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The Analytics of Performance Measurement Using a Security Market Line

PHILIP H. DYBVIG and STEPHEN A. ROSS*

ABSTRACT

Security market line (SML) analysis, while an important tool, has never been fully justified from a theoretical standpoint. Assuming symmetric information and an inefficient index, we show that SML analysis can be grossly misleading, since, in general, efficient and inefficient portfolios can plot above and below the SML. On a more positive note, if SML analysis uses the return on a marketed riskless asset for the zero-beta rate, efficient portfolios must plot above the SML. Nonetheless, arbitrarily inefficient portfolios also plot above the SML.

MEAN-VARIANCE THEORY IS STILL the premier theory in finance. It continues to survive theoretical criticism[1] and empirical rejection[2], and it does so in no small part because of the popularity of using deviations from the security market line (SML) in performance measurement (of portfolio managers) and in event studies (of the price impact of information). The SML is the relationship between risk and return predicted by the mean-variance model, namely that risk, as measured by beta, and expected return are linearly related. SML analysis interprets deviations in expected return from the SML as abnormal returns above or below what is appropriate or warranted for the amount of risk taken on. However, in the mean-variance model, all assets and portfolios lie on the SML ex ante, and any ex post deviations from the SML are attributable solely to statistical measurement errors. It is for this reason that Roll [22–25] has argued that justification for SML analysis must lie outside the original mean-variance model. The purpose of this paper and its companion paper (Dybvig and Ross [8]) is to study whether SML analysis is valid in several plausible theoretical frameworks, i.e., whether theory justifies intuition.

Specifically, in this article we examine the validity of using SML analysis when the index portfolio is inefficient. In the companion paper, we look at performance

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[1] The most important critique of the mean-variance Capital Asset Pricing Model (CAPM) was mounted in a series of papers by Roll [22–25]. Some earlier researchers had been troubled by mean-variance theory, but Roll was the first to clearly understand the weaknesses of the theory and the empirical implications of these deficiencies.

[2] An exhaustive bibliography of empirical tests of mean-variance theory would be larger than this paper. The reader is referred to Gibbons [11], a major empirical piece rejecting the efficiency of market indices, which is also a good source of references to earlier work.

of a manager with superior information; in this paper, there is no asymmetry of information. This distinction means that using the SML as a pragmatic performance measure will encounter the problems discussed here, plus an additional statistical problem, since, in general, we cannot condition on a portfolio manager's information. In this paper, when we speak about performance measurement we are explicitly referring to the case where the observer and the manager have the same information concerning returns, and the manager is being evaluated only on ability to find an efficient portfolio.

Unlike most previous results in mean-variance theory, our results hinge on whether or not there is a riskless asset. In the presence of a riskless asset, we find some justification for using SML analysis in one-sided event studies' tests or for performance measurement. When there is no riskless asset, abnormal returns analysis may be misleading.

One task of the paper is to lay out rather completely the relationship between the mean-variance analysis and the SML. We deal only with the *ex ante* problem—ignoring *ex post* statistical measurement errors. In other words, we are studying the question of whether expected abnormal returns are good, not the question of how best to measure expected abnormal returns in small samples, as studied by Brown and Warner [2, 3].

Section I begins by describing the relevance of efficiency and SML analysis for performance measurement. We then present a simple geometric tool that clarifies the analytic relationship between excess returns and mean and variance. This tool leads to many new results about deviations from the SML when the market index is inefficient. In particular, we find some support for using SML analysis in the presence of a riskless asset. But without a riskless asset there is, in general, no relation between SML analysis and efficiency. In Section II, the argument is extended to examine the virtues of the security market line as a tool for testing whether a new security could be used to improve an inefficient portfolio. Section III applies the analysis to resolve Roll's assertion that for any inefficient index another can be found which reverses the ranking of all portfolios. After changing slightly the mathematical statement of Roll's assertion, we essentially verify it. However, when there exists a riskless asset, Roll's assertion is false. This result resolves the contradiction between our positive result and Roll's assertion. Section IV concludes and summarizes the paper and calls attention to some of the shortcomings of our work.

I. Misspecification of the Index

This section concerns itself with the implications of evaluating returns from the perspective of an inefficient index. To motivate our analysis of such situations, consider a performance measurement problem. To make matters simple, we will assume that the manager of the portfolio and the observer of his or her performance share the same information concerning the underlying mean and covariance matrix of returns, E and V. What the observer does not know is whether or not the reference portfolio is efficient. If it is, then the manager must plot on the observer's SML. But, if it is not, i.e., if the index is misspecified and if the observer uses the SML analysis to investigate the performance of the manager, then that is precisely the problem analyzed in this section. (Notice that since the observer and the manager have the same, possibly imperfect, information concerning returns, we are testing whether the manager can find an efficient portfolio.)

We will denote the possibly inefficient index or market proxy by α_p. Formally, α_p is an n-vector whose i^{th} component denotes the proportionate investment in asset i. All portfolios, α, have the property that $\alpha e = 1$, where e is a vector on ones, i.e., $e = (1, \cdots, 1)$.[3] Asset return means and covariances are given by E and V. Since we assume that E and V are known, our results are biased towards justifying SML analysis; we are assuming away identification problems and significant measurement error. We will also ignore the degenerate cases in which all assets have the same expected return and cases which admit arbitrage. Furthermore, we will assume that there are at least three assets with linearly independent returns, or else any index, α_p, would be efficient. (In what follows, *efficient* portfolios are those with minimum variance at a given level of return. *Positively efficient* portfolios also maximize return subject to a fixed variance.)

