THE JOURNAL OF FINANCE • VOL. XL, NO. 2 • JUNE 1985

On Determination of Stochastic Dominance Optimal Sets

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ABSTRACT

Applying Fishburn's [4] conditions for convex stochastic dominance, exact linear programming algorithms are proposed and implemented for assigning discrete return distributions into the first- and second-order stochastic dominance optimal sets. For third-order stochastic dominance, a superconvex stochastic dominance approach is defined which allows classification of choice elements into superdominated, mixed, and superoptimal sets. For a choice set of 896 security returns treated previously in the literature, 454, 25, and 13 distributions are in the first-, second-, and third-order convex stochastic dominance optimal sets, respectively. These optimal sets compare with admissible first-, second-, and third-order stochastic dominance sets of 682, 35, and 19 distributions, respectively.

The applicability of superconvex stochastic dominance for continuous distributions defined over a bounded interval is then shown. The difficulties in identifying the elements of the superdominated set for distributions defined over the entire real line are demonstrated in the determination of the dominated choices for a set of normally distributed mutual fund returns previously examined by Meyer [9]. Specifically, we find that the dominated set determined by Meyer is too large.

FINANCIAL DECISION MAKING UNDER uncertainty boils down to the following two elements:

(i) characterization of the choice set of securities by a joint probability distribution of returns, and
(ii) a preference ordering that ranks the above alternatives by a utility function defined over the probability distribution characterizing the choice set.

Simply stated, decision making under uncertainty may be viewed as ranking alternative probability distributions of returns, based on a consistent set of preferences.

A significant breakthrough for decision making was the development of the von Neumann-Morgenstern expected utility paradigm. This choice criteria posits that the decision maker chooses the strategy which maximizes the expected

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utility of return given the nature of the return distributions and the functional form of the utility function.

In a general form, no specification is made about the return distribution, and the observed discrete distribution is posited to represent the underlying distribution. Simultaneously, the utility function is postulated in terms of observable behavior of individuals. As additional restrictions are placed on the decision maker's utility function, a smaller and smaller number of individuals are embraced by the results derived.

Generally accepted observable behavior has led to the following classes of utility functions, $u(\cdot)$:

(i) nonsatiation axiom: $u' > 0$
(ii) risk aversion: $u' > 0$, $u'' < 0$
(iii) decreasing absolute risk aversion: $u' > 0$, $u'' < 0$, $(-u''/u')' \leq 0$

These mathematical constraints and their interpretation are described and well-accepted in the finance literature.

Starting with return distributions as choice elements and classes of utility functions characterized by observable behavior, various authors have derived stochastic dominance (SD) admissible sets and inadmissible sets. The inadmissible set is the subset of the choice set that contains elements which are pairwise dominated by another particular distribution in the choice set for the hypothesized class of utility functions. Elements of the choice set which are not in the inadmissible set constitute the admissible set for this class (e.g., Quirk and Saposnik [11], Fishburn [3], Hadar and Russell [6, 7], Hanoch and Levy [8], Whitmore [13], and Bawa [1]). However, some members of the admissible set may never be chosen by any individual having a utility function within the hypothesized class. Instead, the optimal set, which is a subset of the admissible set, is the set of alternatives which *will* be chosen by some individual with a utility function in the hypothesized class. In turn, the dominated set is the set of choice elements which will not be chosen by any individual with a utility function in this class.

Fishburn [4, 5] was the first to determine conditions for identifying members of the optimal and dominated choice sets. These conditions he labelled "convex stochastic dominance" (CSD). The notion of CSD can be used to eliminate from further consideration alternatives that are SD-admissible in the usual sense but that are not optimal in the sense of CSD—being dominated by a convex combination of other alternatives. Alternatives which are dominated by a convex combination of other alternatives are classified to be in the dominated set. For the given choice set, any element of the dominated set will never be chosen by any investor having utility functions in the relevant stochastic dominance class. In turn, the optimal set is defined as the subset of the choice set which cannot be dominated by a convex combination of the other elements of the choice set. For the given choice set, every element of the optimal set will be chosen by some investor having utility functions in the relevant stochastic dominance class. A strength of Fishburn's results is his proof that a distribution which is dominated by a mixture of other alternatives will never be chosen by any individual manifesting utility functions within the class which lead to the respective dominance rule (i.e., nonsatiation and first-order stochastic dominance, nonsatiation and risk aversion, and second-order stochastic dominance).

In this work, we define a procedure which will identify elements of first-, second-, and third-order stochastic dominance choice sets as dominated or optimal. Given a choice set made up of discrete distributions under first- and second-order convex stochastic dominance, we show that classification of a particular distribution into either the optimal or dominated set can be reduced to a simple linear program. Determination of optimal sets is a semi-infinite programming problem for continuous distributions and for discrete distributions with third-order dominance. Nonetheless, we show that elements of the dominated and optimal sets can be identifed by a linear programming-based methodology. These topics are discussed in Section I.

In Section II, we apply these techniques to a particular data set examined by Bawa, Lindenberg, and Rafsky [2] in determination of the first-, second-, and third-order admissible sets. In Section III, the problem of determining stochastic dominance optimal sets for continuous distributions is introduced. The straightforward extension of the superconvex stochastic dominance procedures for continuous distributions defined over a bounded interval is motivated. Identification of the elements of the superoptimal, mixed, and superdominated sets for this class of distributions is discussed.

