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Debt and Taxes and Uncertainty

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ABSTRACT

With a graduated personal tax schedule, Miller showed that there could be an equilibrium debt supply for the corporate sector as a whole. In the presence of uncertainty there is also a unique debt/equity ratio for each individual firm, and this ratio is related to the firm's operational risk characteristics. However, if firms merge and spin off in response to tax incentives, the identity of firms is ambiguous and only the corporate sector is a meaningful construct. These arguments are developed in both discrete and continuous models that employ extensions of the arbitrage-free pricing theory.

IN HIS FAMOUS PAPER, "Debt and Taxes" (1977), Merton Miller [11] showed that the original Miller-Modigliani propositions on the irrelevance of capital structure were not necessarily overturned by considerations of taxation. Miller argued that with a broad enough span of progressivity in personal income taxation, the corporate tax advantage to the issuance of debt would be offset by the disadvantage to investors of receiving ordinary income on personal account. In equilibrium, the differential rates of return on equity and debt would be such that the marginal investor would be indifferent between purchasing debt instruments or equity. The margin would be determined by the total amount of debt issued by the corporate sector, but in a competitive world it would be unaffected by the debt issued by any one firm. As a consequence, the total supply of corporate debt would be determined in equilibrium, increasing to the point where aggregate corporate value was not enhanced by further debt financing, but any individual firm would be looking at prices set at the margin and would find itself indifferent between issuing debt or equity.

The purpose of this article is to extend Miller's argument by recognizing the differences between debt and equity when the environment is not certain. We will not be concerned with what might be called corner questions, i.e., with whether or not there is a marginal investor or an internal equilibrium. To do so would obscure the basic points we have to make. Instead, we will examine models that have internal equilibria and ordinary neoclassical marginal interpretations. In related work, De Angelo and Masulis [5, 6] have also extended Miller's analysis, but their primary concern was with the effects of a more detailed treatment of the tax code and with the impact of bankruptcy. We will deal with such matters only in passing; our central concern is with the impact of uncertainty.

Our main result is the finding that there is a relationship between the capital structure of the firm—as induced by tax effects—and ordinary (beta) measures

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of risk. This relationship is particularly surprising because we will find that firms have optimal internal financial mixes in the absence of any bankruptcy costs or any agency, asymmetric information or signaling effects. It is a consequence solely of the interplay between uncertainty and taxes. This link has both empirical and theoretical implications. By tying the corporate financial structure to observable characteristics of the firm, we have a tool for relating financial structure to product market structure. This permits us to use the constructs of asset pricing, e.g., beta measures, to address such questions as whether, say, high-tech firms have high debt/equity ratios.

Unfortunately, though, the link we construct is not as strong as we would want; it really only throws the irrelevance question back a bit, but it does not eliminate it. If firms have nontax-related reasons for the particular forms they take, then our result holds, but if they merge and spinoff in an effort to minimize taxes or if intermediaries can accomplish the task for them, then Miller's irrelevance proposition will be reinstated. Furthermore, in some models, there are simple and appropriate modifications of the tax system itself that lead back to irrelevance. In particular, real indexing of taxation breaks the link.

The first section of the paper describes the tools and notation we will use and develops the basic model of pricing. The second section examines the optimal financial choice, while the third develops a diffusion model and uses it to study the cross-sectional relations. The fourth section describes some general limitations and caveats to the analysis and the fifth and final section briefly summarizes and concludes the paper with some suggestions for future research.

I. The Model

We will begin with a one-period world. In this world, equilibrium is reached today, at time 0, and cash flows are received one-period later, at time 1. (In Section III, we will examine a version with a richer dynamic structure.) A firm in such a world has cash flows at time 1 labelled as $x(\theta)$ or, simply, x. The cash flows, x, depend upon the state of nature, θ, that will be realized at time 1. Thus, x is a random variable, and θ is an element of an appropriately defined probability space.

Debt issued by the firm carries a face value of F. In the absence of taxes, the total payoff to the debt at time 1 will be given by

$$\min(F, x), \tag{1}$$

and the equity will receive the residual,

$$\max(x - F, 0), \tag{2}$$

which is equivalent to a call on the cash flows (or next period value) with an exercise price equal to the face value of the debt.

If we let D denote the current, time 0, market value of the debt and S the current value of the equity, then the market value of the firm, V, is given by

$$V \equiv D + S. \tag{3}$$

It is well known that in the absence of arbitrage there exists a positive linear valuation operator which values securities. Ross [12, 13] proved that such an operator exists and subsequent work, notably by Harrison and Kreps [9], Hansen and Richard [8], and Dybvig et al. [7] develop the martingale representation of this operator. Letting E^* denote the operator (and ignoring spanning issues) we have

$$\begin{aligned} V &\equiv D + S \\ &= E^*(\min(F, x) + \max(x - F, 0)) \\ &= E^*(x), \end{aligned} \tag{4}$$

independent of the amount of debt issued, F. This verifies the Miller-Modigliani theorem in this simple world.

To see what happens when taxes are introduced we will assume that the tax code takes a form that is simple enough to permit analysis, but rich enough to capture the essential elements.

We will first assume that taxes are collected from both firms and individuals on gross returns. At the corporate level, taxes will be paid on the returns to equity, and all debt payments will be deductible. Thus, if τ_c denotes the average tax rate of the firm, then the after tax returns to equity will be given by

$$(1 - \tau_c)\max(x - F, 0). \tag{5}$$

Notice that no tax credits are given in bankruptcy. Notice, too, that we have avoided the complication of separating out the interest and principal repayment portions of the total debt payment, F. We will assume a constant corporate tax rate, τ_c. In general, though, we would expect firms—like individuals—to be inframarginal as well as marginal.