The zero-beta return associated with α_p is denoted by r_0. It is important to realize that if α_p is inefficient and there is no riskless asset, then r_0 is not uniquely determined by simply specifying E, V, and α_p, since the portfolios uncorrelated with α_p do not all have the same expected return.[4] For each portfolio α_p (efficient or not) and zero-beta rate r_0, there is an associated SML_{p,r_0}. A portfolio α is said to lie on the SML_{p,r_0} if

$$\alpha E - r_0 - \frac{\alpha V \alpha_p}{\alpha_p V \alpha_p}(\alpha_p E - r_0) = 0. \tag{1}$$

(See Sharpe [31], Merton [20], Roll [22], or Ross [29] for derivations of mean-variance analysis.) When α_p is efficient, this is satisfied identically by all α. When α_p is inefficient, excess returns $(E_i - r_0)$ and covariances with the proxy $((V\alpha_p)_i)$ are not proportional, and the SML_{p,r_0} will not be the set of all portfolios, but will, instead, be a space of dimensionality one less than the full dimension of the portfolio space.[5] To write the SML condition more economically, we define the vector of deviations from the market line,

$$\delta \equiv E - r_0 e - (\alpha_p E - r_0)\frac{V\alpha_p}{\alpha_p V \alpha_p}. \tag{2}$$

[3] We use the convention that $xy \equiv \sum x_i y_i$. Of course, we could have followed Chamberlain and Rothschild [4] and Hansen and Richard [13], and used Hilbert space notation to state our results. In Hilbert space notation, covariances are viewed as the inner product relation on a Hilbert space. Using Hilbert space notation would not have changed the statement or proof of any of our results.

[4] See Roll [26] and Kandel [18]. Furthermore, it is easily shown that choosing r_0 from the cross-sectional relation between risk and return is completely arbitrary. Such an r_0 can be changed arbitrarily by making the otherwise irrelevant change of repackaging securities without changing spanning or the market's payoff across states.

[5] Even if α_p is efficient, if r_0 is not chosen to be the return on a portfolio uncorrelated with α_p, then (1) will not hold identically for all α. This is easy to see, e.g., if α is chosen to be uncorrelated with α_p, then the left-hand side of (1) will be the difference between the zero-beta return for α_p and the chosen r_0.

The condition that α is on the SML_{p,r_0} may then be written as

$$\alpha\delta = 0. \tag{3}$$

Similarly, a portfolio lies above or below the SML_{p,r_0} as $\alpha\delta > 0$ or $\alpha\delta < 0$.

We begin our study with a simple theorem that is the core of the analysis. The theorem is probably familiar as a folk theorem in the literature on performance measurement. A portfolio's deviation from the SML, which we have denoted as $\alpha\delta$ above, was first introduced by Jensen [17] as a measure of that portfolio's performance, and we refer to it as Jensen's measure. Sharpe [31] obtained another measure of performance by looking at the return-risk tradeoff in mean-variance space. Sharpe's measure is given by the difference between the mean-standard deviation tradeoff for the portfolio

$$\frac{\alpha E - r_0}{\sqrt{\alpha V \alpha}},$$

and that for the index

$$\frac{\alpha_p E - r_0}{\sqrt{\alpha_p V \alpha_p}}.$$

(Both Sharpe and Jensen originally assumed that r_0 was the return on a riskless asset, but we will make no such requirement.) Our theorem demonstrates that superior performance in Sharpe's sense implies superior performance by the Jensen measure as well. As we shall see, however, the converse of this result is not true.

THEOREM 1. *If α dominates α_p under the Sharpe measure,*

$$\frac{\alpha E - r_0}{\sqrt{\alpha V \alpha}} > \frac{\alpha_p E - r_0}{\sqrt{\alpha_p V \alpha_p}} > 0, \tag{4}$$

then α plots above the SML, $\alpha\delta > 0$. Also, if the inequalities in (4) *are reversed, then $\alpha\delta < 0$.*

Proof: From (2),

$$\begin{aligned}\alpha\delta &= \alpha E - r_0 - \frac{\alpha V \alpha_p}{\alpha_p V \alpha_p}(\alpha_p E - r_0)\\ &= (\alpha E - r_0)\left[1 - \rho\,\frac{\sqrt{\alpha V \alpha}}{\sqrt{\alpha_p V \alpha_p}}\,\frac{\alpha_p E - r_0}{\alpha E - r_0}\right]\end{aligned}$$

where

$$\rho \equiv \frac{\alpha V \alpha_p}{\sqrt{\alpha V \alpha}\,\sqrt{\alpha_p V \alpha_p}},$$

is the correlation coefficient between α and α_p. Since the correlation coefficient, ρ, lies between -1 and 1 and the expression multiplying the correlation coefficient lies between 0 and 1 by condition (4) of this theorem (or its reversal when $\alpha_p E -$

$r_0 < 0$), we have

$$\alpha\delta \gtrless 0, \quad \text{as} \quad \alpha_p E - r_0 \gtrless 0. \quad \text{Q.E.D.}$$

Condition (4) of the theorem is basically that the market price of risk, the ratio of the mean excess return to the standard deviation, be less for α_p than for α. (The second inequality is simply an assumption that the expected return on α_p exceeds r_0.) Figure 1 illustrates the theorem for the two cases, one for mean returns above r_0 and the other with mean returns below. The plus and minus signs in Figure 1 refer to portfolios which lie above and below SML_{p,r_0}, respectively. In other words, all portfolios in a particular labeled region lie on one side of the SML_{p,r_0}.