Finally, in the same section, i.e., Section III, the difficulty apparent in applying our superconvex stochastic dominance procedures for distributions defined over an unbounded interval is motivated by the case of the normal distribution. For normally distributed choice alternatives, the elements of the mean-variance admissible set can only be assigned to the mixed or superoptimal set by our approach. Our results also point out that the optimal and dominated sets obtained by Meyer's [9] solution to this problem are not correct. Concluding remarks are in Section IV.

I. Procedures for Identifying Discrete Distribution Optimal Sets

A. Notation and Preliminaries

Adopting the notation of Bawa [1], let the uncertain prospects be characterized by random variables x_i, $i = 1, 2, \ldots, n + 1$, with known right continuous probability distribution functions $F_i(\cdot)$ defined over a closed interval R given by $[a, b]$, $a < b$.

Let the following progressively restrictive set of utility functions $u(\cdot)$ describe the decision maker's preferences. The utility functions are defined over the space R of realizations of a random variable x:

$$U_1 = \{u(x) \mid u(x) \quad \text{is finite,} \quad u'(x) > 0, \quad \text{for all} \quad x \in R\},$$

$$U_2 = \{u(x) \mid u(x) \in U_1,\ u''(x) < 0, \quad \text{for all} \quad x \in R\},$$

$$U_3 = \{u(x) \mid u(x) \in U_2,\ u'''(x) > 0, \quad \text{for all} \quad x \in R\}.$$

It may be noted that U_3 includes the class of utility functions defining decreasing absolute risk aversion. We have the following well-known stochastic dominance theorems.

THEOREM 1 (First-Order Stochastic Dominance, FSD). *For any two cumulative distributions F_i and F_j, F_i is (strictly) preferred to F_j for all utility functions in U_1 if and only if*

$$F_i(x) \leq F_j(x)\ \forall\ x \in R \quad (\text{and} < \text{for some } x \in R) \tag{1}$$

THEOREM 2 (Second-Order Stochastic Dominance, SSD). *For any two cumulative distributions F_i and F_j, F_i is (strictly) preferred to F_j for all utility functions in U_2 if and only if*

$$\int_a^x F_i(t)\,dt \leq \int_a^x F_j(t)\,dt\ \forall\ x \in R \quad (\text{and} < \text{for some } x \in R) \tag{2}$$

THEOREM 3 (Third-Order Stochastic Dominance, TSD). *For any two cumulative distributions F_i and F_j, F_i is (strictly) preferred to F_j for all utility functions in U_3 if and only if*

(i) $\mu_i \geq \mu_j$

(ii) $$\int_a^x \int_a^y F_i(t)\,dtdy$$

$$\leq \int_a^x \int_a^y F_j(t)\,dtdy \quad \forall\ x \in R \quad (\text{and} < \text{for some } x \in R) \tag{3}$$

There are two equivalent representations of the stochastic dominance rules for the class of utility functions denoted by U_1, U_2, and U_3. One representation, as described above, is in terms of the original distribution function of the random variable and the finite integrals of these functions.

The alternative representation is in terms of lower partial moments of the probability distribution functions. The equivalence of these two approaches was described by Bawa [1]. This representation of the lower partial moment which will be used in Section I.*C* is:

$$L_{mi}(x) = \int_a^x (x - y)^m\, dF_i(y) \tag{4}$$

Integration by parts yields the following, which shows the equivalence of the two approaches.

Relation	Criterion for
$L_{0i}(x) = F_i(x)$	FSD
$L_{1i}(x) = \int_a^x F_i(y)\,dy$	SSD
$L_{2i}(x) = 2\int_a^x \int_a^y F_i(t)\,dtdy$	TSD (proportional)

We now turn to the problem of determining convex stochastic dominance optimal sets.

B. Convex Stochastic Dominance and Superconvex Stochastic Dominance

The notion of stochastic dominance as it applies to simple mixtures or a convex combination of probability distributions is called "convex stochastic dominance" or CSD. Fishburn [4, 5] was the first to introduce the concept of convex stochastic dominance. Illustrating for three distribution functions, namely F, G, and H, he proved that if a convex combination of F and G dominates H, then H is dominated by one of the distributions, F *or* G, for all individuals with utility functions in that class. The notion of CSD can be used to eliminate from further consideration alternatives that are SD-admissible in the usual sense, but that are not optimal in the sense of CSD, being dominated by a mixture of other alternatives.

