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Towards a Semigroup Pricing Theory

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ABSTRACT

In an arbitrage-free economy, there will always exist a set of linear operators which map future contingent dividends of securities into their current prices. It happens that such operators will also form an "evolution semigroup" as a consequence of intertemporal analysis of the no-arbitrage restriction. This paper summarizes some of the major implications of the semigroup properties, but avoids almost all of the technical discussion which underlies them. Instead, several practical examples are presented. Some well-known continuous-time results are replicated by this alternative method, and certain new developments are explored.

IT IS WELL KNOWN that in an arbitrage-free economy, there exists a nonnegative linear operator mapping dividend streams into current prices.[1] What is less well known is that these operators, as a consequence of *intertemporal* no-arbitrage, must additionally possess a semigroup property. This paper is an overview of work in progress which deals with the semigroup features, but which is contained in rather longer and more complicated papers.[2] Here I shall attempt to make the main results (and prospective results in some cases) accessible to a broader financial economics readership.

At first blush, the terminology itself is intimidating: we shall be dealing with the apparently difficult notions of (1) linear operators; (2) infinitesimal generators; and (3) evolution semigroups. These terms become less daunting when it is realized that almost everyone has encountered a simple example of each via the "discounting" of future payments. Consider a world of certainty where the discount function between dates t and τ is given as ${}_tQ_\tau$. That is, if the current time is t and a future payment is to be received at time τ in the amount A_τ, then the present value of the future payment would be given as $V_t = {}_tQ_\tau A_\tau$. We see that ${}_tQ_\tau$ may be considered an *operator*, since it maps future payments into current values; it is *linear*, since two (contemporaneous) payments may be added together first and then discounted, or discounted individually and added, with the same result; and finally it is *nonnegative*, since no positive future payment can ever have a present value of less than zero. Thus, the (certainty) discount function is an instance of a nonnegative linear operator. The *infinitesimal generator* may be thought of as the time derivative of the operator, in our certainty case the rate of change of the discount function. In other words, we would want to characterize what happens to the operator ${}_tQ_\tau$ as τ approaches t (i.e., the time interval shortens). If ${}_tQ_\tau$ is "well-behaved" then we would expect it

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[1] See, e.g., Ross [19].

[2] Principally [8].

to approach 1.0, which is another way of saying that the present value of payments made a short time hence will approach their nominal payment amount as the time interval shortens. This in turn leads to the rate-of-change concept, or infinitesimal generator. Consider the limit $\tau \rightarrow t$ of $({}_tQ_\tau - 1)/(\tau - t)$, which looks very much like a derivative calculation, except that that it is performed on an operator. For the discount operator, this can be recognized as (minus) the instantaneous rate of interest.[3] The corresponding infinitesimal generator then measures "how fast" the present values decline with time; in a small period of time, such a decline will approximately equal the instantaneous interest rate multiplied by the time period length in our certainty example. Finally, the discount operators ${}_tQ_\tau$ must form an *evolution semigroup.*[4] This simply means that their term structure properties must be consistent, or ${}_tQ_\tau = {}_tQ_{uu}Q_\tau$ for all $t \leq u \leq \tau$. For instance, in our world of certainty, the five-period discount function must be the product of successive two- and three-period discount functions; it must also be the product of the successive one- and four-period discount functions, etc. The trust of our efforts in semigroup valuation theory is merely to place economies with uncertainty on an equal footing, that is, with analogous properties of term structure consistency. Thus, for the moment (until we get to the discussion of probability and translation operator semigroups), the reader may safely think of "discount functions" whenever we mention "linear operators," "time derivatives of discount functions" when we speak of "infinitesimal generators," and "term structure consistency" when "semigroup" properties are invoked.

The fundamental point of arbitrage theory is to demonstrate from first principles the existence of such nonnegative linear operators mapping future dividends into current prices. The first principles typically involve the formation of arbitrage portfolios and the assuming the absence of solutions to these. Now classical in this regard is Ross'[5] [19] one-period (two-date) description: Let

p_j = the price at date 0 of security j, $j = 1, 2, \cdots, J$
$a_{sj} = p_{sj} + d_{sj}$ = the payoff to security j at date 1 if the prevailing state then is s, $s = 1, 2, \cdots, S$, where p_{sj} and d_{sj} are the conditional price and dividend, respectively, at date 1.

Consequently, the no-arbitrage condition may be stated as

No Arbitrage, One Period. There exists no portfolio $\{x_j\}$ such that

$$\Sigma_j x_j p_j < 0$$

and

$$\Sigma_j x_j a_{sj} \geq 0 \quad \text{for all states} \quad s = 1, 2, \ldots, S \qquad \text{(NA-1P)}$$

where x_j is unconstrained.

[3] Use a function like ${}_tQ_\tau = e^{-r(\tau-t)}$ to see this.

[4] "Evolution system" is the more usual term in time-inhomogeneous environments, while "semigroup" is used in homogeneous ones; we have chosen to combine the terms to capture the sense of both.

[5] See also the precursor development by Beja [2].