The case of a negative excess return on the proxy is not included as mere caprice. The analysis we are employing is to be applied to ex post estimates and it is not unusual to find that the ex post excess return on a proxy is negative in a given subperiod. More importantly, if an event under study includes aggregate information, it may be true that the expected return to the market conditional on that event is less than the riskless rate. We will, however, relegate this case to footnotes and parenthetical remarks to avoid clutter.

The following result is a simple corollary of Theorem 1 and is also illustrated implicitly by Figure 1.

COROLLARY 1. *If* $\alpha E > \alpha_p E > r_0$ *and* $\alpha V \alpha < \alpha_p V \alpha_p$, *then* $\alpha\delta > 0$. (*Also, if* $\alpha E < \alpha_p E < r_0$ *and* $\alpha V\alpha < \alpha_p V \alpha_p$, *then* $\alpha\delta < 0$.)

Proof: Obvious, given Theorem 1. Q.E.D.

We can use these simple results to specify the exact relationship between SML analysis and the mean-variance diagram. To do this, we will need to take a look at the geometry of the problem in portfolio space,

$$S \equiv \{\alpha \mid \alpha e = 1\}.$$

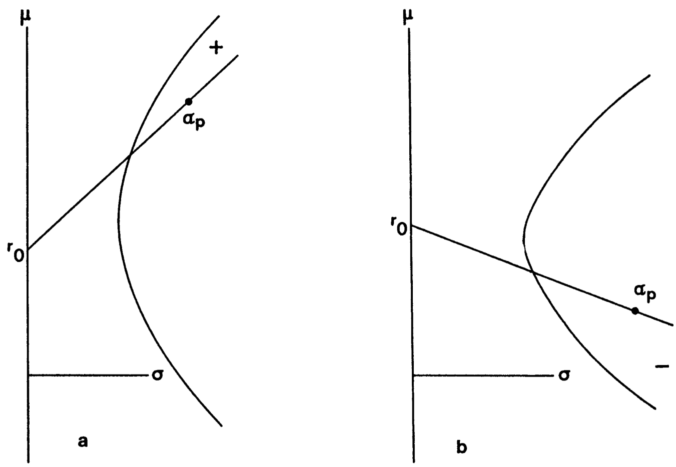

Figure 1. Theorem 1 tells us that the deviations from the SML are positive in the area marked + and negative in the area marked −.

the $n-1$ dimensional space of all feasible portfolios. The space, S, is an $n-1$ dimensional hyperplane in R^n. Within S are three subspaces of interest: the global efficient frontier,

$$G^* \equiv \{\alpha \mid \alpha \text{ is efficient given } (E, V)\}, \tag{5}$$

the space of all portfolios on the SML_{p,r_0},

$$F \equiv \{\alpha \mid \alpha \in S \text{ and } \alpha\delta = 0\}, \tag{6}$$

and the frontier of F,

$$F^* = \{\alpha \mid \alpha \text{ is efficient given } (E, V) \text{ and } \alpha \in F\}. \tag{7}$$

Notice that for α_p inefficient, F is an $n-2$ dimensional subspace of S.[6]

The space G^* is the space of mean-variance efficient portfolios. A well-known result in mean-variance theory holds that G^* satisfies two fund separation. In other words, there are two portfolios such that each element of G^* is composed of the span of these two. This implies that G^* is a line in portfolio space.[7] As we will see shortly, the same argument is valid for F^* as well.

Since G^* is a line in S and F is a hyperplane in S, they must intersect each other at one point, or cross nowhere, or the line must be embedded in the plane (which would imply α_p is efficient since $\alpha_p \in F$). This observation about the portfolio space has the following implication for mean-variance analysis.

THEOREM 2. *Let α_p be inefficient with $\alpha_p E \neq r_0$. Then F^* (F's frontier) is a hyperbola in mean-standard deviation space and is either tangent to G^* (the global frontier) at a single point, or never touches G^*.*

Proof: That F^* is a hyperbola follows from the standard argument that the frontier is a hyperbola, once we choose $n-1$ portfolios that generate F and view these portfolios as assets. To prove that F^* and G^* are tangent at a single point or never intersect, note that any intersection between F^* and G^* in mean-standard deviation space implies that F and G intersect in portfolio space,[8] since there is only one efficient portfolio with a given mean. Lastly, F^* and G^* cannot be coincident, since $\alpha_p \in F^*$ but $\alpha_p \notin G^*$. Q.E.D.

From Theorem 2, F^* divides the feasible set into three, or possibly only two, regions. (The three region case is illustrated in Figure 2.) The next theorem proves that all portfolios in a half moon-shaped region between F^* and G^* must lie on the same side of the SML_{p,r_0}.