Now, let $\lambda = (\lambda_1, \lambda_2 \cdots \lambda_n)$, $\lambda \in \Lambda_n$ with $\lambda_i \geq 0$, $i = 1, 2, \cdots n$ and $\sum_{i=1}^{n} \lambda_i = 1$. Then following Fishburn [4], we have the convex generalizations of Theorems 1 to 3.[1]

THEOREM 4 (Convex First-Order Stochastic Dominance, CFSD). *For the set of cumulative distributions, F_i, $i = 1, 2, \cdots, n + 1$, one of the distributions, F_i, $i = 1, 2, \cdots, n$ is (strictly) preferred to F_{n+1} for all utility functions in U_1 if and only if there exists a $\lambda \in \Lambda_n$ such that*

$$\sum_{i=1}^{n} \lambda_i F_i(x) \leq F_{n+1}(x) \quad \forall\, x \in R \quad (\textit{and} < \textit{for some } x \in R) \tag{5}$$

THEOREM 5 (Convex Second-Order Stochastic Dominance, CSSD). *For the set of cumulative distributions F_i, $i = 1, 2, \cdots, n + 1$, one of the distributions F_i, $i = 1, 2, \cdots n$ is (strictly) preferred to F_{n+1} for all utility functions in U_2 if and only if there exists a $\lambda \in \Lambda_n$ such that*

$$\sum_{i=1}^{n} \lambda_i \int_a^x F_i(t)\, dt \leq \int_a^x F_{n+1}(t)\, dt \quad \forall\, x \in R \quad (\textit{and} < \textit{for some } x \in R) \tag{6}$$

THEOREM 6 (Convex Third-Order Stochastic Dominance, CTSD). *For the set of cumulative distributions F_i, $i = 1, 2, \cdots, n + 1$, one of the distributions F_i, $i = 1, 2, \cdots, n$ is (strictly) preferred to F_{n+1} for all utility functions in U_3 if and only if there exists a $\lambda \in \Lambda_n$ such that*

$$\text{(i)}\quad \sum_{i=1}^{n} \lambda_i \mu_i \geq \left(\sum_{i=1}^{n} \lambda_i\right) \mu_{n+1}$$

$$\text{(ii)}\quad \sum_{i=1}^{n} \lambda_i \int_a^x \int_a^y F_i(t)\, dtdy$$

$$\leq \int_a^x \int_a^y F_{n+1}(t)\, dtdy \quad \forall\, x \in R \quad (\textit{and} < \textit{for some } x \in R) \tag{7}$$

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n(1 + \varepsilon)$ with $\lambda_i \geq 0$, $i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} \lambda_i = 1 + \varepsilon$ where $\varepsilon > 0$ is given. Then we define superconvex first (second- and third-) order stochastic dominance [SCFSD (SCSSD or SCTSD)] to hold if (5) (inequal-

[1] Proofs of Theorems 4 and 5 are in Fishburn [4]. Identification of the conditions for CTSD follow these proofs directly (Fishburn [5]).

ity (6) or (7)) holds for some $\lambda \in \Lambda_n(1+\varepsilon)$. Note that the definition of SCFSD (SCSSD or SCTSD) assumes that $\varepsilon > 0$ is specified. Clearly SCFSD (SCSSD or SCTSD) implies CFSD (CSSD or CTSD). This follows from the fact that if $\lambda_i^* \geq 0$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \lambda_i^* = 1 + \varepsilon$ satisfies (5) (inequality (6) or (7)), then clearly $\bar{\lambda}_i = \lambda_i^*/(1+\varepsilon) \geq 0$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \bar{\lambda}_i = 1$ satisfies (5) (inequality (6) or (7)).

C. Discrete Distributions

Let the i^{th} discrete probability function $p_i(\cdot)$ be defined at points $a_{i1}, a_{i2}, \cdots, a_{iv_i}$, so that $p_i(a_{ij}) = p_{ij}$ for $j = 1, 2, \cdots, v_i$. In other words, $p_i(\cdot)$ represents the probability function associated with the uncertain prospect characterized by the random variable x_i. Suppose we have $n + 1$ discrete probability functions corresponding to $i = 1, 2, \cdots, n + 1$. Let $D_i = \{a_{ij}, j = 1, 2, \cdots, v_i\}$. Let $D = \cup_{i=1}^{n+1} D_i$ and denote by d_i, $i = 1, 2, \cdots, h$ the elements of D, where $h = |D|$. Throughout this section, we are interested in verifying whether (5) or (6) or (7) holds, i.e., whether the random variable x_{n+1} is convex stochastically dominated. Consequently, if $\lambda_i > 0$ for any i, $1 \leq i \leq n$, then $a_{i1} \geq a_{n+1,1}$. Hence, we can, without any loss in generality, assume that $d_1 = a_{n+1,1}$.

For the i^{th} probability function, the discrete lower partial moment of order m corresponding to the definition in (4) is given by

$$L_{mi}(t) = \textstyle\sum_{j\,|\,a_{ij}\leq t} (t - a_{ij})^m p_{ij} \quad \text{for} \quad d_1 \leq t \leq d_h$$

$$\text{for} \quad m = 0, \qquad \text{we have} \quad L_{0i}(t) = F_i(t)$$

$$\text{for} \quad m = 1, \qquad \text{we have} \quad L_{1i}(t) = tF_i(t) - \textstyle\sum_{j\,|\,a_{ij}\leq t} a_{ij}p_{ij} \tag{8}$$