Ross' conclusion (by appealing to Farkas' lemma [10]) is then that there must exist a nonnegative set of quantities $\{k_s\}$, *independent of j*, such that

$$p_j = \Sigma_s k_s a_{sj} \tag{R-1P}$$

if and only if (NA-1P) holds. This "representation" result essentially rationalizes the existence of an implicit (Arrow-Debreu-type) price structure $\{k_s\}$ even when markets are incomplete. The linear operator associated with this uncertainty world is then "$\Sigma_s k_s$." In other words, if each state-contingent future payoff of a security is "discounted" by the implicit price k_s and the sum is taken over all future states, the current security price results.

Subsequently, Rubinstein [21] and others[6] have formulated an infinite-period dividend-stream approach, followed by further work[7] to generalize the multiperiod approach to the existence of future spot markets and the possibility of arbitrage within these. Harrison and Kreps [14] also develop a multiperiod setting but go on to show that the implicit prices may be rescaled into a martingale distribution which will be associated with every arbitrage-free market.

Curiously, it seems that none of the multiperiod formulations have viewed no-arbitrage as a *continuing* feature of the economy, and so the corresponding semigroup property (or term structure consistency) which is so evident in certainty worlds was not uncovered in those efforts.

I. Finite Economies

The basic points are easiest to comprehend in a finite economy. A finite economy will be understood to possess a finite number of securities, a finite number of discrete trading dates, and a finite number of possible states at each date.[8] It is quite important that all "value-relevant" states be identified at each date, so that security prices and dividends will depend only upon the states. Moreover, we will combine all the possible states at each date into one big superset of states, for convenience. (Clearly, this can always be done in finite economies.) Call this set $\Omega = \{1, 2, \cdots, S\}$, where the number of states S is assumed finite.

As is shown elsewhere[9], the fundamental "representation" result of intertemporal no-arbitrage in finite economies is that there exists a set of linear operators (matrices) $\{_tK_\tau\}$ such that for security prices P_{jt} and dividends D_{jt} (both vectors), we have

$$P_{jt} = \textstyle\sum_{m=t+1}^{\tau} {}_tK_m D_{jm} + tK_\tau P_{j\tau} \tag{R-IT}$$

for all securities $j = 1, 2, \cdots, J$ and, in addition, the matrices $_tK_\tau$ possess the property that

$$_tK_\tau = {}_tK_{vv}K_\tau \quad \text{for all} \quad t \le v \le \tau,$$

[6] See, Cox and Ross [5].

[7] Garman [13].

[8] If the state space demands it, e.g., through the beliefs or tastes of agents as state descriptors, a finite number of agents may also be implied.

[9] Duffie and Garman [8].

i.e., the evolution semigroup property. This result is demonstrable through the repeated application of the one-period, two-period, etc., no-arbitrage problems and, importantly, articulating these all simultaneously. ${}_tK_\tau$ may be thought of as a matrix of Arrow-Debreu prices, where the rows represent states at the current date t, and the columns represent states at the future date τ. The prices P_{jt} and dividends D_{jt} may be thought of as column vectors of state-contingent values. The operator thus "discounts" future contingent dividends and (because future spot markets exist) future contingent prices, back to today's contingent (on the known current state) price. An example may serve to clarify.

Example 1: Suppose there are three dates, 0, 1, 2, with exactly two states available at each date. Assume further that only one security exists, and that its conditional price and dividend vectors are given as

$$P_0 = \begin{bmatrix} 300 \\ 240 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 250 \\ 210 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 200 \\ 200 \end{bmatrix}$$

and

$$D_1 = \begin{bmatrix} 200 \\ 90 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 100 \end{bmatrix}.$$

It is quickly verified that the set of positive implicit price matrices

$${}_0J_1 \equiv \begin{bmatrix} 0.20 & 0.70 \\ 0.40 & 0.20 \end{bmatrix}, \quad {}_1J_2 \equiv \begin{bmatrix} 0.80 & 0.30 \\ 0.60 & 0.30 \end{bmatrix}, \quad {}_0J_2 \equiv \begin{bmatrix} 0.31 & 0.45 \\ 0.11 & 0.40 \end{bmatrix}$$

will simultaneously solve all the representation equations $P_0 = {}_0J_1D_1 + {}_0J_1P_1$, $P_0 = {}_0J_1D_1 + {}_0J_2D_2 + {}_0J_2P_2$, and $P_1 = {}_1J_2D_2 + {}_1J_2P_2$. But it is also clear that $\{{}_0J_1, {}_1J_2, {}_0J_2\}$ do not constitute a semigroup, since ${}_0J_2 \neq {}_0J_1\, {}_1J_2$. Nevertheless, the existence and nonnegativity of these implicit price matrices assure the absence of arbitrage opportunities. Therefore, it is also guaranteed that at least one implicit price structure having the semigroup property exists, for instance

$${}_0K_1 = \begin{bmatrix} 0.40 & 0.40 \\ 0.20 & 0.50 \end{bmatrix}, \quad {}_1K_2 = \begin{bmatrix} 0.50 & 0.50 \\ 0.30 & 0.50 \end{bmatrix}, \quad {}_0K_2 = \begin{bmatrix} 0.32 & 0.40 \\ 0.25 & 0.35 \end{bmatrix}.$$

The latter matrices can be shown both to solve the representation equations and to possess the semigroup property, as can be quickly verified.