THEOREM 3. *All portfolios in a given region between F^* and G^* must lie on one side of the SML_{p,r_0}, i.e., all have either $\alpha\delta > 0$ or $\alpha\delta < 0$.*

[6] This would not be true if δ were proportional to e since all α satisfy $\alpha e = 1$, but since α_p is on SML_{p,r_0}, $\alpha_p\,\delta = 0$, so δ cannot be (nontrivially) proportional to e.

[7] There is a possibility of redundancy if V is singular of rank less than $n-1$. (Rank $n-1$ can occur if there exists a riskless portfolio.) In this case, some of the assets are redundant, and we will simplify our analysis by eliminating them and assuming that V is of rank at least $n-1$. Recall that this is possible since we have assumed that there are no arbitrage opportunities.

[8] It also follows immediately from the first order conditions that α_p is in F^* generated by α_p, and also, that the line defined by r_0 and α_p is tangent to the frontier for F at the α_p point.

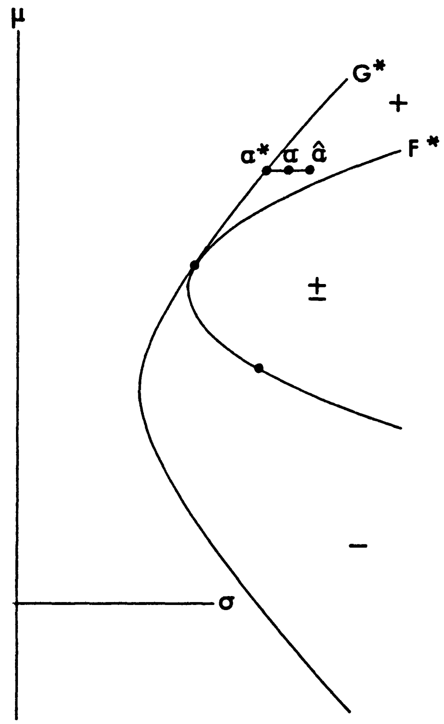

Figure 2. Theorems 2 and 3 tell us that each connected region between F^* and G^* consists exclusively of portfolios with SML deviations that are either all positive (+) or all negative (−). The area marked (±) has at each point portfolios with both signs, as shown in Theorem 4. (The points α^*, α, and $\hat{\alpha}$ are used in the proof of Theorem 2.)

Proof: The argument is illustrated in Figure 2. Since there is only one point (at most), call it α^c, on the global efficient frontier, G^*, where $\alpha\delta = 0$, by continuity the points on G^* and on the same side of α^c must also lie on the same side of the SML_{p,r_0}. Consider the region P of mean variance space which lies between F^* and that portion of G^* for which all portfolios α have $\alpha\delta > 0$. To prove that all portfolios in P have $\alpha\delta > 0$, suppose instead that some $\hat{\alpha} \in P$ has $\hat{\alpha}\delta < 0$. Let α^* be the portfolio on G^* with the same mean as $\hat{\alpha}$. Since $\alpha^*\delta > 0$, some convex combination of α^* and $\hat{\alpha}$, call it α, is on SML_{p,r_0}. But this convex combination lies between F^* and G^* (by the triangle inequality, since α^* and $\hat{\alpha}$ have the same mean), which contradicts the definition of F^*. The argument is symmetric for P defined with $\alpha\delta < 0$. Q.E.D.

Our final theorem of Section I shows that in the region above F^*, there is always an ambiguous relationship with the SML_{p,r_0}.

THEOREM 4. *Whenever α_p is inefficient, at each point (μ, σ) that is inefficient relative to F^*, there is a portfolio α^0 that lies above SML_{p,r_0} (i.e., with $\alpha^0\delta > 0$) and another portfolio α^1 below SML_{p,r_0} (i.e., with $\alpha^1\delta < 0$).*

Proof: The argument is illustrated in Figure 3. Let α^* be the portfolio in G^* with the same mean as α_p. Since α_p is inefficient, $\alpha^*\delta \neq 0$ (by Corollary 1). If $\hat{\alpha}$

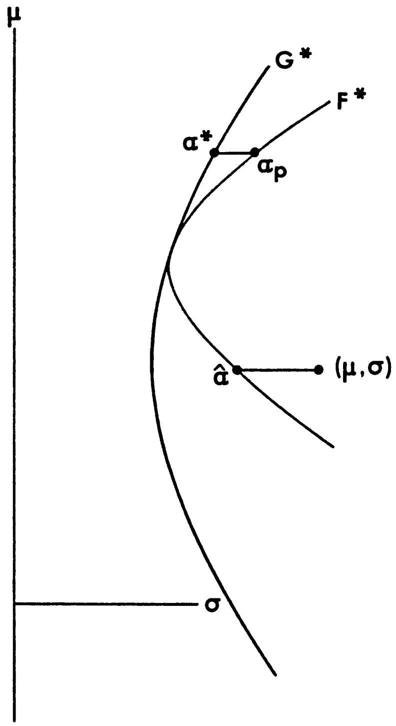

Figure 3. This figure is used to illustrate the construction in the proof of Theorem 3.

is the portfolio on F^* with mean μ, then $\hat{\alpha} + \lambda(\alpha^* - \alpha_p)$, is a portfolio with the same mean μ for all λ. Furthermore, for some $\lambda > 0$, this portfolio has variance σ^2, and since $(\hat{\alpha} + \lambda(\alpha^* - \alpha_p))\delta = \lambda\alpha^*\delta$, it lies on the same side of SML_{p,r_0} as α^*. Similarly, for some $\lambda < 0$, it has variance σ^2 and lies on the opposite side of SML_{p,r_0} as α^*. Q.E.D.