As for individual taxation, we will assume that equity returns and debt returns are taxed differentially with the latter being treated as ordinary income. For the moment, we can defer a more detailed discussion of exactly how this is done and move directly to the firm valuation. In Ross [14], it was shown that in the absence of arbitrage and with taxes there would exist a tax adjusted "before (personal) tax" margingale operator, E^*, or that could be used to value securities. In general, this operator would assign different values to cash flows with different tax characteristics. Thus, in the absence of arbitrage, the value of one firm will be given by

$$\begin{aligned} V(F) &\equiv D + S \\ &= E_0^*(\min(x, F)) + E_S^*((1 - \tau_c)\max(x - F, 0)), \end{aligned} \tag{6}$$

where E_0^* is the martingale operator used to value ordinary income receipts and E_S^* is the martingale operator used to value equity receipts. In general, though, these operators are not the same and they cannot be combined, as we did previously, to eliminate the influence of the debt level, F. To make this possibility explicit we have written the value, V, as $V(F)$.

Under a wide variety of circumstances, the valuation operator takes the form of tax-adjusted Arrow-Debreu state contingent claims prices, and this permits us to rewrite (6) in a more familiar form. Letting $Q_{g\theta}$ be the cumulative pricing distribution function for capital gains, i.e., for equity returns, and $Q_{0\theta}$ be the price distribution function for ordinary income, (6) becomes

$$\begin{aligned} V(F) &= E_0^*(\min(x, F)) + E_g^*((1 - \tau_c)\max(x - F, 0)) \\ &= \int \min(x, F)\, dQ_{0\theta} + \int (1 - \tau_c)\max(x - F, 0)\, dQ_{g\theta} \\ &= \int_{\{\theta\,|\,x(\theta)\leq F\}} x(\theta)\, dQ_{0\theta} + F \int_{\{\theta\,|\,x(\theta)>F\}} dQ_{0\theta} \\ &\quad + \int_{\{\theta\,|\,x(\theta)>F\}} (1 - \tau_c)(x(\theta) - F)\, dQ_{g\theta}. \end{aligned} \tag{7}$$

If the state space is finite then (7) has the simple form

$$V(F) = \textstyle\sum_{\{\theta|x(\theta)\leq F\}} x(\theta)\, q_{0\theta} + F \sum_{\{\theta|x(\theta)>F\}}^{q_{0\theta}} + \sum_{\{\theta|x(\theta)>F\}} (1 - \tau_c)(x(\theta) - F)q_{g\theta}, \tag{8}$$

where $q_{0\theta}$ and $q_{g\theta}$ are, respectively, the before personal tax prices of pure contingent claims to ordinary income and to capital gains.

Looking at the valuation formula (7) or its specialization to (8), it is at least plausible that the debt issued, F, will influence value, V. Indeed, since every set of positive before tax prices, $q_{0\theta}$ and $q_{g\theta}$, is consistent with (some) equilibrium (see Ross [14]), we need only assume some particular set of such prices and examine (7) or (8) to see how V changes with F in such a world. To say more about the qualitative properties of this dependence, though, we will have to specify the individual tax code in more detail. This will enable us to derive a particularly easy to interpret set of before tax state contingent prices.

To do so we will consider a very simple consumer problem with complete state markets for both ordinary income and capital gains. We are doing this primarily to aid our interpretation of the state prices. Such a model is not necessary for their existence—indeed they exist simply from the absence of arbitrage—rather, it is useful for interpreting our results. This distinction is an important one since we do not have to assume the completeness of markets for state contingent claims, let alone claims partitioned by tax characteristics.

Let c_θ denote consumption of the single consumption good in state θ, and let y_θ and g_θ denote the purchases of contingent claims entitling the holder to one unit of ordinary income or capital gains, respectively, in state θ. Let $U(\cdot)$ denote the von Neumann-Morgenstern utility function of the representative individual under consideration and let Π_θ denote the probability distribution of the states. (We will assume that the functions we use are smooth enough to validate our manipulations with them.)

We will represent the tax code by a function $T(\cdot)$ which is state independent, and maps adjusted gross income into taxes. Taxable income is defined as ordinary income minus a capital gain deduction of $1 - \gamma \in [0, 1]$ of capital gains. Thus, taxes in state θ are given by

$$T(y_\theta + \gamma g_\theta). \tag{9}$$

We will assume that $(\forall z)T(z) < z$, i.e., taxes are not confiscatory, and also that taxes are monotone, i.e., $T' > 0$.

The basic problem facing the consumer/investor in this world can be stated as

$$\max \int U(c_\theta)\, d\Pi_\theta$$

subject to

$$\int y_\theta\, dQ_{0\theta} + \int g_\theta\, dQ_{g\theta} = w$$

and

$$c_\theta = y_\theta + g_\theta - T(y_\theta + \gamma g_\theta), \tag{10}$$

where w is the individual's wealth.

Differentiating with respect to y_θ and g_θ, yields the first order conditions

$$\pi_\theta U'(c_\theta)(1 - \gamma T') = \lambda q_{g\theta},$$

and

$$\pi_\theta U'(c_\theta)(1 - T') = \lambda q_{0\theta}, \tag{11}$$

where λ is the marginal utility of wealth and π_θ, $q_{g\theta}$, and $q_{0\theta}$ are the densities of their respective distributions at θ.

Eliminating λ, we have

$$q_{g\theta} = \left[\frac{1 - \gamma T'}{1 - T'}\right] q_{0\theta}. \tag{12}$$

For any individual consumer facing prices $q_{g\theta}$ and $q_{0\theta}$, the interior conditions may not be satisfied, but, of course, given Inada conditions ($U'(0) = \infty$, $U'(\infty) = 0$) in the economy with an individual representative agent, prices will adjust at equilibrium to assure that (11) holds. Given relative prices

$$\frac{q_g}{q_0}.$$

though, depending on adjusted gross income an individual will have

$$\psi \equiv \frac{1 - \gamma T'}{1 - T'} \gtreqless \frac{q_{g\theta}}{q_{0\theta}}.$$

With progressive taxes, i.e., T' increasing in taxable income and common utility functions, presumably there will be a cutoff wealth level, W_θ, associated with each state such that above that level

$$\psi \equiv \frac{1 - \gamma T'}{1 - T'} > \frac{q_{g\theta}}{q_{0\theta}},$$

and the individual will purchase only equity, with the converse result below W.

For every state, too, all individuals cannot be on one side of the margin since, then, prices would adjust, although it is possible for some to be on the margin and some off.