Example 1 illustrates a number of salient points. First, the market in this example is not "complete" since there are fewer securities than states at each date.[10] This gives rise to nonuniqueness in the implicit prices. Second, there may be numerous sets of implicit price matrices which simultaneously solve the representation equations, but only a subset of these will form semigroups. Yet it happens that we are guaranteed of the latter's existence.

There are a few corollaries to the basic semigroup result which are of some

[10] This is somewhat imprecise; see Duffie's [7] discussion of "martingale multiplicity" for a precise characterization of multiperiod market completeness.

interest. First, it may be shown that if arbitrage is available over a period of time spanning several trading dates, it is available in shorter periods; indeed, it will be available between adjacent dates. This means that it is sufficient to view no-arbitrage restrictions in a "myopic" fashion, i.e., that no-arbitrage between all adjacent dates assures no-arbitrage globally over longer periods. It also implies that "dynamic" trading strategies add nothing to the attainment of arbitrage profits: since all strategies are buy-and-hold between adjacent dates, there is nothing to be gained in considering longer-horizon strategies.[11]

II. Continuous Time

The semigroup results reported for finite economies in the previous section may be extended in a number of ways. First, the consideration of more general state spaces at each date may be undertaken; this involves the specification of appropriate measures on such spaces and the articulation of these from date to date (which was finessed in the finite economy via the state "superset" construct). Second, a (countably) infinite number of trading dates might be considered; this involves, in addition to some new difficulties in constructing the appropriate "superset" of states, the necessity of providing nonspeculation conditions. Finally, the trading dates may be expanded to the point where trading is continuously available, i.e., so-called "continuous time." Along with the foregoing difficulties, there is then the need to provide for continuity properties in the linear operators over time. This appears to devolve upon on whether values at one point in time are "nearly substitutable" for values at "nearby" points in time.[12] These matters are in various stages of resolution (and some may never have satisfactory resolution), and so in the interests of space I do not confront them in the present paper. Rather, I shall address the following questions: Assuming that continuous-time economies indeed possess the semigroup aspects which appear to be so comfortable and reasonable in finite economies, will their implications be consistent with the extant continuous-time literature of financial economics? Are there new implications which do not appear in that literature? The answer to both of these questions seems to be affirmative.

In the "continuous-time" economies, we shall now simply suppose that there exists an evolution semigroup of nonnegative (implicit price) operators which solve the representation Equation (R-IT) (with the summation replaced by an integral) for all continuous trading dates t and $\tau \geq t$. Interpreted in this fashion, the representation result (R-IT) is restated as

$$P_{jt} = {}_tK_\tau P_{j\tau} + \int_{u=t}^{\tau} {}_tK_u D_{ju}\, du \qquad \text{(R-IT}')$$

[11] Caveat: in continuous time, this result does *not* generally hold. See, e.g., Kreps' [16] "doubling strategy."

[12] A dollar received now is, potentially, entirely different in value from a dollar received a microsecond from now; indeed, they may be thought of as entirely different numeraires with no necessary connection. Yet most of us would see these as "close" substitutes. The precise characterization of such substitution is the heart of the issue.

for all securities $j = 1, 2, \cdots, J$ where now D_{ju} is construed as the conditional dividend *rate* and integration replaces the previous summation of dividend values. For convenience, define Δ_u as the "translation" operator, i.e., $\Delta_u f(t) \equiv f(t+u)$. A sequence of key manipulations then commences by rewriting (R-IT′) as

$$P_{jt} = {}_tK_\tau \Delta_{\tau-t} P_{jt} + \int_{u=t}^{\tau} {}_tK_u D_{ju}\, du.$$

Rearranging terms and dividing through by $(\tau - t)$, we have

$$0 = \frac{({}_tK_\tau \Delta_{\tau-t} - I)}{(\tau - t)} P_{jt} + \frac{1}{(\tau - t)} \int_{u=t}^{\tau} {}_tK_u D_{ju}\, du$$

$$= \frac{(\Delta_{\tau-t} - I)}{(\tau - t)} P_{jt} + \frac{({}_tK_\tau - I)}{(\tau - t)} \Delta_{\tau-t} P_{jt} + \frac{1}{(\tau - t)} \int_{u=t}^{\tau} {}_tK_u D_{ju}\, du.$$

Taking the limit as $\tau \rightarrow t$, we arrive at our central continuous-time result:

Differential Representation. If no arbitrage is permitted and the infinitesimal generator of ${}_tK_\tau$ exists, then we have the "differential representation"

$$\left(\frac{\partial}{\partial t} + \Psi_t\right) P_{jt} + D_{jt} = 0, \tag{R-D}$$

where Ψ_t is the infinitesimal generator of $\{{}_tK_\tau\}$.

To see why this is true, we note first that the translation operator $\{\Delta_{\tau-t}\}$ forms a semigroup, with infinitesimal generator $\partial/\partial t$, which is the limit of the first term above. The operators ${}_tK_\tau$ also form a semigroup, by virtue of our maintained supposition; the limit $\tau \rightarrow t$ of $({}_tK_\tau - I)/(\tau - t)$ is thus merely the definition of Ψ_t, the infinitesimal generator[13] of this latter semigroup. Clearly, the limit $\tau \rightarrow t$ of $\Delta_{\tau-t} = I$, the identity. And finally, the integral term involving the dividend can be shown[14] to reduce to simply the dividend rate D_{jt}.