There are eight possible cases, four of which—those with a positive excess return on the index—are illustrated (for no risky asset) in Figure 4.[9]

In the case of a riskless asset, shown in Figure 5, the entire positively efficient frontier lies above the line through r_0 and α_p. By Theorem 1, we have, therefore, that efficient portfolios have positive deviations from the SML. This result—that good performance implies positive deviations—is our modest justification for using SML analysis in the presence of a riskless asset. On the other hand, many other portfolios also have positive deviations from the SML, including some that are arbitrarily inefficient (see Figure 5). Here is a summary of what happens in the presence of a riskless asset. If we see a portfolio with negative deviations from the SML, we can reject the possibility of superior performance. (This is our positive result.) However, a portfolio with positive deviations from

[9] If the excess return on the index is negative, then the regions are still the same as those marked for the positive case, e.g., the right half-moon region in case (a) is still positive. In the degenerate case where F^* and G^* do not interesect, the region between them has the same sign as the excess return on the index.

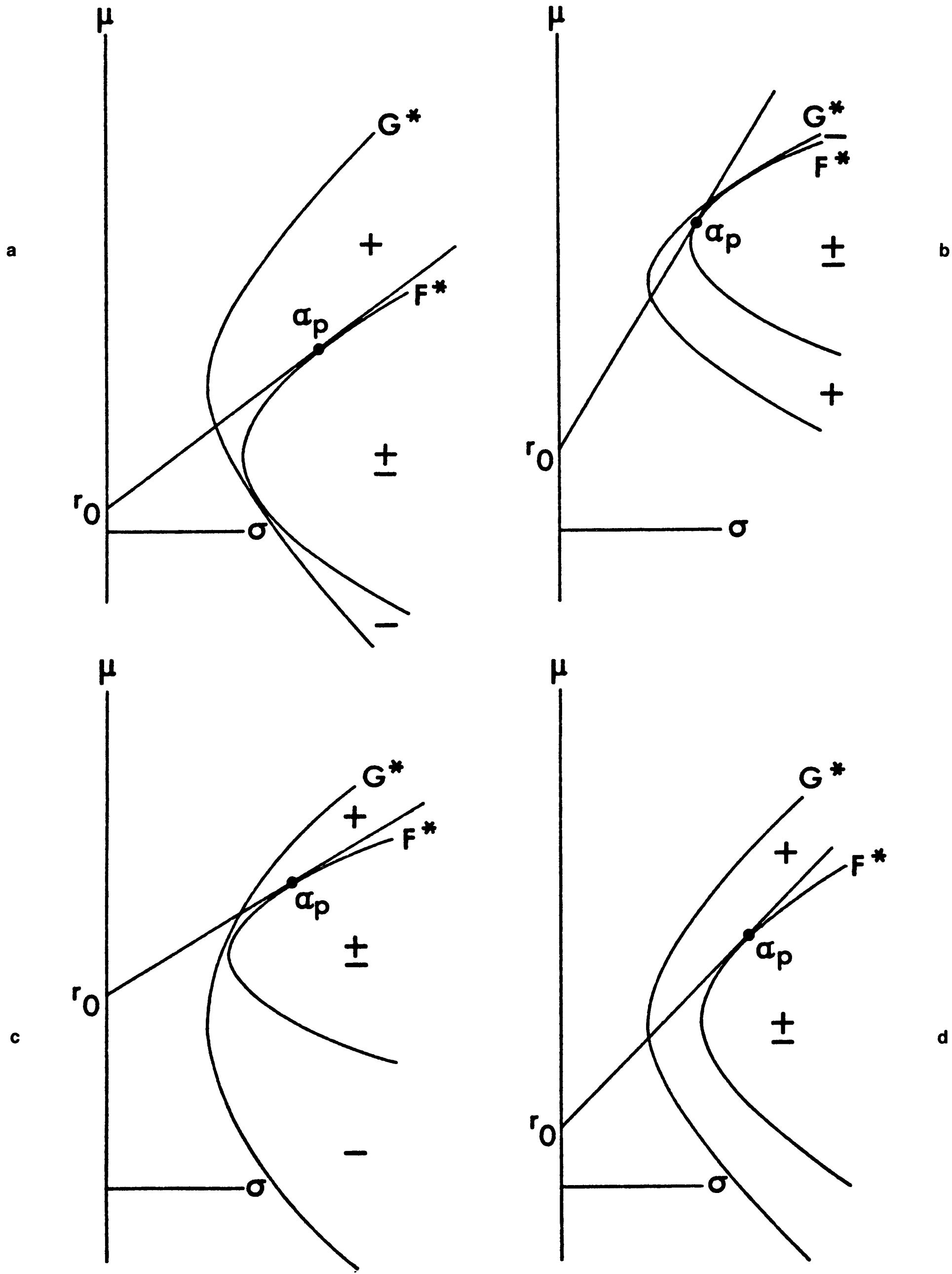

Figure 4. There are eight possible patterns of positive and negative SML deviations and placements of the index portfolio. Shown here are the four cases corresponding to a mean return on the index in excess of the zero-beta rate, without a riskless asset.

the SML may be efficient or arbitrarily inefficient. All of these results are, of course, dependent on using the riskless asset's return for r_0.

