The Cumulative Distributions F_c and F_s where $F_s(r) < F_c(r)$ and $P_c < P_s$

combination, Y_β, taken from $\{G_\beta\}$, there is at least one linear combination, X_α, taken from $\{F_\alpha\}$ which dominates it.

Levy and Kroll [15] proved that for any two risky options, F and G, $\{F_\alpha\}$ dominates the set $\{G_\beta\}$ for all risk averters, that is, for all $u \in U_2$ ($u' \geq 0$, $u'' \leq 0$), if and only if

$$\mathrm{Inf}_{0 \leq p < p_F} \frac{\int_0^p [Q_G(t) - r]\, dt}{\int_0^p [Q_F(t) - r]\, dt} \geq \mathrm{Sup}_{p_F < p \leq 1} \frac{\int_0^p [Q_G(t) - r]\, dt}{\int_0^p [Q_F(t) - r]\, dt} \tag{A2}$$

where p_F is the value which solves the equation, $\int_0^{p_F} [Q_F(t) - r]\, dt = 0$, and Inf and Sup stand for the lowest and the highest values obtained in the appropriate ranges, respectively.

If the above inequality holds, we say that F dominates G by SSDR, which stands for second degree stochastic dominance with riskless asset. Namely, if inequality (A2) holds, we can conclude that, for every $G_\beta \in \{G_\beta\}$, there is at least one combination, $F_\alpha \in \{F_\alpha\}$, such that $E_{F_\alpha} u(X) \geq E_{G_\beta} u(X)$ for all utility functions, u, with $u' \geq 0$ and $u'' \leq 0$. Applying (A2) to our specific case (and using definition (A1)), we assert that C dominates S if

$$\mathrm{Inf}_{0 \leq p < p_c}\, \gamma_c(p) \geq \mathrm{Sup}_{p_c < p \leq 1}\, \gamma_c(p) \tag{A3}$$

Before we turn to the proof, we need the following:

LEMMA. *Let F and G be cumulative distributions such that $P_F < P_G$ (where P_F is defined as above and P_G is defined similarly, with Q_G replacing Q_F). If $\gamma_F(p)$ has an internal minimum in the range $0 \leq p < P_F$, say at p^* and internal maximum the range $P_F < p \leq 1$, say at p^{**} (where $p^{**} > p^*$), then at these two extreme points, we have $\gamma(p^*) = \delta(p^*)$ and $\gamma(p^{**}) = \delta(p^{**})$.*

For proof, see Kroll and Levy [12].

Having these definitions and the above Lemma, we turn to prove the sufficiency side of Theorem 2.

Distinguish between two cases, $F_c(r) < F_s(r)$ and $F_c(r) \geq F_s(r)$.

(i) $F_c(r) < F_s(r)$. In this case, F_c dominates F_s by FSDR (see Theorem 1) and since FSDR implies SSDR, F_c also dominates F_s by SSDR.

(ii) $F_c(r) \geq F_s(r)$. Thus, what is left to prove is that in case $F_c(r) \geq F_s(r)$, $p_c \leq p_s$ implies dominance of F_c over F_s by SSDR. This situation is described in Figure A1. We prove below that in this specific case, $\gamma(p)$ has an internal minimum in the range, $0 \leq p < p_c$, and an internal maximum in the range, $p_c < p \leq 1$, and that $p_c < p_s$ implies that the minimum is above the maximum, and hence Equation (A3) holds, which implies dominance of C over S.

Let us first examine the behavior of $\gamma_c(p)$ (as defined in Equation (A1)) in the range, $0 \leq p < p_c$. At $p = 0$, we apply L'Hôpital's rule to obtain $\gamma_c(p) = 1$. As p increases, $\gamma_c(p)$ at first falls below 1. For example, at the point $p = \tilde{p}$, $\gamma_c(p) = -S_1/(-S_1 - S_2) = S_1/(S_1 + S_2) < 1$ (see Figure A1 and definition (A1)). However, as p approaches p_c, the denominator of $\gamma_c(p)$ approaches zero from below, but since $p_c \le p_s$, the numerator of $\gamma_c(p)$ is still negative and hence $\gamma_c(p)$ approaches $+\infty$. Thus, in the range $0 \le p < p_c$, $\gamma_c(p)$ starts from 1, falls below 1 and at p_c approaches $+\infty$, which implies that in the range $0 \le p < p_c$, $\gamma_c(p)$ has an internal minimum.