It follows, then, that there is a marginal investor for each state of nature and in these more complicated worlds we can interpret ψ as depending on the adjusted gross income of the marginal investor with the understanding that the identity of this investor may change with the state of nature.

Another point is worth mentioning. It is not the case that arbitrage would force

$$\psi \equiv 1,$$

nor does equilibrium require this. Of course, if firms can respond to differences in state prices by issuing state contingent claims then, in general, we must have

$$q_{0\theta} = (1 - \tau_c) q_{g\theta}.$$

De Angelo and Masulis [5] present this case, and while it is of theoretical interest, there are a variety of reasons why firms will not issue complete state contingent claims. For one thing, such claims require excessive monitoring expenditures. Perhaps less obviously and more interestingly, the tax code is not well defined for all such hybrid instruments. For example, how the IRS would treat state-dependent securities which collectively made up a classical bond instrument is not at all clear. Certainly the corporation's treatment as debt payments and, therefore, deductible of claims which began "if such and such a state occurs then holder is entitled to. . . ." would not go unchallenged. We will return to a somewhat more interesting version of this argument in Section IV.

The relative price of capital gains in terms of ordinary income,

$$\psi \equiv \frac{q_{g\theta}}{q_{0\theta}} = \frac{1 - \gamma T'}{1 - T'}, \tag{13}$$

will play an important role in our analysis. Notice that ψ is an increasing function of the marginal tax rate, T', and that

$$\psi \geq 1, \tag{14}$$

i.e., capital gains are no less valuable than ordinary income in each state.

In a general equilibrium, the taxes will be given back to consumers—assuming no government investment—in a lump sum redistribution, and we will assume that individual actions are unaffected by such transfers. Individuals take the lump sum transfers as part of their wealth, but we will assume that they ignore any responsiveness of the transfers to their own actions. In a representative agent world, the transfer will equal the taxes and we could take c_θ to be aggregate consumption. Of course, the breakdown into the y_θ and g_θ components is a function of the supply of these forms of income and that is the issue of interest in this paper.

To simplify (7) we can project all of the state-dependent variables onto the price standard—c in the above example. Let

$$G(x \mid c) \equiv \text{conditional distribution of } x \text{ given } c \tag{15}$$

and

$$dQ_c \equiv E\{dQ_0 \mid c\}. \tag{16}$$

This permits us to rewrite (7) as

$$V(F) = \int \int_0^F x \, dG(x \mid c) \, dQ_c + F \int \int_F^\infty dG(x \mid c) \, dQ_c + \int \int_F^\infty (1 - \tau_c)(x - F)\psi \, dG(x \mid c) \, dQ_c, \tag{17}$$

where we have used the definition of ψ in (13) to eliminate dQ_g, and where we have also made use of the following assumption.

Assumption 1 (monotonicity): Taxable income is a monotonely increasing function of the pricing standard, with $T' \leq 1$.

In Assumption 1, we are using consumption as the pricing standard, but if we were to use prices themselves then, of course, the monotonicity would be reversed. Assumption 1 will be satisfied except in unusual circumstances. For example, suppose that there are only two firms, A and B, in the economy. Firm A is bankrupt and B is not. Now, suppose that a small increase in aggregate consumption is accompanied by a big increase in B's cash flows and an offsetting decline in A's flow. Since taxable income will be increased by only a fraction $\gamma < 1$ of the increase in B's capital gains while it decreases by the full amount of A's loss, the net effect can be that taxable income declines.

When the bulk of the cash flows respond positively to increases in consumption this unusual case cannot occur and it is what we are ruling out by our assumption.

II. The Debt/Equity Choice

To prevent arbitrage, the debt level of the firm must be set so as to maximize the value of the firm. This ignores spanning issues, but they are of limited interest in this analysis. Of course, it does not really matter whether the firm or its managers actually choose the optimum debt level themselves or whether the firm is refinanced by outsiders to accomplish this. We are also avoiding the somewhat ill-specified question of what the value will be if the firm chooses a debt level. F, which does not maximize V as defined by (17). Might not the value be bid up to the optimum anyway in anticipation of a refinancing? Depending on how we model the bidding and trading process it might well, in which case the choice of F will be, at least at the early stage of such a process, irrelevant. But, we will assume that to prevent tax losses the debt level is optimally set by the end of the process of trade. Presumably, too, equilibrium is achieved quite quickly and this is the value we would actually observe.

To analyze how $V(F)$ depends upon F, we begin by differentiating (17) with respect to F,

$$V'(F) = \int \int_F^\infty \left[1 - \frac{(1 - \tau_c)(1 - \gamma T')}{(1 - T')} \right] dG(x \mid c) \, dQ_c. \tag{18}$$

The integrand in (18) has been written out in full to make it easily recognizable as the criterion function derived by Miller [11]. Miller showed that at the margin, in the absence of uncertainty,

$$\varphi \equiv 1 - \frac{(1-\tau_c)(1-\gamma T')}{(1-T)} = 0, \tag{19}$$

and the marginal investor finds after tax ordinary income of $1 - T'$ equivalent to after tax equity income of $(1-\tau_c)(1-\gamma T')$. Before tax, then, the investor is indifferent between $(1-\tau_c)$ units of equity income or 1 unit of debt income and, therefore, the firm is indifferent as well. With uncertainty, though, the firm is not at the margin in all states.

We will assume that the tax structure is progressive in the sense that marginal rates increase.

Assumption 2 (progressivity): T is convex, i.e., $T'' \geq 0$.

It follows that in states associated with low levels of consumption—or whatever pricing standard is used—debt is relatively more valuable than equity, and, conversely, when consumption is high, equity income becomes relatively more valuable.