In the absence of any riskless asset, shown in Figure 4, the situation is less favorable for SML analysis. In this case, the entire positively efficient frontier

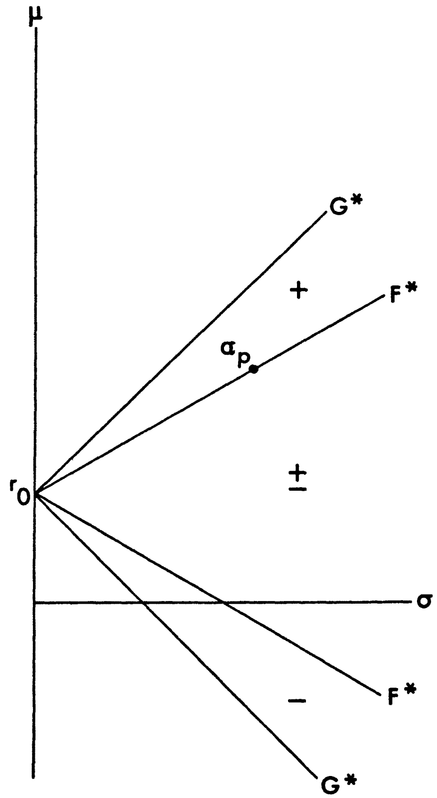

Figure 5. This figure illustrates our main positive result, namely that when there is a riskless asset, all efficient portfolios plot above the SML. Note that some very inefficient portfolios also plot above.

no longer lies above the line through r_0 and α_p, and we cannot appeal to Theorem 1 to show that all efficient portfolios have positive deviations from the SML. In Figure 4, part of the frontier corresponds to portfolios plotting below the SML. Furthermore, it is true that in all cases, many arbitrarily inefficient portfolios have positive deviations from the SML. Here is a summary of what happens in the absence of a riskless asset. If we see a portfolio with negative deviations from the SML, we cannot reject the possibility that it is mean-variance efficient, but we can reject that it mean-variance dominates the index. Furthermore, a portfolio with positive deviations from the SML may be efficient or arbitrarily inefficient. Therefore, in the absence of a riskless asset, there is no general relation between efficiency and SML deviations.

In general, therefore, efficient and inefficient portfolios may plot above or below the SML_{p,r_0}. The only positive result is that if we take r_0 as the return to a riskless asset, assets on the frontier must plot above the SML, although others, including some that are arbitrarily inefficient, will plot above it, too.

II. The SML and Marginal Comparisons

What about the usual marginal interpretation of the SML_{p,r_0}? Our contribution here is to provide a formal proof that the portfolio, α_p, can be improved by adding or shorting a small investment in α, depending upon whether $\mathrm{SML}_{p,r_0} \lesseqgtr 0$, and provided that $\alpha_p E > r_0$. By a zero-beta portfolio, α_0, we will mean any portfolio uncorrelated with α_p (i.e., $\alpha_0 V \alpha_p = 0$), with expected return r_0 (i.e., $\alpha_0 E = r_0$).

Thus, in our performance problem the observer would be able to improve his or her portfolio α_p, by a marginal investment in α if $\mathrm{SML}_{p,r_0} > 0$, or by a disinvestment if $\mathrm{SML}_{p,r_0} < 0$. Similarly, if the research in an event study found that $\mathrm{SML}_{p,r_0} \lesseqgtr 0$, he or she would be able to construct a trading rule conditional on the event that improved the reference portfolio, α_p.

Our first result shows how we can use the SML analysis to construct a portfolio which is marginally superior in a mean variance sense to the inefficient index, α_p.

THEOREM 5. *Let α_p be an inefficient index with an associated zero-beta portfolio, α_0, and suppose that $\alpha\delta > 0$ (alternatively that $\alpha\delta < 0$) and that $\alpha_p E - r_0 > 0$. A mean-variance superior portfolio to α_p can be constructed by holding α_p plus a marginal position into (out of) α and out of (into) α_p, with a balancing position in α_0 to preserve the mean return.*

Proof: Let the candidate superior portfolio be given by

$$\xi = (1 - \lambda - \lambda K)\alpha_p + \lambda K \alpha_0 + \lambda \alpha,$$

where

$$K = \frac{\alpha E - \alpha_p E}{\alpha_p E - r_0},$$

to ensure that $\xi e = 1$ and $\xi E = \alpha_p E$. Since $\alpha_0 V \alpha_p = 0$, straightforward substitution gives the variance as

$$\xi V \xi = [1 - \lambda(1 + K)]^2 \alpha_p V \alpha_p + 2\lambda[1 - \lambda(1 + K)]\alpha V \alpha_p + \lambda^2 (K\alpha_0 + \alpha) V (K\alpha_0 + \alpha).$$

Consequently,

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial \lambda}\, \xi V \xi_{\lambda=0} &= -(1 + K)\alpha_p V \alpha_p + \alpha V \alpha_p \\
&= -\left[\frac{\alpha E - r_0}{\alpha_p E - r_0}\right]\alpha_p V \alpha_p + \alpha V \alpha_p \\
&= \alpha_p V \alpha_p \left[\frac{\alpha V \alpha_p}{\alpha_p V \alpha_p} - \frac{\alpha E - r_0}{\alpha_p E - r_0}\right] \\
&\lesseqgtr 0,
\end{aligned}$$

for $\alpha_p E - r_0 > 0$ as $\alpha\delta \gtreqless 0$. Q.E.D.