By a similar argument, it can be shown that $\gamma_c(p)$ has an internal maximum in the range $p_c < p \le 1$. Thus, what is left to prove is that $\gamma_c(p^*) \ge \gamma_c(p^{**})$, where $\gamma_c(p^*) \equiv \text{Inf } \gamma_c(p)$ and $\gamma_c(p^{**}) \equiv \text{Sup } \gamma_c(p)$ in the appropriate ranges. However, since, at the extreme points, we have (see Lemma) $\gamma_c(p^*) = \delta_c(p^*)$ and $\gamma_c(p^{**}) = \delta_c(p^{**})$, if $\delta_c(p)$ is not increasing for all $p^* \le p \le p^{**}$, then $\gamma_c(p^*) \ge \gamma_c(p^{**})$ and F_c dominates F_s by SSDR. We prove below that in this specific case, $\delta_c(p^*)$ is indeed not increasing.

First note that in the neighborhood of $F_c(r)$, $\gamma_c(p)$ is still declining. The reason is that at the point $p = F_c(r)$, $Q_c(p) - r = 0$, while $Q_s(p) - r > 0$. Hence at the point $p = F_c(r)$, there is zero change in the denominator of $\gamma_c(p)$, while the numerator becomes less negative (since $Q_s(p) - r > 0$ at this point), which implies that $\gamma_c(p)$ is still declining at the point $p = F_c(r)$. Thus, the minimum of $\gamma_c(p)$ is obtained at some point $p^* > F_c(r)$, and since by definition, $p^{**} > p^*$, we must have the following relationships: $p^{**} > p^* > F_c(r)$. Thus, in order to prove the theorem, it is sufficient to show that $\delta_c(p)$ is declining in the range $p^{**} > p > F_c(r)$.

Let us analyze $\delta_c(p)$ (see (A1)) in this range: when $p = F_c(r)$, $Q_c(p) = r$ and $Q_s(p) - r > 0$ (see Figure A1), hence $\delta_c(p)$ approaches $+\infty$. For any value $p > F_c(r)$ we have

$$\frac{\partial \delta_c(p)}{\partial p} = \frac{[Q_c(p) - r]Q_s'(p) - [Q_s(p) - r]Q_c'(p)}{[Q_c(p) - r]^2}$$

Plugging into the above term the quantiles definitions from Equations (4) and (5),

$$Q_c(p) = \frac{1}{C}\,[Q(p) - X] \quad \text{and} \quad Q_s(p) = \frac{1}{S}\,Q(p)$$

the numerator of the derivative can be written as

$$\left[\frac{1}{C}(Q(p) - X) - r\right]\frac{1}{S}Q'(p) - \left[\frac{1}{S}Q(p) - r\right]\frac{1}{C}Q'(p) = \frac{Q'(p)}{SC}[rS_0 - rC - X]$$

Since $Q'(p) > 0$, what is left to show is that $(rP_0 - rC - X)$ is nonpositive. However, since we deal in the case where $F_c(r) \ge F_s(r)$, we have

$$F_c(r) = \text{pr}\left[\frac{1}{C}\,(\tilde{S}_T - X) \le r\right] = \text{pr}(\tilde{S}_T \le X + rC) = F(X + rC)$$

$$F_s(r) = \text{pr}\left[\frac{1}{S}\,\tilde{S}_T \le r\right] = \text{pr}(\tilde{S}_T \le rS) = F(rS)$$

Recall that $\tilde{S}_T$ is the stock price whose distribution is F. Thus $F_c(r) \ge F_s(r)$ implies that $X + rC \ge rS$, which in turn implies that $rS - X - rC \le 0$, hence

$\partial\delta_c(p)/\partial(p)$ is indeed nonpositive in the range, $F_c(r) < p \leq p^{**}$. Thus, $\delta_c(p)$ is a nonincreasing function in this range, which completes the proof.

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