The above differential representation result appears to be rather abstract in terms of its economic meaning. How, for example, should we compare it to the famous Black-Scholes/Merton ([4], [18]) differential equations which are so well known to financial economists? A significant point to note is that no probability distributions were ever mentioned in the development of (R-D), while the Black-Scholes/Merton differential equations depend crucially on these. This of course suggests that probability distributions have yet an important role to play in connecting the two.

I would maintain that, in securities market modelling, probability distributions play at least three separate roles. First, every agent in an economy under uncertainty presumably maintains certain probability beliefs regarding future states. These agent beliefs are distinguished by the fact that they will normally influence prices in the economy. A second role for probabilities is played by the

[13] Again, we have simply assumed this exists; [9] provides sufficient conditions from a mathematical viewpoint, but the economic meaning of these remains unclear at the current writing.

[14] See [9], Lemma 7, p. 619.

beliefs of an outside observer of the economy, e.g., a researcher whose models contain hypotheses regarding probability distributions; the Black-Scholes [4] assumption of lognormal price changes is a good example of this. Finally, there is a pervasive notion of an "objective" probability distribution associated with securities markets. In earlier literature (perhaps starting with Bachelier [1] in 1900), this was simply taken for granted, but more recent literature recognizes that if such a notion is to be taken seriously, it must be connected to equilibrium features through the tastes, beliefs, endowments, etc., of economic agents.[15] Certainly one high-water mark in this development would be Harrison and Kreps' [14] demonstration that certain martingales are associated with arbitrage-free markets. In one sense, the basic construction of an "objective" probability distribution is easy to arrive at: start with the implicit prices and rescale these so that they sum to 1.0. The latter quantities have all the properties of a probability distribution, since they represent a function of states which is nonnegative and which sums to unity. I shall term a distribution derived in such a fashion a "market distribution." One way of thinking about a market distribution is the following: suppose two pure-exchange capital markets economies exist side by side. The first economy is arbitrage-free but contains heterogeneous investors who are perhaps risk-averse. The second economy contains only risk-neutral homogeneous investors. If the same securities inhabiting both economies commanded the same prices, then the uniform beliefs of the second economy's agents would constitute a market distribution for the first economy. Note that the semigroup property applies to both economies, but in the second economy it appears primarily through a "Markovian" feature of the distribution.

Thus, our recharacterizing (R-D) in terms of probability distributions will take two paths. The first path is via the market distribution, and the second is via the beliefs of an observer. For the market distribution, it may be shown[16] through straightforward manipulations that (R-D) may be rewritten[17] as

$$\left\{\frac{\partial}{\partial t} + \Lambda_t - r\right\} P_{jt} + D_{jt} = 0 \tag{R-M}$$

where Λ_t is the infinitesimal generator of the market distribution and $r = r(s, t)$ is the instantaneous interest rate in state s at time t.

The alternative path is to adopt the observer's beliefs and use these directly in (R-D). It appears that not just any arbitrary belief is admissible, but instead these must be "compatible" with the market distribution inasmuch as they should be Markovian (to preserve the semigroup feature), and they must not assign zero probabilities to states that the market distribution asserts are possible (otherwise the observer will perceive arbitrage possibilities). Intuitively speaking, a "compatible" distribution can be reshaped by the introduction of a new factor which is extracted from the implicit prices in such a fashion that only the observer's

[15] See Rubinstein's [20] "consensus beliefs," for example.

[16] In [8].

[17] Intuitively, this is again the rescaling device: the implicit prices are "scaled up" to create a distribution with generator Λ_t and subtracting the instantaneous interest rate then "compensates" for such rescaling.

probability function remains. The result will be that (R-D) may be rewritten in terms of the observer's beliefs as

$$\left\{\frac{\partial}{\partial t} + \Phi_t\right\} (G_t P_{jt}) + D_{jt} = 0 \tag{R-O}$$

where Φ_t is the infinitesimal generator of the observer's (Markovian) beliefs and $G_t = g(s_t, t, s_\tau, \tau)$ is the new "reshaping" factor, which can be interpreted as the observer's imputed rate of market value substitution between current state s_t and date t and future states s_τ at date τ. (Note on notation: the generator is applied against the future state and date arguments, then evaluation of the result takes place by substituting the current state and date for the future ones, similar to an ordinary derivative calculation.)

Comparing the two approaches, we see that asserting knowledge of the market distribution through (R-M) is a much stronger assumption than presuming relative ignorance via (R-O). Generally speaking, any empirical content to the linear-operator methodology should rely on the weaker (R-O). The remainder of this paper deals with the practical application of the last two approaches. In particular, I shall now discuss a series of examples which illustrate not only that the usual continuous-time results can be reproduced, but also introduce some new developments as well.