Proceeding analytically, from progressivity the integrand, φ, is a declining function of c. For c, sufficiently high φ will be negative provided that the marginal bracket can rise high enough. Letting c^* be defined as the lowest value of c which satisfies,

$$\varphi(c^*) = 1 - \frac{(1-\tau_c)(1-\gamma T')}{(1-T')} = 0, \tag{20}$$

we have

$$\varphi \gtreqless 0 \quad \text{as} \quad c \lesseqgtr c^*. \tag{21}$$

If c^* fails to exist, then φ is always positive and the optimal debt level is infinite. Even in this case, though, and generally, $V'(F)$ approaches zero as F increases and,

$$V'(\infty) = 0. \tag{22}$$

From (17) we also have that

$$V(0) = \int \int_0^\infty (1-\tau_c)\psi \, dG(x \mid c) \, dQ_c, \tag{23}$$

and

$$V(\infty) = \int \int_0^\infty x \, dG(x \mid c) \, dQ_c, \tag{24}$$

the values of the all equity and all debt firms, respectively.

Alternatively, we could have $c^* = 0$, in which case φ is never positive and even at the lowest brackets, equity income is preferred. If the marginal bracket for low levels falls to zero, though, we have

$$\varphi = 1 - (1 - \tau_c) = \tau_c > 0. \tag{25}$$

The interesting case to analyze, then, is one where the range of the tax code is broad enough to permit $c^* > 0$ to exist and where φ is positive for c below c^* and negative for c greater than c^*.

From (18), since φ does not depend on the firm's cash flow, x, but, rather, is an aggregate economic variable we have

$$\begin{aligned} V'(F) &= \int \varphi \int_F^\infty dG(x \mid c)\, dQ_c \\ &= \int \varphi \operatorname{Prob}(x > F \mid c)\, dQ_c \\ &= \int_F^\infty \left(\int \varphi(c) g(x \mid c)\, dQ_c \right) dx, \end{aligned} \tag{26}$$

where $g(x \mid c)$ is the density function of $G(x \mid c)$.

In analyzing the firm's choice of a financial structure we will distinguish between two special cases.

Case 1: The "positive beta" case.

In this case, we assume that the conditional density function, $g(x \mid c)$ has the monotone likelihood ratio property (MLRP), i.e.,

$$c' > c'' \quad \text{implies that} \quad \frac{g(x \mid c')}{g(x \mid c'')} \tag{27}$$

is monotone increasing in x.

Milgrom [10] analyzes the MLRP is some detail. It is equivalent to the property that the larger c is, the more likely x is to be larger, i.e., the conditional distribution of x given c rises stochastically as c increases. This means that $(\forall K)$

$$\int_0^K dG(x \mid c') < \int_0^K dG(x \mid c''). \tag{28}$$

Conversely, the MLRP also implies that for $x' > x''$

$$\frac{g(x' \mid c)}{g(x'' \mid c)}$$

is monotone increasing in c, and that the same monotonic properties hold for $g(c \mid x)$.

We refer to this condition as the positive beta case because it is sufficient to ensure that the regression coefficient of x on c,

$$\beta_{xc} \equiv \frac{\operatorname{cov}(x, c)}{\operatorname{var} c} > 0. \tag{29}$$

Continuing with the analysis, from (26) we have

$$V''(F) = -\int \varphi(c)g(F \mid c)\, dQ_c. \tag{30}$$

This enables us to verify the following result.

LEMMA 1. *If $V''(\hat{F}) > 0$, then $(\forall F > \hat{F})V''(F) > 0$, and if $V''(\hat{F}) < 0$, then $(\forall F < \hat{F})V''(F) < 0$.*

Proof: Suppose that $V''(\hat{F}) > 0$. From the MLRP $(\forall F > \hat{F})$

$$\frac{g(F \mid c)}{g(\hat{F} \mid c)} \equiv h(c) \tag{31}$$

rises monotonically in c. Hence, from (30),

$$\begin{aligned} V''(F) &= -\int \varphi(c)g(F \mid c)\, dQ_c \\ &= -\int \varphi(c)g(\hat{F} \mid c)h(c)\, dQ_c \\ &= -\int_0^{c^*} \varphi(c)g(\hat{F} \mid c)h(c)\, dQ_c - \int_{c^*}^{\infty} \varphi(c)g(\hat{F} \mid c)h(c)\, dQ_c \\ &> h(c^*)\left(-\int_0^{\infty} \varphi(c)g(\hat{F} \mid c)\, dQ_c\right) \\ &> 0. \end{aligned} \tag{32}$$

A similar argument proves the remainder of the lemma. Q.E.D.

This lemma has two immediate corollaries.

THEOREM 1. *The value function, V, has one of the following three forms*:

(i) *V is everywhere concave and rises monotonically,*
(ii) *V is everywhere convex and falls monotonically, and*
(iii) *$(\exists \hat{F} > 0)$ such that V is concave on $[0, \hat{F}]$, convex on $(\hat{F}, \infty)$ and falls monotonically on the convex segment.*

Figure 1 illustrates the third form.

Proof: Follows immediately from the lemma and (22), $V'(\infty) = 0$. Q.E.D.

It follows, then, that we have shown that a unique optimal F^* exists. In form (i) $F^* = \infty$, in form (ii) $F^* = 0$, and in form (iii) F^* is either 0 or it is characterized by

$$V'(F^*) = 0, \tag{33}$$

for some $F^* < \infty$, as is illustrated in Figure 1.

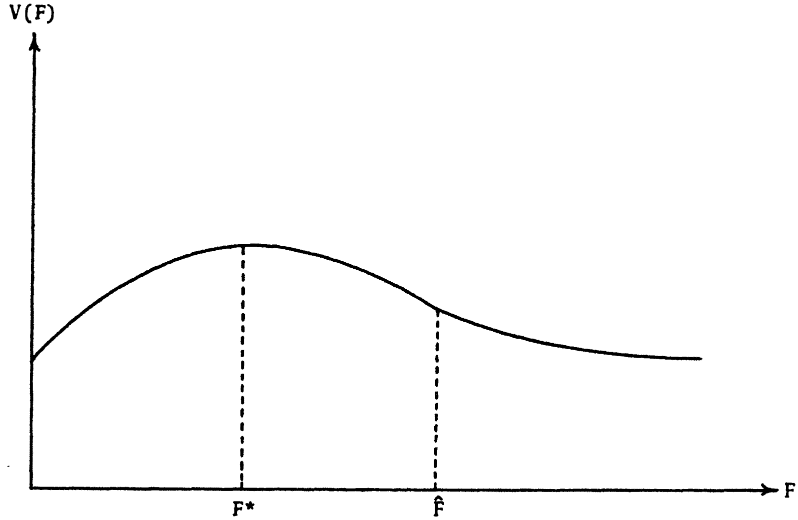

Figure 1. Positive Beta

Case 2: The "negative beta" case.