Thus $L_{0i}(t)$ is a step function with the steps occurring at a_{ij}, $j = 1, 2, \cdots, v_i$. It is easily verified that $L_{1i}(t)$ is a nonnegative, continuous nondecreasing piecewise linear function of t with the slope changing at the points a_{ij}, $j = 1, 2, \cdots, v_i$. Furthermore, $L_{1i}(t)$ is a convex function since the slope, $F_i(t)$, is nondecreasing. For $m = 2$, we have

$$L_{2i}(t) = t^2F_i(t) - 2t \textstyle\sum_{j\,|\,a_{ij}\leq t} a_{ij}p_{ij} + \textstyle\sum_{j\,|\,a_{ij}\leq t} a_{ij}^2 p_{ij} \tag{9}$$

It follows that $L_{2i}(t)$ is a nonnegative continuous nondecreasing piecewise quadratic function of t with appropriate quadratic functions defined over the intervals $[a_{ij}, a_{i,j+1}]$, $j = 1, 2, \cdots, v_{i-1}$. The quadratic functions are twice differentiable over the open intervals $(a_{ij}, a_{i,j+1})$, $j = 1, 2, \cdots, v_{i-1}$. For $t \in (a_{ij}, a_{i,j+1})$, since $F_i(t) = \sum_{k=1}^{j} p_{ik}$ is independent of t, the first two derivatives are given by:

$$L_{2i}'(t) = 2tF_i(t) - 2\textstyle\sum_{k=1}^{j} a_{ik}p_{ik}$$

and

$$L_{2i}''(t) = 2F_i(t) > 0$$

Note that $L_{2i}'(t) > 0$ since $t \in (a_{ij}, a_{i,j+1})$, $F_i(t) = \sum_{k=1}^{j} p_{ik}$ and $a_{ik} < t$ for $k = 1, 2, \cdots, j$. Thus, $L_{2i}(t)$ is a nondecreasing convex function.

C.1. Convex First- and Second-Order Stochastic Dominance. For the random variable x_{n+1}, we have convex m^{th} ($m = 1$ or 2)-order stochastic dominance if and only if the following holds:

$$\sum_{i=1}^{n} \lambda_i L_{m-1,i}(t) \le L_{m-1,n+1}(t) \quad \text{for} \quad d_1 \le t \le d_h \tag{10}$$

$$\sum_{i=1}^{n} \lambda_i = 1 \tag{11}$$

$$\lambda_i \ge 0 \qquad i = 1, 2, \cdots, n \tag{12}$$

Inequalities (10) to (12) can be verified by solving the following linear program:

$$Z(m) = \text{Max} \sum_{i=1}^{n} \lambda_i \tag{PI.1}$$

subject to

$$\sum_{i=1}^{n} \lambda_i L_{m-1,i}(d_j) \le L_{m-1,n+1}(d_j) \qquad j = 1, 2, \cdots, h$$

$$\lambda_i \ge 0 \qquad i = 1, 2, \cdots, n$$

We then have convex m^{th}-order ($m = 1$ or 2) stochastic dominance if and only if $Z(m) \ge 1$. Note that if $Z(m) < 1$, then there is no feasible solution to the set of restrictions (10), (11), and (12). On the other hand, if λ_i^*, $i = 1, 2, \cdots, n$ solves (PI.1), with $Z(m) = \sum_{i=1}^{n} \lambda_i^* \ge 1$, then $\bar{\lambda}_i = \lambda_i^*/Z(m)$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^{n} \bar{\lambda}_i = 1$ is feasible to the set of restrictions (10), (11), and (12). This follows from the fact that $L_{0i}(t)$ is a step function with the steps occurring at a_{ij}, $j = 1, 2, \cdots, v_i$, while $L_{1i}(t)$ is a nonnegative continuous piecewise linear function of t with the slope changing at the points a_{ij}, $j = 1, 2, \cdots, v_i$.

Since $L_{0i}(t) = F_i(t)$ is a step function for CFSD, the inequalities in (PI.1) can be further reduced as follows:

$$\text{Let } \delta_i = \text{Min}_{2 \le j \le v_i} (a_{ij} - a_{i,j-1}) \qquad i = 1, 2, \cdots, n+1$$

$$\text{Let } \delta = \text{Min}_{1 \le i \le n+1} \delta_i \qquad \text{note that } \delta > 0$$

Then for CFSD, it is sufficient to solve the following linear program

$$Z = \text{Max} \sum_{i=1}^{n} \lambda_i$$

subject to

$$\sum_{i=1}^{n} \lambda_i F_i(a_{n+1,j} - \delta) \le F_{n+1}(a_{n+1,j-1}) \quad j = 2, 3, \cdots, v_{n+1}$$

$$\lambda_i \ge 0 \qquad i = 1, 2, \cdots, n$$

For strict convex m^{th}-order ($m = 1$ or 2) stochastic dominance, in addition to the inequalities (10) to (12), we must have (10) holding as a strict inequality for some t in the interval $[d_1, d_h]$. This can be verified by solving the following linear program:

$$Z(m) = \text{Max} \sum_{j=1}^{n} s_j$$

subject to:

$$\sum_{i=1}^{n} \lambda_i L_{m-1,i}(d_j) + s_j = L_{m-1,n+1}(d_j) \qquad j = 1, 2, \cdots, h$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \ge 0 \qquad i = 1, 2, \cdots, n$$

$$s_j \ge 0 \qquad j = 1, 2, \cdots, h; \qquad \text{where } s_j \text{ is a slack variable}$$

If $Z(m) > 0$, we have strict convex stochastic dominance of order m.