First, what examples can be given for Φ_t? Under conditions of certainty, this generator is merely the state derivative, i.e., $\Phi_t \equiv \mu(s, t) \dfrac{\partial}{\partial s}$, where $s \equiv s_t$ is the (deterministic) state variable, and $\mu(s, t)$ is the "drift" in s. (When s_t is implied, we will often drop the subscript.) In a similar vein, diffusion processes in one state variable (again "s") possess the infinitesimal generator $\Phi_t = \dfrac{\sigma^2(s, t)}{2} \dfrac{\partial^2}{\partial s^2}$ (with $\sigma^2(s, t)$ the "volatility" of s). Infinitesimal generators exist for other processes, e.g., "jump" processes, and the like.[18]

Example 2: *Option Pricing Revisited.* Here, we rederive the basic option pricing model, slightly generalized from that of Black and Scholes [4]. The point of the analysis here is to show how the relative pricing of the option pricing model is related to the underlying absolute pricing of the three securities involved (stock, bond, and option). That is, we show the relative pricing relationships are also a consequence of applying (R-O) to the individual securities, without directly relying on a "hedge portfolio" derivation. For the example, let:

s = the stock price, the only state variable
$r(s, t)$ = the interest rate of a constant-price (riskless) asset, or "bond"
$p(s, t)$ = option (or other derivative asset) price.

The combined generator consisting of translation, certainty, and diffusion com-

[18] Dothan and Williams [6] also employ a form related to (R-O) for processes which are mixtures of "jump" (Poisson) and diffusion components. See [8] for other generators.

ponents (in that order) is then

$$\Phi \equiv \frac{\partial}{\partial t} + \mu(s, t)\frac{\partial}{\partial s} + \frac{\sigma^2(s, t)}{2}\frac{\partial^2}{\partial s^2},$$

where $\mu = \mu(s, t)$ is the "drift" and $\sigma^2 = \sigma^2(s, t)$ is the diffusion "volatility." Hence, we have three "absolute" pricing relationships which follow from the differential representation (R-O) applied to each of the three assets:

Bond: $\Phi G + r(s, t) = 0,$

Stock: $\Phi(Gs) = s\Phi G + \Phi s + \sigma^2(s, t)\dfrac{\partial G}{\partial s} = 0,$

Option: $\Phi(Gp) = p\Phi G + \Phi p + \sigma^2(s, t)\dfrac{\partial G}{\partial s}\dfrac{\partial p}{\partial s} = 0.$

Combining these three absolute pricing relationships in the relative amounts of $[-p + s\partial p/\partial s, -\partial p/\partial s, 1]$ eliminates all terms involving G (which are of course specific to the economy and observer in question), therefore yielding the relative pricing relationship

$$\Phi p - rp + [rs - \mu(s, t)]\frac{\partial p}{\partial s} = \frac{\partial p}{\partial t} + \frac{\sigma^2(s, t)}{2}\frac{\partial^2 p}{\partial s^2} - rp + rs\frac{\partial p}{\partial s} = 0.$$

This is a slightly generalized version of the Black-Scholes [4] equation, since here the riskless interest rate r may depend upon the stock price s, and the equation applies to all univariate diffusions, not just lognormal Brownian motion.

Example 2 demonstrates that certain well-known results are encompassed by (R-O); we now apply our methodology to derive some new results in the next two examples.

Basically, two types of stochastic processes have been examined in the literature of continuous-time finance: the diffusions and the simple jump processes. Example 2 provides a diffusion example, and it would be strightforward here to replicate the "jump" process valuation of securities using the methods herein. But are there any other significant processes? The answer is affirmative. Kunita and Watanabe [17] show in essence that all stochastic processes can be considered to be the sum of three types of components: a diffusion (continuous) component; a jump (Poisson) component; and a "discontinuous" (our term) component. Completeness demands that an appropriate theory of valuation should encompass all three types of processes. Since the first two processes are reasonably well-developed in the finance literature, we shall explore several generators of the "discontinuous" type to show that these are encompassed also. Our first illustration parallels the option-pricing development given in Example 2.

Example 3: *Option Valuation under a Gamma Process.* Consider again the case of a single state variable, s, which may be taken to be the price of a "stock". Assume that the increments of s are governed by a gamma process, that is, the conditional distribution of the state variable at time t_1 given its value at time t_0

is given by

$$f_{t_1}(s_1 \mid s_0, t_0, \vartheta) = \frac{e^{-\vartheta(s_1-s_0)}\vartheta^{(t_1-t_0)}(s_1 - s_0)^{(t_1-t_0-1)}}{\Gamma(t_1 - t_0)},$$

for $s_1 > s_0$, $t_1 > t_0$, $\vartheta > 0$ where ϑ is the "intensity" parameter, and Γ is the usual gamma function, i.e.,

$$\Gamma(z) \equiv \int_0^\infty e^{-w}w^{z-1}\,dw.$$

This process is one of the "discontinuous" processes mentioned above. It is nondecreasing and discontinuous[19] everywhere. Perhaps surprisingly, its infinitesimal generator involves an integral, i.e., for the gamma process we have (setting $\vartheta = 1$ for brevity):

$$\Phi u(s, t) \equiv \frac{\partial u}{\partial t} + \mu\frac{\partial u}{\partial s} + \int_0^\infty \frac{u(s-y, t) - u(s, t)}{y}\,e^{-y}\,dy.$$