This case can be dealt with in precise analogy with the positive beta case. We simply reverse the monotonicity of (27), which implies that

$$h(x) \equiv \frac{g(x \mid c')}{g(x \mid c'')} \tag{34}$$

declines monotonically in x for $c' > c''$. It is easily verified that

$$\beta_{xc} < 0. \tag{35}$$

The conditions of the lemma are now reversed and we have

LEMMA 2. *If $V''(\hat{F}) < 0$, then $(\forall F > \hat{F})V''(F) < 0$, and if $V''(\hat{F}) > 0$, then $(\forall F < \hat{F})V''(F) > 0$.*

Proof: See the proof of Lemma 1, and use the monotonicity of (34). Q.E.D.

Figure 2 illustrates the general shape of $V(F)$ in the negative beta case and the following characterizes the possibilities.

THEOREM 2. *The value function, V, has one of the following three forms:*

(i) *V is everywhere concave and rise monotonically,*
(ii) *V is everywhere convex and falls monotonically, or*
(iii) *$(\exists \hat{F} > 0)$ such that V is convex on $[0, \hat{F}]$, concave on $(\hat{F}, \infty)$ and rises monotonically on the concave segment.*

Proof: Immediate. Q.E.D.

We see, then, that the negative beta case must be characterized by an extreme

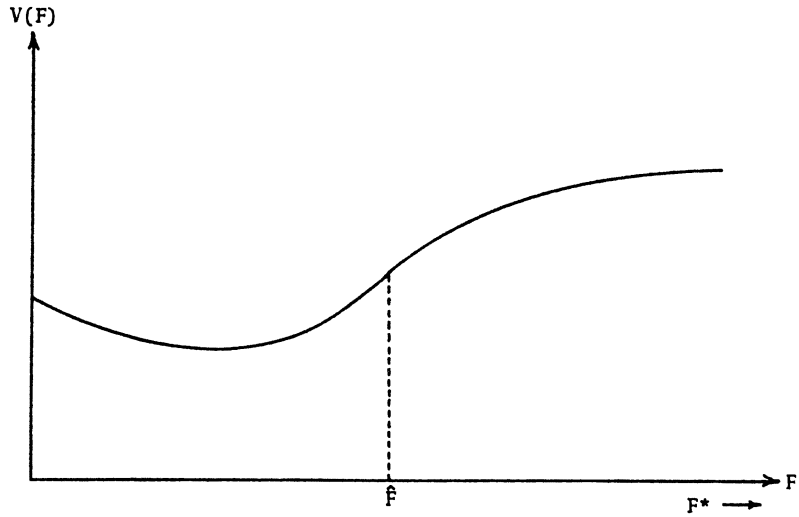

Figure 2. Negative Beta

solution, i.e., either

$$F^* = 0$$

or

$$F^* = \infty.$$

Concluding this section, we have shown that as a general matter, the financial choice is not indifferent and that, in the positive beta case, an internal structure can be optimal. This occurs because firms face valuations for debt and equity income that favor equity income in "good" states of nature. Consequently, depending on the firm's responsiveness to (priced) general economic conditions, a firm that issues too little debt will find itself giving excessive equity income to its stockholders in moderate states where they place too little value on it. If debt is too high, though, they will give inadequate amounts of equity income, unnecessarily excluding states where it would be well valued by the market. In the next section, we will see how the optimal choice of a financial structure depends upon the firm's responsiveness or beta.

III. Cross-Sectional Relationships in a Diffusion Model

To examine the comparative statics properties of the debt choice, we will switch our analytical gears. While it would be possible to analyze the properties of the above one-period problem directly, it is extremely tedious to do so. Furthermore, it is useful to put the analysis into a more dynamic context so as to make the linkage with observables more direct.

To accomplish this, we first observe that the earlier section is a perfectly valid analysis of a firm which is making a debt decision for a fixed time period, T, where, in continuous time, x denotes the time T value of its assets. We will assume the same tax code and that there are no interim cash flows in the period $[0, T)$.

In continuous time, it is known that there is a continuous time martingale pricing standard, q, with the property that q follows a diffusion,

$$\frac{dq}{q} = -rdt + \sigma_q \, dz_q, \tag{36}$$

(where r is the riskless (constant) interest rate)[1] and such that the value of any asset with payoff of p_t is given by

$$p_o = E\{q_t p_t\}; \quad q_0 = 1. \tag{37}$$

For the analysis of this see Dybvig et al. [7].

With taxation, the modification of the analysis is obvious (see Ross [14]). Letting interest income be ordinary and taxed at a rate T' where T' is independent on q,

$$T' = T'(q), \tag{38}$$

we can think of q as the pricing standard for ordinary income and

$$g = \psi q, \tag{39}$$

as the standard for capital gains income.

In our example above,

$$q_t = \frac{e^{-\delta t}(1 - T'(c_t))u'(c_t)}{u'(c_0)}, \tag{40}$$

where δ is a subjective rate of time preference. (This is clearly monotonely declining in c_t which verifies that T' is monotonely falling in q and that, by our previous analysis, ψ also falls monotonely in q.)

Using these results, we can write a dynamic form for the value of debt and equity as

$$D_t = E\{q_T \min(x_T, F)\}$$

and

$$S_t = E\{q_T \psi_T \max(x_T - F, 0)\}. \tag{41}$$

Writing these out in integral form with $h(q)$ denoting the densities of q yields

$$D_t = \int \int_0^F xg(x \mid q) \, dxqh(q) \, dq + F \int \int_F^\infty g(x \mid q) \, dxqh(q) \, dq$$

and

$$S_t = \int \int_F^\infty (1 - \tau_c)(x - F)g(x \mid q) \, dx\psi(q)qh(q) \, dq, \tag{42}$$

which are formally identical to Equation (17).