C.2. Convex Third-Order Stochastic Dominance (CTSD). For the random variable x_{n+1}, we have CTSD, if and only if the following holds:

$$\textstyle\sum_{i=1}^{n} \lambda_i L_{2i}(t) \leq L_{2,n+1}(t) \quad \text{for} \quad d_1 \leq t \leq d_n \tag{13}$$

$$\textstyle\sum_{i=1}^{n} \lambda_i(\mu_{n+1} - \mu_i) \leq 0 \tag{14}$$

$$\textstyle\sum_{i=1}^{n} \lambda_i = 1 \tag{15}$$

$$\lambda_i \geq 0 \qquad i = 1, 2, \cdots, n \tag{16}$$

By dropping (15) and maximizing $\sum_{i=1}^{n} \lambda_i$, we have a semi-infinite programming problem. We can find an approximate solution by appropriate discretization of t and solving the associated linear program. It would, of course, be computationally more efficient to solve the dual of the associated linear program. But for an arbitrary discretization of t, if $\sum_{i=1}^{n} \lambda_i \geq 1$, we cannot be certain that CTSD holds. We next give a discretization of t, so that given $\varepsilon > 0$, the solution and the optimal objective function value Z^* to the associated finite linear program have the desirable property that if $Z^* \geq 1 + \varepsilon$, then CTSD holds and obviously if $Z^* < 1$, then CTSD does not hold.

Let $t_1 = d_2$. Define t_j, $j = 2, 3, \cdots, q$ with $t_j > t_{j-1}$, $j = 2, 3, \cdots, q$ and $t_q = d_h$ so that:

$$L_{2,n+1}(t_j) \leq (1 + \varepsilon)L_{2,n+1}(t_{j-1}) \qquad j = 2, 3, \cdots, q \tag{17}$$

Note that since $L_{2,n+1}(t)$ is a continuous increasing function of t, the points t_j, $j = 2, 3, \cdots, q$ can always be chosen so that (17) holds as an equality for $j = 2, 3, \cdots, q - 1$. In fact, the points t_j, $j = 2, 3, \cdots, q - 1$ can be chosen so that q is the smallest integer such that

$$(1 + \varepsilon)^{q-1} L_{2,n+1}(d_2) \geq L_{2,n+1}(d_h)$$

Consider now the following finite linear program

$$Z^* = \text{Max} \textstyle\sum_{i=1}^{n} \lambda_i \tag{PI.2}$$

subject to

$$\textstyle\sum_{i \mid a_{i1} = d_i} \lambda_i p_{i1} \leq p_{n+1,1}$$

$$\textstyle\sum_{i=1}^{n} \lambda_i L_{2i}(t_j) \leq L_{2,n+1}(t_j) \quad \text{for} \quad j = 1, 2, \cdots, q$$

$$\textstyle\sum_{i=1}^{n} \lambda_i(\mu_{n+1} - \mu_i) \leq 0$$

$$\lambda_i \geq 0 \qquad i = 1, 2, \cdots, n.$$

Clearly, if $Z^* < 1$, we cannot have CTSD and the $n + 1$ distribution which satisfies this condition assigned to the superoptimal set. For the given choice set and a given $\varepsilon > 0$, the superoptimal set is a subset of the optimal set. Every element of the superoptimal set will be chosen by some investor having utility functions in the third-order stochastic dominance class.

As shown in the theorem below, if $Z^* \geq 1 + \varepsilon$, CTSD holds. If $1 \leq Z^* < 1 + \varepsilon$, we can be certain that CTSD holds only if the solution to (PI.2), λ_i^*, $i = 1, 2, \cdots, n$, or more precisely the solution λ_i^*/Z^*, $i = 1, 2, \cdots, n$, happens to satisfy

(13). For such cases, the $n + 1$ distribution is in the superdominated set. For the given choice set, and a given $\varepsilon > 0$, the superdominated set is a subset of the dominated set. Every element of the superdominated set will never be chosen by an investor in the third-order stochastic dominance class.

If $1 \leq Z^* < 1 + \varepsilon$ and the solution λ_i^*/Z_i^*, $i = 1, 2, \cdots, n$, does not satisfy (13), the $n + 1$ distribution is assigned to the mixed set. The mixed set is the subset of the total choice set which for the given ε cannot be identified as a member of the superoptimal set or the superdominated set. Thus, elements of the mixed set can be elements of either the dominated or the optimal set. As ε approaches zero, this mixed set will approach the null set, and the superoptimal and superdominated sets will approach the true optimal and dominated sets.