Repeating the analysis of Example 2 for a bond, stock, and option, we have the following:

$$\text{Bond: } \Phi G + r = \frac{\partial G}{\partial t} + \mu\frac{\partial G}{\partial s} + \int_0^\infty \frac{G(s-y) - 1}{y}\,e^{-y}\,dy + r = 0,$$

$$\text{Stock: } \Phi(Gs) = -rs + \mu - \int_0^\infty G(s-y)e^{-y}\,dy = 0,$$

$$\text{Option: } \Phi(Gp) = \frac{\partial p}{\partial t} + \mu\frac{\partial p}{\partial s} - rp + \int_0^\infty \frac{G(s-y)[p(s-y, t) - p(s, t)]}{y}\,e^{-y}\,dy = 0.$$

Combining the above equations in the same fashion as Example 2, we have the integro-differential equation

$$\frac{\partial p}{\partial t} + \left(rs + \int_0^\infty G(s-y)e^{-y}\,dy\right)\frac{\partial p}{\partial s} - rp + \int_0^\infty \frac{G(s-y)[p(s-y, t) - p(s, t)]}{y}\,e^{-y}\,dy = 0,$$

which is the analogue of the Black-Scholes differential equation for gamma processes. Note that like the Black-Scholes development, the equation is independent of the "drift" term μ. However, unlike the Black-Scholes differential equation, it is impossible to eliminate the economy-specific term G. In other words, the bond and the stock do *not* span the option in a gamma-process world; the process moves "so fast" in an infinitesimal period that it is impossible to

[19] See [11], p. 321, for a necessary and sufficient condition to demonstrate this property.

perform the replication of the option through a portfolio of the other two securities.[20] (Cf. Kreps' [16] approach to "prices determined by arbitrage.")

The next example treats a discontinuous process of a class that was once quite popular in the finance literature.

Example 4: *Cauchy Stochastic Process.* The "stable Paretian" model for stock price changes enjoyed a number of years of popularity in the finance literature. Interestingly, little articulation between the stable Paretian model and modern continuous-time finance has taken place. This seems odd, because the Gauss-Wiener process upon which the modern continuous-time approach is based, is in fact a member of the stable Paretian class of probability distributions (with "characteristic exponent" $\alpha = 2$). Are there differential equations paralleling the Black-Scholes differential equation for non-Gaussian stable Paretian processes? At first blush, it might seem not; after all, the Black-Scholes differential equation was built upon a "hedging" argument, and the other stable Paretian processes appear to "move too fast" for a hedging argument to suitably apply. While it turns out that the latter statement is correct, (R-O) and (R-M) tell us that the existence of differential representations may not depend upon the "hedgeability" of the process. We now illustrate this in the case of the Cauchy process, which is one of the stable Paretian processes (with "characteristic exponent" $\alpha = 1$). We shall find that the relevant differential representation involves an elliptical partial differential equation, as opposed to the parabolic form that stems from the Black-Scholes analysis.

The generator associated with the Cauchy continuous-time process is (e.g., [11, p. 297]):

$$\Phi u(s, t) = \frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial s} + \int_0^\infty \frac{u(s-y, t) - u(s, t) + (\partial u/\partial s)\xi(y)}{y^2}\, dy$$

where

$$\xi(y) = \begin{cases} y & \text{if} \quad |y| \le 1 \\ 1 & \text{if} \quad y > 1 \\ -1 & \text{if} \quad y < -1 \end{cases}.$$

Instead of repeating the relative-pricing analysis of the previous examples, we take advantage of certain special properties to examine a simple valuation problem. Let $\Theta \equiv \dfrac{\partial^2}{\partial s^2}$ be the associated generator of the standardized normal process, ignoring the certainty terms. Let Ξ be the corresponding operator associated with the standardized Cauchy process, i.e. the last term of Φ defined above. Then in an operator sense, it can be shown[21] that $\Xi = -\sqrt{-\Theta}$, where the notation "$\sqrt{\ }$" means $\Xi^2 = -\Theta$, i.e. Ξ applied twice as an operator is equivalent to applying the operator $-\Theta$.

[20] As the integro-differential equation implies, an exception to this spanning property arises when the option is a linear function of the stock price, e.g., as when the "option" has strike price zero (a forward contract).

[21] See [15].

Now consider the following absolute valuation problem: an asset pays off \$1 at maturity T if a continuous-time Cauchy process s takes on a value between A and B (i.e., $A \leq s \leq B$), otherwise it expires worthless. What differential equation governs its valuation? We could repeat the analysis of Example 3 to show that relative pricing again fails, i.e., the value of this contingent claim is dependent upon G. As an alternative, however, we shall suppose that the Cauchy distribution is the market distribution in the sense of (R-M). In this case (R-M) implies that

$$\left\{\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial s} + \Xi - r\right\} V = 0$$

with the boundary condition

$$V(s, T) = \begin{cases} 1 & \text{if } A \leq s \leq B \\ 0 & \text{otherwise} \end{cases}$$