While we could analyze (42) directly, a more interesting approach is to return to the martingale analysis of (36) as modified by (40) to explicitly derive the local equations which describe the changes in the value for debt and equity.

[1] The constancy of the interest rate would generally have to be forced by the technology.

We will assume for purposes of analysis that all bond returns are taxed as ordinary income and that all expected returns, including those on stocks, are also taxed at ordinary rates. In this sense, it is as though the firm had adopted a dividend policy of paying out what would have been the deterministic portion of capital gains. The stochastic portion of capital gains, however, is taxed at the favorable capital gains rates. Finally, at the termination date, T, the payment on bonds is untaxed and the payment on equity is taxed at the corporate rate, τ_c.

To obtain explicit solutions, we will assume that the firm's terminal cash flow at time T is derived from the equation

$$\frac{dx}{x} = \mu\, dt + \sigma\, dz. \tag{43}$$

Under the assumptions, it is easy to see that the return of any asset must satisfy

$$(1 - T')E(dp) + (1 - T')c\, dt - (1 - T')rp\, dt = -E(dn\, dp), \tag{44}$$

where p is the asset's price, c is its cash flow, and n is the pricing standard. (For a different approach to the derivation of valuation equations with taxes, see Constantinides [1, 2] and Constantinides and Ingersoll [3].) The change in n, dn, is given by dq, from (40), for bonds and dg from (39) for stocks.

It is important to notice that while at any time t,

$$q_t = 1 - T'(c_t), \tag{45}$$

is the relative price of taxable ordinary income in terms of numeraire (after tax) consumption, the change in q for valuation purposes is given by (40) and includes the change in marginal utility as well as in tax rates.

Using (39) and (40) and taking Ito differentials, we have that the stochastic changes in the relevant pricing standards are given by

$$d\tilde{q} = \left[(1 - T')\frac{u''}{u'} - T''\right]d\tilde{c},$$

and

$$d\tilde{g} = \left[(1 - \gamma T')\frac{u''}{u'} - \gamma T''\right]d\tilde{c}. \tag{46}$$

To parametrize this system in the simplest possible fashion, we will make the following assumptions.

Assumption 3: The covariance between cash flow and the pricing standard for ordinary income is constant, i.e.,

$$E\left\{\frac{d\tilde{q}}{q}\, d\tilde{z}\right\} = \text{constant}. \tag{47}$$

This assumption is useful because it permits us to eliminate state variables from the analysis which would otherwise be needed to describe the development of the covariances. The next assumption serves a similar purpose.

Assumption 4: The covariance between cash flow and the pricing standard for capital gains is proportional to ψ^{-1}, i.e.,

$$E\left\{\frac{d\tilde{g}}{g}\,d\tilde{z}\right\} = \text{constant}\cdot\psi^{-1}. \tag{48}$$

This latter assumption, then, implies that

$$E\left\{\frac{d\tilde{g}}{q}\,d\tilde{z}\right\} = \text{constant}. \tag{49}$$

From (46), Assumptions 3 and 4 establish a parametric relationship between the tax code and the utility function, i.e.,

$$\frac{(1 - \gamma T')u''/u' - \gamma T''}{(1 - T')u''/u' - T''} \equiv k, \text{ a constant}, \tag{50}$$

which, integrating, tells us that

$$T' = \left[\frac{1 - k}{\gamma - k}\right] + \frac{b}{u'} \tag{51}$$

where b is an integration constant. Notice, too, that

$$k > 0 \tag{52}$$

and that from (46)

$$\frac{d\tilde{g}}{q} = k\,\frac{d\tilde{q}}{q}. \tag{53}$$

From (44), we now know that x is the only state variable for the system of equations governing the valuation changes for debt and equity. Letting β denote the cash flow beta of the firm,

$$\beta \equiv -\frac{\operatorname{cov}\left(\dfrac{d\tilde{q}}{q}\;\sigma\,d\tilde{z}\right)}{\operatorname{var}\left(\dfrac{d\tilde{q}}{q}\right)}$$

$$\equiv -\frac{\sigma\operatorname{cov}\left(\dfrac{d\tilde{q}}{q}\;d\tilde{z}\right)}{\sigma_q^2} \tag{54}$$

and assuming that q is also lognormal we have the following result.[2]

THEOREM 3. *The following equations characterize debt, D, and equity, S.*

Debt

$$\tfrac{1}{2}\sigma^2x^2D_{xx} + (\mu - \beta\sigma_q^2)xD_x - D_\tau - rD = 0$$

$$D(x, 0) = \min(x, F), \tag{55}$$

[2] The reader may be worried that the system is overparametrized at this stage. However, since the consumption process is still unspecified, it is easy to see that this is not yet a problem.

and

Equity

$$\tfrac{1}{2}\sigma^2x^2S_{xx} + (\mu - k\beta\sigma_q^2)xS_x - S_\tau - rS = 0$$
$$S(x, 0) = (1 - \tau_c)\max(x - F, 0), \tag{56}$$

where τ is the time to receipt of the cash flow, x_T.

Proof: Follows immediately from (44) and Assumptions 3 and 4. Q.E.D.

Since these equations have been designed to be variations on simple diffusions (see Cox and Ross [4]), their solutions are variants of a well-known option formula. Let

$$C \equiv c(\mu, x, F, \tau, \sigma^2, r)$$
$$\equiv e^{-r\tau}\{xe^{\mu\tau}N(d_+) - FN(d_-)\}, \tag{57}$$

where $N(\cdot)$ is the cumulative standard normal distribution function, and

$$d_+ \equiv \frac{\ln\dfrac{x}{F} + \left(\mu + \dfrac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}},$$

and

$$d_- \equiv d_+ - \sigma\sqrt{\tau}. \tag{58}$$

THEOREM 4. *The respective solutions to* (55) *and* (56) *are given by,*

$$D(x, \tau) = xe^{(\mu_D - r)\tau} - c(\mu_D)$$

and

$$S(x, \tau) = (1 - \tau_c)c(\mu_S), \tag{59}$$

where

$$\mu_D \equiv \mu - \beta\sigma_q^2,$$

and

$$\mu_S \equiv \mu - k\beta\sigma_q^2, \tag{60}$$

and where we have suppressed undifferentiated variables in this notation for c.