THEOREM. *If $Z^* \geq 1 + \varepsilon$, then CTSD holds.*

Proof: Let λ_i^*, $i = 1, 2, \cdots, n$, be an optimal solution to (PI.2) so that $Z^* \geq 1 + \varepsilon$. Let $\overline{\lambda}_i = \lambda_i^*/Z^*$, $i = 1, 2, \cdots, n$. Clearly, $\overline{\lambda}_i$, $i = 1, 2, \cdots, n$, satisfies (14) to (16). We want to show that $\overline{\lambda}_i$, $i = 1, 2, \cdots, n$, also satisfies (13). Consider t such that $t_{j-1} < t < t_j$ for any j, such that $2 \leq j \leq q$. We then have:

$$\textstyle\sum_{i=1}^{n} \overline{\lambda}_i L_{2i}(t) < \sum_{i=1}^{n} \overline{\lambda}_i L_{2i}(t_j) \leq (1/Z^*) L_{2,n+1}(t_j) \leq L_{2,n+1}(t_{j-1}) < L_{2,n+1}(t) \tag{18}$$

The first and the last strict inequalities in (18) follow from the fact that $L_{2i}(t)$, $i = 1, 2, \cdots, n + 1$ is an increasing function of t. The second inequality in (18) follows from the definition of λ_i, $i = 1, 2, \cdots, n$ and the fact that λ_i^* is feasible to (PI.2). The third inequality in (18) follows from (17) together with the fact that $L_{2i}(t) \geq 0$ for all i and t, and the assumption that $Z^* \geq 1 + \varepsilon$. Consequently, $\overline{\lambda}_i$, $i = 1, 2, \cdots, n$ satisfies (13) also and hence we must have CTSD. The approach used in this section and the discretization of t as specified by (17) will henceforth be referred to as superconvex third-order stochastic dominance procedure.

II. Experimental Results for Discrete Distributions

Following previous work on identifying the first-, second-, and third-order stochastic dominance admissible set of choice elements (Porter, Wart, and Ferguson [10] and Bawa, Lindenberg, and Rafsky [2]), we will study 896 distributions of monthly equity returns from the CRSP tape from January 1960 to December 1965. Unlike Porter, Wart, and Ferguson, who study choice of a single portfolio return, we will follow Bawa, Lindenberg, and Rafsky and consider the choice problem of choosing one and only one equity security from this set of 896 returns distributions. This example is used for illustrative purposes. The optimal sets derived could be viewed as the true optimal sets if the investors were constrained to earn all income from wealth invested in one and only one of the equities in the choice set. More realistically, the equity returns can be viewed as representing feasible portfolio returns. However, examination of the optimal set for portfolios of these returns and other asset returns or income is beyond the scope of this paper.

Application of the Bawa, Lindenberg, and Rafsky [2] algorithm to this data set yields first-, second-, and third-order admissible sets of 682, 35, and 19 elements, respectively.[2] These sets contain the distributions that cannot be dominated by another single element of the choice set for all investors in the particular class identified.

Determination of the optimal sets requires solution of a large number of linear programming problems. For convex first- and second-order dominance, no more than one linear program is solved for each distribution. Given a choice of ε, the same is true for superconvex third-order stochastic dominance. However, it is possible to limit the number of linear programs run and the size of these problems.

In all cases, the distribution with the highest mean will be in the optimal set. This is because no linear combination of lower partial moments defined by the other distributions in one choice set can lie below the lower partial moment associated with the distribution having the highest mean at all points along the real line for all orders of dominance. Hence, running a linear program with the distribution associated with the highest mean as the $n + 1$ distribution will yield an optimal solution less than one.

Similarly, any distribution with a lower partial moment of order m which lies below the lower partial moments of order m for all other distributions in the choice set at any point cannot be dominated by a linear combination of these distributions for dominance of order $m + 1$. Therefore, to further limit the number of linear programs run, we assign any distribution with a lower partial moment of order m, that is the lowest of all distribution lower partial moments of order m at any point on the real line directly to the optimal set. Again, running a linear program with this distribution as the $n + 1$ distribution will yield on optimal solution less than one, and this distribution is a member of the optimal set.

Additionally, distributions that are in the dominated set for orders of dominance lower than the one being solved for are also dominated for the higher dominance class. For example, members of the second-order stochastic dominance admissible set which are dominated by convex first-order stochastic dominance do not have to be tested by the second-order convex stochastic dominance procedure. These distributions are directly assigned to the second-order dominated set. This assignment of choice distributions to the dominated set and the two previous ways of directly identifying members of the optimal set are used to limit the number of linear programs run.

To further ease the computational intensity of solving the CSD problems, transitivity of von Neumann-Morgenstern expected utility is utilized. Under this axiom, any dominated distribution is not necessary for dominating other distributions. To see this, recognize that should distributions F and G dominate distribution H, while distributions I and G dominate distribution F, then the choice of distributions I and G will also dominate distribution H. Therefore, once a given distribution is dominated, it is eliminated from the set of distributions making up the tableau of the linear program for the associated convex stochastic dominance problem. This means of decreasing the size of the associated linear

[2] The Bawa, Lindenberg, and Rafsky [2] work does not report the size of the first-order admissible set. Also, our second-order admissible set size is larger than theirs. This difference arises due to rounding error introduced by using different computers, and a small programming error in their original algorithm. We thank the authors of this algorithm for its use.

programs, coupled with the three ways of directly assigning distributions to the optimal and dominated sets, greatly reduces the computer time required to solve our problem.