As we have seen in Section I, though, this result must be interpreted with great care. It only ensures us of a *marginal* improvement; it says nothing about the desirability of a global switch from α_p to α. Two alternative statements of what amounts to the same result were given in an earlier version of this paper. For brevity, we just mention them here. In both cases, we assume that the expected return on the market exceeds the zero-beta rate. Here are the results. First, if a portfolio plots above (below) the SML, then moving from the index towards (away from) the portfolio yields a position that dominates some points in F^*, in mean and variance. Second, this new position will be an improvement for any individual (with increasing differentiable preferences over mean and variance) who optimally chooses the index when constrained to F.

III. Roll's Conjecture

In an important paper, Roll [23] asserted that for any inefficient index α_p and corresponding zero-beta rate r_0, it is possible to find another index $\alpha_{p'}$ and corresponding zero-beta rate r_0' that linearly reverse the security market deviations. In other words, if we index δ by the choice of proxy and the riskless rate, then $\delta_{p,r_0} = -\delta_{p',r_0'}$. This is equivalent to saying that the two indices, p and p', evaluate each portfolio α exactly opposite to each other. This is an interesting way of looking at the misspecification problem, and Roll's assertion seems to contradict the positive result we obtained in Section I, for the case when there is a riskless asset. We will use the apparatus developed in Section I to examine Roll's assertion.

To start, we can show simply that the assertion, as it stands, is false when the global minimum variance portfolio is on the SML_{p,r_0}, as it is when there is a riskless asset. This will explain the contradiction between Roll's assertion and our positive result. Let $\alpha_{p'}$ be a portfolio which exactly reverses the orderings of α_p. Since $\alpha_{p'}$ will lie on its own SML, $\text{SML}_{p',r_0'}$ and since it exactly reverses α_p, then it must also lie on SML_{p,r_0} (i.e., it is in F)—see Figure 6. Furthermore, $\alpha_{p'}$ must be efficient relative to α_p, i.e., $\alpha_{p'} \in F^*$. If $\alpha_{p'}$ were not in F^*, then the relatively efficient portfolio with the same mean would be on the SML_{p,r_0}, but (by Theorem 1) it would plot either above or below SML_{p,r_0}, depending on whether r_0' is less than or greater than the expected return on $\alpha_{p'}$.

But, having now shown that only a portfolio of F^* can exactly reverse the orderings of α_p, we see from Figure 6 and the analysis of Section I that we have a contradiction, since whether the new index has a mean greater or less than the minimum variance portfolio, the sign of the SML deviations of all portfolios lying under F^* is unaltered, i.e., there can be no reversals.[10] This completes our demonstration that Roll's assertion is wrong when there is a riskless asset, or more generally when F^* and G^* meet at the global minimum variance portfolio.

Roll's assertion is true, however, in the interesting case where F^* intersects G^* at a return in excess of the return on the global minimum variance portfolio. In this case, for every portfolio α_p in a neighborhood of the intersection point,

[10] In the absence of a riskless asset, a less pathological counterexample can be constructed if we require the reversing proxy, $\alpha_{p'}$, to have a positive excess return. Now no reversal will be possible whenever F^* intersects G^* at a point with an expected return less than the return on the global minimum variance portfolio. This case is as in Figure 4 (top).

By the same argument as in the text, any portfolio, $\alpha_{p'}$ which linearly reversed α_p's rankings would also have to be an element of F^*. But, since we require α_p to have a positive excess return, its return must exceed the return on the global minimum variance portfolio. By Figure 4 and the analysis of Section I, though, both α_p and $\alpha_{p'}$ will give the same sign to assets below F^*. Once again, no reversal is possible.

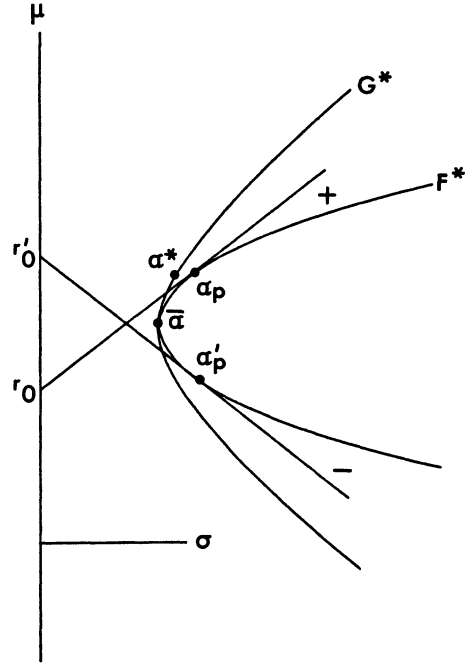

Figure 6. This is an illustration of the case in which Roll's assertion fails, and no reversal is possible. The riskless asset case illustrated in Figure 5 is a special subcase.

α_m, a portfolio $\alpha_{p'}$ can be found on the other side of the intersection which exactly reverses the rankings of α_p. (The analysis below will verify this point.) This case is illustrated by using Figure 4 (top and bottom) with r_0 labeled as r_{0p} and $r_{0p'}$ respectively, and with the intersection point of F^* and G^* fixed at α_m in both.