In operator terms, then, we have the equality

$$-\Xi = \sqrt{-\Theta} = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial s} - r$$

which implies that

$$-\Theta = \frac{\partial^2}{\partial t^2} + \mu^2 \frac{\partial^2}{\partial s^2} + 2\mu \frac{\partial^2}{\partial s \partial t} - 2 r\mu \frac{\partial}{\partial s} - 2r \frac{\partial}{\partial t} + r^2$$

also in operator terms, which means that price of this asset must obey the differential equation

$$(1 + \mu^2) \frac{\partial^2 V}{\partial s^2} + \frac{\partial^2 V}{\partial t^2} + 2\mu \frac{\partial^2 V}{\partial s \partial t} - 2r\mu \frac{\partial V}{\partial s} - 2r \frac{\partial V}{\partial t} + r^2 V = 0.$$

Note that this is an elliptical partial differential equation, whereas the standard Black-Scholes analysis leads to a parabolic partial differential equation. To illustrate a solution, we suppose now that the interest rate r is a constant and that (for simplicity) $\mu = 0$. We then compute the solution to the differential equation with the above boundary condition as

$$V(s, t) = \frac{e^{-r(T-t)}}{\pi} \left\{\arctan\left(\frac{B - s}{T - t}\right) - \arctan\left(\frac{A - s}{T - t}\right)\right\},$$

as may be directly verified by taking derivatives. This solution may be further validated by integrating the boundary condition with respect to the Cauchy distribution, since according to the terms of (R-M), assets will clearly take on the value of their expected payoffs.

The "jump processes" extant in the finance literature tend to be of the simple Poisson variety and do not take full advantage of certain lattice properties inherent in some proceses. A more general viewpoint must include at least the birth-and-death processes as also representative of typical continuous-time, discrete-state processes which may make transitions to multiple states from any given state. A simple illustration of these is given in the next example.

Example 5: *A Birth-and-Death Process.* In the Land of Oz, it is either rainy (state 1) or clear (state 2). Oz is a continuous-time economy, and transitions from rainy to clear weather have a Poisson intensity of 2 per year, while transitions from clear to rainy have a Poisson intensity of 3 per year. Consequently, the relevant infinitesimal generator is given[22] by the matrix

$$M = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}.$$

Let us again ignore the risk aspects of the economy by appealing to (R-M) and supposing that the given distribution is in fact the market distribution, and assuming further that the interest rate r in Oz is constant. Then we have the differential representation

$$\left\{\frac{\partial}{\partial t} + M - r\right\} P_{jt} + D_{jt} = 0.$$

Let us suppose that two assets exist: asset A has a dividend rate of \$10 per year while the weather is rainy, \$20 per year while it is clear, in perpetuity. Asset B matures at time T and pays off \$300 if the weather is rainy then and \$200 if it is clear; it pays no dividends. What are the values of these two securities?

Following standard matrix analysis (see e.g., [3, p. 165]), we first compute the matrix $e^{M\tau} = I + M\tau + M^2\tau^2/2! + M^3\tau^3/3! + \cdots$. In the present example, this is found[23] to be

$$e^{M\tau} = I + M(1 - e^{-5\tau})/5 = \begin{bmatrix} \frac{3}{5} + \frac{2}{5}e^{-5\tau} & \frac{2}{5} - \frac{2}{5}e^{-5\tau} \\ \frac{3}{5} - \frac{3}{5}e^{-5\tau} & \frac{2}{5} + \frac{3}{5}e^{-5\tau} \end{bmatrix}.$$

The valuation solution to the above differential equation is then computed for security A, assuming a nonspeculative condition in price formation, as (e.g. [3; p. 169]):

$$P_{At} = \begin{bmatrix} P_{A1} \\ P_{A2} \end{bmatrix}_t = \int_{u=0}^{\infty} e^{-ru} e^{Mu} \begin{bmatrix} 10 \\ 20 \end{bmatrix} du = \begin{bmatrix} \frac{14}{r} - \frac{4}{r+5} \\ \frac{14}{r} + \frac{6}{r+5} \end{bmatrix}$$

Note that the solution is stationary (i.e., does not depend upon t), as we would expect for a "perpetual." In a slightly different fashion (and without assuming a nonspeculation condition, since a terminal boundary condition exists), the value

[22] The off-diagonal elements of the generator are the transition intensities from the row state; the on-diagonal elements are the negative sum of all transition intensities out of that state.

[23] By the Cayley-Hamilton theorem, M obeys its own characteristic equation, giving rise to a recursive substitution for the powers of M in the matrix exponential series.

of asset B is given as

$$P_{Bt} = \begin{bmatrix} P_{B1} \\ P_{B2} \end{bmatrix}_t = e^{-r(T-t)} e^{M(T-t)} \begin{bmatrix} 300 \\ 200 \end{bmatrix} = \begin{bmatrix} 260e^{-r(T-t)} + 40e^{-(r+5)(T-t)} \\ 260e^{-r(T-t)} - 60e^{-(r+5)(T-t)} \end{bmatrix}.$$

While the Land of Oz might seem to be a rather frivolous example, it must be pointed out that it illustrates an important class of continuous-time, discrete-state processes that have not been adequately treated by the "jump processes" of the current literature to date. In particular, it provides the basic framework for integrating the literature on "market microstructure" into modern continuous-time finance. That is, like the Land of Oz, the market microstructure models[24] assume a birth-and-death process, with Poisson-distributed events involving stock price changes of $\pm 1/8$, $\pm 1/4$, etc. These microstructure models will therefore possess similar infinitesimal generators as their most fundamental descriptions, and thus can potentially be treated by the methods illustrated.