Applying our exact first- and second-order convex stochastic dominance procedures to identify the optimal set which is the set of alternatives that will be chosen by some investor in the admitted class of utility functions, we find:

	FCSD	SCSD
Choice Set	896	896
Admissible Set	682	35
Optimal Set	454	25

To reduce the third-order admissible set to the third-order optimal set, the superthird-order convex stochastic dominance procedure is executed.[3] We find:

	STCSD
Choice Set	896
Admissible Set	19
Superdominated Set (= dominated set)	6
Mixed Set	0
Superoptimal Set (= optimal set)	13

[3] The actual solution technique used for solving the semi-infinite programming problem is as follows:

(a) Consider the linear programming problem: Max $\sum_{i=1}^{n} \lambda_i^*$ subject to (13), (14), (15), and (16) defined for $t = d_j, j = 1, 2, \cdots h$.
(b) Solve the defined linear program. Let λ_i^*, $i = 1, 2, \cdots n$ be an optimal solution with objective function value Z^*.
(c) If $Z^* < 1$ for distribution $n + 1$, assign distribution $n + 1$ to the superoptimal set.
(d) If $Z^* \geq 1$, determine if any constraints are violated by searching between data points d_k and d_{k+1}, where $k = 1, 2, \cdots h - 1$.

For each $k = 1, 2, \cdots h - 1$, this is accomplished by solving the following problem:

$$\text{Max } Q_{n+1}(k) = \sum_{i=1}^{n} (\lambda_i^*/\lambda^*) L_{2i}(t) - L_{2,n+1}(t)$$

$$\text{subject to } \quad d_k \leq t \leq d_{k+1} \qquad \text{where } \quad \lambda^* = \sum_{i=1}^{n} \lambda_i^*$$

In view of (9), this reduces to

$$\text{Max } Q_{n+1}(k) = \sum_{i=1}^{n} (\lambda_i^*/\lambda^*)\{t^2 F_i(d_k) - 2t \sum_{j \mid a_{ij} \leq d_k} a_{ij} p_{ij} + \sum_{j \mid a_{ij} \leq d_k} a_{ij}^2 p_{ij}\}$$

$$- \{t^2 F_{n+1}(d_k) - 2t \sum_{j \mid a_{n+1,j} \leq d_k} a_{n+1,j} p_{n+1,j} - \sum_{j \mid a_{n+1,j} \leq d_k} a_{n+1,j}^2 p_{n+1,j}\} \quad d_k \leq t \leq d_{k+1}$$

Since $Q_{n+1}(k)$ is a quadratic function of t, the above problem can be solved by standard calculus. Denoting the maximum value as $Q_{n+1}^*(k)$, we have

$$Q_{n+1}^* = \text{Max}_{1 \leq k \leq h-1} Q_{n+1}^*(k).$$

(i) If $Q_{n+1}^* \leq 0$, the distribution $n + 1$ is assigned to the superdominated set.
(ii) If $Q_{n+1}^* > 0$, add to the linear programming problem the constraint defined by (13) for $t = t^*$ where t^* maximizes $Q_{n+1}(p)$ and $Q_{n+1}^*(p) = Q_{n+1}^*$. (Go to Step (b)).

Although the above procedure, in general, is not a finite procedure for the distributions tested here, this procedure was not only finite but, in fact, step (ii) was never executed.

We find that application of convex stochastic dominance procedures does significantly reduce the number of distributions that must be considered by investors with utility functions in classes U_1, U_2, or U_3 for the specified choice problem.

III. Continuous Distribution Optimal Sets

When distributions are continuous, special properties of the particular distribution will have to be utilized to make the problem of identifying the optimal set finite. For the random variable, x_{n+1}, we will have stochastic dominance of order $m + 1$, $\{m = 0, 1, 2\}$ if the following holds:[4]

$$\textstyle\sum_{i=1}^{n} \lambda_i \mathrm{LPM}_{mi}(t_i) \leq \mathrm{LPM}_{m,n+1}(t_i) \quad \text{for} \quad d_1 \leq t \leq d_h \tag{19}$$

$$\textstyle\sum_{i=1}^{n} \lambda_i \mu_i \geq \left(\sum_{i=1}^{n} \lambda_i\right) \mu_{n+1} \tag{20}$$

$$\textstyle\sum_{i=1}^{n} \lambda_i = 1 \tag{21}$$

$$\lambda_i \geq 0 \qquad i = 1, \cdots, n \tag{22}$$

For a continuous distribution, defined over a bounded interval, solution of the above problem over a discrete grid by a linear programming procedure will only serve to classify elements of the admissible set of order $m + 1$ into either the superoptimal set or the mixed set.

Nevertheless, application of the superconvex stochastic dominance procedure (the procedure in Section I.C.2 for any order greater than or equal to 3) will again allow classification of the superdominated, mixed, and superoptimal sets of the appropriate order. This follows directly from the proof in Section I.C.2 for SCTSD when the discrete lower partial moment of order 2 is replaced by the appropriate continuous LPM.

However, the finiteness of this procedure (for given $\varepsilon > 0$) holds only when d_1 and d_h bound the distribution. When no bounds exist on possible realizations, SCTSD will not be finite. To motivate this issue, we will discuss such a case, the normal distribution.

As a member of the general location-scale family of distributions, Bawa [1] has shown that selection of a single normal distribution from a set of normal distributions as a member of the SSD admissible set can be reduced to a simple mean-variance decision rule. Any alternative with equal or lower mean and higher variance than another distribution will not be chosen by a decision maker with a utility function in U_2.