Notice that this case will arise whenever the market portfolio is efficient relative to the index and is, therefore, at the intersection point. [It is also interesting to observe from Figure 4 (top and bottom) that in every neighborhood of this intersection point there are pairs of inefficient indices arbitrarily near the efficient frontier, G^*, which give exactly opposite orderings.[11]]

Returning to the analysis above, though, we must conclude that an exact linear reversal of any index is not generally possible. But by weakening the requirement a bit, we can still preserve its spirit. A portfolio $\alpha_{p'}$ with its associated zero-beta rate r_0' reverses the ordering of α_p *affinely* if

$$\delta_{p',r_0'} = \phi\delta_{p,r_0} + \lambda e, \tag{8}$$

for some $\phi < 0$. Notice that the constant term plays no role in the ordering since

$$\xi e = 1$$

[11] For an index whose zero-beta rate makes the index close to the frontier intersection point in Figure 4 (top), move along F^* to a new index past the intersection point, and choose r_0 to be tangent to the original index's frontier in mean-standard deviation space. Clearly F^* stays the same, but there is a linear reversal of the SML deviations, and we have the case in Figure 4 (bottom). To get exact reversals clearly requires only a more careful choice of the reversing index.

for all portfolios, ξ. To find a portfolio $\alpha_{p'}$ that reverses the ordering of α_p, we can solve the above equation for $\phi < 0$ and any λ. Previously, we required an exact linear reversal (as Roll did), i.e.,

$$\phi = -1, \quad \lambda = 0.$$

Now we are requiring an affine reversal, i.e., linear plus a constant. Notice that it will indeed be the case that for any two portfolios, ξ and ξ^*,

$$\delta_{p',r_0}\xi > \delta_{p',r_0}\xi^*$$

if and only if $\delta_{p,r_0}\xi > \delta_{p,r_0}\xi^*$ when $\phi > 0$, and
if and only if $\delta_{p,r_0}\xi < \delta_{p,r_0}\xi^*$ when $\phi < 0$.

Letting π and π' be the respective prices of risk,

$$\pi = \frac{\alpha_p E - r_0}{\alpha_p V \alpha_p}$$

and

$$\pi' \equiv \frac{\alpha_{p'} E - r_{0'}}{\alpha_{p'} V \alpha_{p'}}, \tag{9}$$

Equation (8) requires

$$E - r_0' e - \pi' V \alpha_{p'} = \phi[E - r_0 e - \pi V \alpha_p] + \lambda e,$$

which implies (assuming no singularities) that

$$\alpha_{p'} = \left(\frac{\phi\pi}{\pi'}\right)\alpha_p + \left(\frac{1-\phi}{\pi'}\right)V^{-1}E - \left(\frac{\lambda + r_0' - \phi r_0}{\pi'}\right)V^{-1}e, \tag{10}$$

a linear combination of α_p (the inefficient index), $V^{-1}e$ (the global minimum variance portfolio), and $V^{-1}E$ (a portfolio which, with $V^{-1}e$, spans G^*).

Now, for any α_p and π, and any choice of ϕ, we can solve the budget constraint,

$$\alpha_{p'} e = 1, \tag{11}$$

by simply fixing

$$\lambda + r_0' - \phi r_0 = \lambda^*, \tag{12}$$

for some prespecified λ^*. Adjusting π' so that $\alpha_{p'}$ satisfies the portfolio constraint (11) is always possible since, while r_0' changes with π', λ can be a slack variable. Notice that without this freedom, particular values of $\phi < 0$ may not permit a reversing portfolio: that was the problem we had before with the constraints $\lambda = 0$ and $\phi = -1$.

In conclusion, then, although we have altered slightly the mathematical formulation of Roll's assertion, we have verified its intent, at least in the absence of a riskless asset.

IV. Conclusions

We have attempted in this paper to develop the analytic foundations for SML analysis based solely on the mean-variance framework. To do so, we have focused on the case where the index portfolio is inefficient with respect to the global mean-variance frontier. In this case we have found that SML analysis and mean-variance analysis of performance do not coincide, even in the presence of a riskless asset. A security or portfolio that is mean-variance efficient relative to the index must plot above its SML, but mean-variance inferior as well as superior portfolios will also plot above the SML. More generally, efficient and inefficient portfolios can plot above and below the SML. Nevertheless, the SML does provide an accurate guide for marginally improving an inefficient portfolio. As a final problem, we have examined Roll's assertion and essentially verified that the ranking of portfolios by the SML analysis is sensitive to the choice of the index used for the ranking. Indices very close to each other and to a truly efficient index can completely reverse rankings.[12]

Ultimately, though, the usefulness of these or any other analytical tools for performance measurement must turn on their behavior in the presence of differential information. Presumably, this is the major rationale for claims by managers of superior performance. We defer these issues to the companion paper (Dybvig and Ross [8]). For the present, though, we must conclude that even on its own turf, with no information issues at hand, the security market analysis does not provide a reliable global tool for either event studies or performance measurement. To put the matter bluntly, SML analysis tells us more about the index we are using than it tells us about the portfolio we are evaluating.

[12] Of course, magnitudes will be close and the severity of this problem ultimately turns on the choice of a loss function. Implicitly, we are assuming that rank reversals are important. See Green [12].

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