As a final illustration of the efficiencies introduced by the differential-generator approach, we provide a simple conceptual example, which nonetheless provides a novel insight.

Example 6: *A Short Question.*

Q. The diffusion generator involves a second-order derivative with respect to the state variable s, i.e., $\partial^2/\partial s^2$, the impact of which is proportional to the "volatility" parameter. Is there any stochastic process for which a third-order derivative will be descriptive (thus with impact proportional to a "skewness" parameter)?

A. No. $\partial^3/\partial s^3$ is the generator of no (appropriate) semigroup (see, e.g., [9, p. 656]). *This seems to suggest that "skewness is irrelevant in the small" for all continuous-time processes.* This result is a direct consequence of the semigroup property and seems to have no counterpart in utility-based valuation.

III. Conclusions

In this paper I have attempted to summarize certain aspects of semigroup valuation theory that arise as a consequence of intertemporal no-arbitrage restrictions. We find that in a finite economy, no-arbitrage as a local condition (i.e., between adjacent trading dates) implies the global absence of arbitrage. In this case, absence of arbitrage obtains if and only if there exists a set of nonnegative implicit prices exhibiting the semigroup property. Finally, when the semigroup property is assumed to characterize also the continuous-time economies, we find that the differential equations arising in the standard literature of continuous-time finance (e.g., of the Black-Scholes and related types) are also consequences of the differential representation (R-O), when the corresponding generators exist. Generators for the discontinuous processes are added to those already explored in the literature, i.e., the simple jump and diffusion processes, and present new implications. The approach ties together absolute and relative pricing in such economies, and provides insights on valuation questions.

[24] For example, [12].

REFERENCES

1. L. Bachelier. "Theory of Speculation (1900)." In P. Cootner (ed.), *The Random Character of Stock Market Prices.* Cambridge: MIT Press, 1964.
2. A. Beja, "Capital Markets with Delayed Learning." Ph.D. dissertation, Engineering Dept., Stanford, 1967.
3. R. Bellman. *Introduction to Matrix Algebra.* New York: McGraw-Hill, 1960.
4. F. Black and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* (May/June 1973), 637–54.
5. J. Cox and S. Ross. "The Valuation of Options for Alternative Stochastic Processes." *Journal of Financial Economics* 3 (January/March 1976), 145–66.
6. U. Dothan and J. Williams. "Valuation of Assets on a Markov State Space." *Economic Letters* 1 (1978), 163–66.
7. D. Duffie. "Stochastic Equilibra: Existence, Spanning Number, and the 'No Expected Gain from Trade' Hypothesis." Working paper, Stanford, 1984.
8. ———. and M. Garman. "Intertemporal Arbitrage and the Semigroup Valuation of Markov Securities." Notes, Berkeley, 1984.
9. N. Dunford and J. Schwartz. *Linear Operators.* New York: Interscience Publishers, 1958.
10. J. Farkas, "Ober die Theorie der Einfachen Ungleichungen." *Journal fur die Reine und Angewandte Mathematik* 124 (1902), 1–27.
11. W. Feller. *An Introduction to Probability Theory and Its Applications*, vol. 2. New York: Wiley, 1966.
12. M. Garman. "Market Microstructure." *Journal of Financial Economics* 3 (June 1976), 257–75.
13. ———. "A General Theory of Asset Valuation under Diffusion State Processes." Working paper no. 50, School of Business, University of California, Berkeley, 1976.
14. M. Harrison and D. Kreps. "Martingales and Arbitrage in Multiperiod Securities Markets." *Journal of Economic Theory* 20 (1979), 381–408.
15. E. Hille and R. Phillips. *Functional Analysis and Semigroups.* Providence, RI: American Mathematical Society, 1957.
16. D. Kreps. "Three Essays on Capital Markets." Institute for Mathematical Studies in the Social Sciences, Technical Report No. 261, Stanford, 1979.
17. H. Kunita and S. Watanabe. "On Square-Integrable Martingales." *Nagoya Mathematics Journal* 30 (1967), 209–45.
18. R. Merton. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (Spring 1973), 573–615.
19. S. Ross. "Return, Risk, and Arbitrage." In I. Friend and J. Bicksler (ed.) *Risk and Return in Finance.* Cambridge: Ballinger, 1976.
20. M. Rubinstein. "An Aggregation Theorem for Security Markets." *Journal of Financial Economics* 1 (October 1974), 225–44.
21. ———. "The Valuation of Uncertain Income Streams and the Pricing of Options. *Bell Journal of Economics* 7 (Autumn 1976), 407–25.

DISCUSSION

CHI-FU HUANG*: This paper is a typical example of how a seemingly simple observation can be pushed far to show interesting results. It is well known that in an Arrow-Debreu equilibrium with more than one date, the implicit prices for Arrow-Debreu securities over time are linked in a special way to prevent arbitrage. Professor Garman goes a step further to recognize that those implicit prices form an evolution semigroup. In the continuous time case, given that the infinitesimal generator of the implicit prices semigroup exists, he demonstrates that an arbitrage-free price system must satisfy a differential relation. This results in continuous time is the interesting part of this paper. Professor Garman fails,

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