In order to further reduce the set of distributions that must actually be considered by this class of investors, Meyer has suggested application of Fishburn's conditions for CSSD on the SSD admissible set to determine the optimal set.[5]

[4] Constraint (20) is redundant for first- and second-order dominance, i.e., for $m = 0$ or 1.

[5] Meyer [9] in his work on CSSD failed to recognize that a mean-variance criterion for mixtures of normal distributions is too strong for dominance, rejecting distributions, which will actually be chosen by some possible investor with a utility function in U_2.

For normal distributions, it can be shown through integration by parts that:

$$\int_{-\infty}^{x_i} F(y)\,dy = (x_i - \mu_i)\Phi\left(\frac{x_i - \mu_i}{\sigma_i}\right) + \sigma_i\phi_i\left(\frac{x_i - \mu_i}{\sigma_i}\right) = \mathrm{LPM}_{1i}(x_i) \qquad (23)$$

where Φ is the standard normal distribution and ϕ is the standard normal density.

Substituting this relation into (19), and setting $d_1 = -\infty$, and $d_h = +\infty$ yields the true problem. However, to identify members of the superoptimal set, we set:[6]

$$d_l = \mu_l + 5.6\sigma_h$$

$$d_h = \mu_h + 5.6\sigma_h$$

$$\varepsilon = 0.005$$

μ_l is the lowest mean for a distribution in the admissible set, μ_h is the highest mean for a distribution in the admissible set, and σ_h is the standard deviation of the distribution with the highest mean. (This is the distribution with the highest standard deviation.)

We will determine the feasibility of the system of inequalities (19) to (22). Once again given the closed interval we have considered, the linear programming based superconvex stochastic dominance procedures can be applied. Any distribution for which the system of inequalities (19) to (22) is not feasible for a subset of $t = \{-\infty, \infty\}$ will be superoptimal.

However, we are unable to test for the feasibility of this system of equations for all $t = \{-\infty, \infty\}$. Therefore, even if our linear programming based methodology implies CSSD over a closed, bounded interval, we cannot be sure that CSSD holds outside of that interval. Any distribution found to be dominated over a closed, bounded interval will only be classified as a candidate for the dominated set. Unless found to be superoptimal, it is classified in the mixed set.

Nonetheless, it is of interest to reexamine the previous work of Meyer [9] on determining CSSD optimal sets for normal distributions. For his problem of examining the admissible set of 12 mutual fund return distributions derived from Sharpe's [12] data on 35 mutual funds we find:

	SCSSD (d_1, d_h)
Choice Set	35
Admissible Set	12
Mixed Set	6
Superoptimal set	6

These results compare with Meyer's finding of seven dominated and five optimal distributions. We see that the two-parameter decision rule he suggests for the determination of the CSSD optimal set is too strong.

[6] The range is defined to be 5.6 standard deviations above and below the mean to be consistent with accuracy at the single precision level for the normal cumulative distribution. The distribution values 5.6 standard deviations below and above the mean are 0 and 1, respectively, at the single precision level of accuracy.

However, our inability to identify members of the dominated set remains. The application is of interest to correct the previous inconsistencies stated in the literature on CSSD for normal distributions, to motivate how this problem can affect other distributions defined over the entire real line, and to show another application of superconvex stochastic dominance.

IV. Conclusions

Exact solution procedures are presented for convex first- and second-order stochastic dominance with discrete distributions. For dominance of orders higher than two with discrete distributions, and for continuous distributions over a bounded interval, a dominance rule, superconvex stochastic dominance, has been proposed. This rule allows identification of some members of the convex stochastic dominance optimal and dominated sets. A superdominated set, a subset of the CSD-dominated set, and a superoptimal set, a subset of the CSD optimal set, are identified. These SCSD sets will not generally equal the corresponding CSD set. An additional set of alternatives, the mixed set, is identified which will contain either dominated elements, optimal elements, or both. By an appropriate choice of the parameter ε, the size of the mixed set may be reduced. It is, of course, entirely possible that for a particular application (as in the case of the 896 security returns from January 1960 to December 1965), the mixed set may be empty. Generally, a particular decision maker with a utility function in a certain class will not have to consider elements of the superdominated set. Members of the mixed and superoptimal set will have to be evaluated.

In the first application of convex stochastic dominance rules for identifying the elements of the optimal set from the full choice set, the optimal sets have been found to be significantly smaller than the previously identified admissible sets. The convex stochastic dominance procedure is a useful tool for limiting the set of alternatives that must be considered by an individual having utility functions in the classes we have examined.

Finally, CSD rules are discussed for distributions defined over the entire real line. For these distributions, we motivate the difficulty of identifying members of the superdominated set. The case of normally distributed choice elements is treated. It is shown that for this associated problem, although a superoptimal set may be identified, members of the superdominated set are not readily identified. Therefore, using our SCSD procedure will not reduce the number of normal distributions that an individual will want to consider from the admissible set. However, our procedure applied to the mutual fund data treated by Meyer [9] shows that the optimal set identified in that paper is incorrect.

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