Proof: Follows directly from (55), (56), and (57). Q.E.D.

The positive beta case, $\beta > 0$, corresponds to our usual intuition about firms whose returns are positively correlated with aggregate consumption, and it is the one we will consider. Despite the different appearance of the model developed in this section, it is actually just a specific parameterization of the general analysis of Sections I and II. As a consequence, it is straightforward to verify that Theorem 1 applies and that if $k < 1$, there can be a determinate internal optimal debt level, F^*. We will analyze this case.

THEOREM 5. *The optimal debt level satisfies*

$$V'(F^*) = e^{-r\tau}\{N(d_-(\mu_D)) - (1-\tau_c)N(d_-(\mu_S))\} = 0, \tag{61}$$

with the attendant second-order condition,

$$V''(F^*) = e^{-r\tau}\frac{1}{\sigma\sqrt{\tau}}\{(1-\tau_c)n(d_-(\mu_S)) - n(d_-(\mu_D))\} < 0, \tag{62}$$

where $n(\cdot)$ is the unit normal density.

Proof: Simply differentiate (59). Q.E.D.

The first-order condition, (61), captures our original intuition describing how the optimal debt level is set. At the margin, an increase in F raises the value of debt by $N(d_-(\mu_D))$, the tax adjusted probability of paying off the debt. The loss to equity is the other side of this, but it is computed using a different tax adjustment for expected returns, $\mu_S > \mu_D$, which is what permits a nontrivial internal solution.

If $k > 1$, then from (61), $V'(F) > 0$ for all F, and an all debt firm is optimal. In this case, the tax code is not sufficiently progressive to confer any tax advantage to equity financing. The tax advantage to equity in this model comes from the difference between the change in its marginal capital gains tax and that on debt, $(1-\gamma)T''$, which acts to offset its higher after tax beta. Interestingly, from (50) we can see that the condition $k < 1$ is equivalent to having the elasticity of taxation exceed risk aversion,

$$\frac{CT''}{T'} > -\frac{Cu''}{u'}. \tag{63}$$

Of course, the negative beta case will reverse this intuition.

We now move to our basic comparative statics results.

THEOREM 6. *The optimal debt level, F^*, is proportional to x, i.e.,*

$$\frac{x}{F^*}\frac{\partial F^*}{\partial x} = 1, \tag{64}$$

and independent of the interest rate, r, and exponential in μ, i.e.,

$$\frac{1}{F^*}\frac{\partial F^*}{\partial \mu} = \tau. \tag{65}$$

Proof: Follows from (61) directly. Q.E.D.

The independence of F^* and r may seem surprising, but keep in mind that μ is the return on a cash flow project and not on a capital asset. In the latter case, as for an option, μ would depend on r and that portion of Theorem 6 need no longer hold.

Lastly we can relate F^* to beta.

THEOREM 7. *In a cross-section of firms with the same total variance, σ^2, those* *with higher (positive) cash flow betas will have lower debt levels, i.e.,*

$$\left.\frac{\partial F^*}{\partial \beta}\right|_{\sigma^2} < 0. \tag{66}$$

Proof: Differentiating (61) and holding σ^2 constant—which means that the residual variability,

$$\sigma^2 - \beta^2\sigma_q^2,$$

must pick up the slack—yields

$$\left.\frac{\partial V'}{\partial \beta}\right|_{\sigma^2} = e^{-r\tau}\left\{n(d_-(\mu_D))\left[-\frac{1}{c\sqrt{\tau}}\sigma_q^2\tau\right] - (1-\tau_c)n(d_-(\mu_S))\left[-\frac{1}{\sigma\sqrt{\tau}}\sigma_q^2 k\tau\right]\right\}$$

$$= e^{-r\tau}\frac{\sigma_q^2\sqrt{\tau}}{\sigma}\{k(1-\tau_c)n(d_-(\mu_S)) - n(d_-(\mu_D))\}$$

$$< 0, \tag{67}$$

by the second-order condition, (62). Since $V''(F) < 0$.

$$\left.\frac{\partial F^*}{\partial \beta}\right|_{\sigma^2} = [-V''(F^*)]^{-1}\left.\frac{\partial V'}{\partial \beta}\right|_{\sigma^2} < 0. \quad \text{Q.E.D.} \tag{68}$$

The intuition underlying this result should be clear. Increasing beta lowers the marginal contribution to value of additional debt because it raises the expected return and, therefore, lowers the value of debt. It has a similar effect on equity, but it is mitigated by the capital gains offset. Hence, at the margin, debt must fall to accommodate its additional disadvantage relative to equity.

Further results are possible, but they remain to be explored. In particular, we would like to directly relate debt/equity ratios to ordinary equity betas,

$$\frac{xS_x}{S}\beta. \tag{69}$$

This would facilitate empirical testing. Nevertheless, we have succeeded in our initial goal of relating the choice of financial structure at the firm level to the characteristics of the firm's earnings stream.

IV. Robustness and Some Caveats

The above results obviously depend on our assumptions concerning taxation. But one particular adjustment for taxation is worth mentioning. The results hinge sensitively on the use of the financial structure to position the firm with respect to changes in marginal tax rates. An unanticipated increase in consumption, for example, is assumed to move the economy into a regime of higher marginal tax rates. Suppose, though, that progressivity in the tax regime is only relative and that the rates themselves are indexed against "bracket creep." In such circumstances, the impact on the firm will depend on the movement of investors across brackets, rather than on the movement in aggregate consumption. In a representative economy, there are no such distributional effects, and marginal tax rates will be fixed. In the above analysis, this means that T' will be fixed.

This brings us back, then, in the world of fixed rates, to Miller's analysis. In effect, by indexing the code the impact of uncertainty upon financial choice has been eliminated. In the absence of such complete real as well as nominal indexing, though, the effects we have examined will still play a role. This will be particularly true to the extent to which endogenous changes in the code itself are contemplated and firms position themselves in anticipation of such events.

Somewhat more interestingly, though, we are also assuming that firms have motives for taking the form we have given them other than simply tax considerations. In particular, if firms can merge and spin off in an arbitrary fashion, then tax considerations will again drive them to a form of Miller's world of indeterminancy at the individual firm level.

To see this suppose that there are two firms with cash flows of x_1 and x_2, and respective optimal debt levels of F_1 and F_2. Suppose that in some state of nature with capital gains more valuable than ordinary income, i.e., $(1 - \tau_c)q_{c\theta} > q_{0\theta}$, firm 1 is bankrupt and firm 2 is still paying its debt off. By transferring the income of firm 1 in this state to firm 2 we can increase their joint value, $V_1 + V_2$. In general, the corporate sector will solve the problem

$$\max_{\{F1, \cdots, Fn\}} \textstyle\sum_i V(x_i, F_i)$$
$$\text{subject to } \textstyle\sum_i x_i(\theta) = x(\theta), \tag{70}$$

the aggregate (before tax) earnings. Since, from (7), V is linear in x, the solution will be either a hyperplane of indifference or a corner. In this regime, the corporate sector has the ability to create pure state contingent claims. The only equilibrium, then, will be one in which each firm is always bankrupt or never bankrupt.

THEOREM 8. *If firms can form and combine in any fashion subject only to the constraint that*

$$\textstyle\sum_i x_i(\theta) = x(\theta), \tag{71}$$

and if taxes are the only consideration, then the equilibrium will be equivalent to one in which firms offer pure tax-contingent claims. Each firm can now be assumed to be always bankrupt or have zero debt.

Proof: Suppose that at the equilibrium for the corporate sector firm 1 has debt level F_1 and firm 2 has F_2. If the firms are not simultaneously bankrupt or solvent in all of the same states, then there will be a state, θ, where, say, firm 1 is bankrupt and firm 2 is not. In this state,

$$(1 - \tau_c)q_{c\theta} \gtreqless q_{0\theta}. \tag{72}$$

If $(1 - \tau_c)q_{c\theta} > q_{o\theta}$, then transferring $x_1(\theta)$ to firm 2 will increase their joint value and if $(1 - \tau_c)q_{c\theta} < q_{0\theta}$, then transferring $x_2(\theta)$ to firm 1 will increase their joint value. Thus, the only equilibrium is where all equity income is realized in states for which $(1 - \tau_c)q_{c\theta} > q_{0\theta}$ and all debt income is realized in states for which

$(1 - \tau_c)q_{c\theta} < q_{0\theta}$. As a consequence, each firm will be either an all equity or an all debt firm. Q.E.D.

Depending on the structure of the market, and, in particular, whether short sales are permitted and what rules prevail on intercorporate offers the state contingent prices, $(1 - \tau_c)q_{c\theta}$ and $q_{0\theta}$ may or may not be equalized. But, even if they are not, in this world the financial structures are determinate but the firms themselves are not. In essence, then, Miller's indeterminancy returns since only the corporate sector as a whole can be identified. In particular, two firms which are all equity could combine or not with no impact on the equilibrium.

V. Conclusions

Using the force of taxation in a world of uncertainty we have linked the cash flows generated by the firm to its choice of a financial structure. We have also shown that such a structure can be determinate at the firm level and not just for the corporate sector as a whole. But, this result was not without exception; it took the industrial structure as given rather than as dependent on taxation itself. Perhaps, the most immediate task for future research, though, is the need to extend the analysis so as to establish a direct link amongst purely financial variables such as, for example, debt/equity ratios and systematic and nonsystematic equity risk measures.

Some other interesting extensions would be to inframarginal firms. It is clear that Miller's analysis can be extended to such a case with the margin defined by the marginal firm as well as the marginal investor. Inframarginal firms in a world of certainty would issue all debt or all equity, but in a world of uncertainty this need no longer be true. It remains to explore the robustness of such results to the considerations raised in Section IV, such as possibilities for redefining firms and the effects of financial intermediaries. It would also be worthwhile to consider the impact of uncertain changes in the code itself. To some extent, the results of this paper may be interpreted as applying to such problems.

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DISCUSSION

GEORGE M. CONSTANTINIDES*: Ross' reexamination of Miller's [3] leverage irrelevance proposition is both economically interesting and technically important. Ross summarizes his contribution as follows: "Our main result is the finding that there is a relationship between the capital structure of the firm—as induced by tax effects—and ordinary (beta) measures of risk. This relationship is particularly surprising because we will find that firms have optimal internal financial mixes in the absence of any bankruptcy costs or any agency, asymmetric information or signalling effects."

Under completeness in both the equity-taxed and debt-taxed markets, De Angelo and Masulis [1] prove that each firm has a unique interior optimal debt-equity ratio if there are leverage-related costs. One source of leverage-related costs is the assumed inability by the firm to sell its tax deductions in excess of earnings in the bankrupt states. We consider these costs as leverage-related because both the potential tax deductions are higher and the probability of bankruptcy is higher, the higher the debt-equity ratio.

Although Ross does not assume completeness in both equity-taxed and debt-taxed markets, he does assume the above leverage-related costs. Therefore, his finding that firms have optimal internal financial mixes should not come as a surprise. Later on, Ross correctly points out that if firms can merge and spin off in an arbitrary fashion, the leverage-irrelevance proposition is restored. Mergers and spinoffs effectively make tax credits marketable and the earlier leverage-related costs disappear.

Theorem 7, the main result, states: "In a cross-section of firms with the same total variance, those with higher (positive) cash flow betas will have lower debt levels." The intuition is simple: two firms with the same distribution of cash flows (but not perfectly correlated) will have the same expected tax credits and, therefore, the same expected bankruptcy costs. However, the present value of the expected bankruptcy cost will be higher for the high-beta firm because these costs are incurred in the states that the marginal utility of consumption is higher than in the low-beta firm. Therefore, the high-beta firm will choose a lower debt-equity ratio.

Whereas the intuition of Theorem 7 is simple, the formal proof is technically challenging. Ross' primary contribution lies in proving this technical result by using the methodology developed in the companion paper by Dybvig et al. [2].

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