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Author(s): David P. Brown and Michael R. Gibbons

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A Simple Econometric Approach for Utility-Based Asset Pricing Models

DAVID P. BROWN and MICHAEL R. GIBBONS*

ABSTRACT

Utility-based models of asset pricing may be estimated with or without assuming a distribution for security returns; both approaches are developed and compared here. The chief strength of a parametric estimator lies in its computational simplicity and statistical efficiency when the added distributional assumption is true. In contrast, the nonparametric estimator is robust to departures from any particular distribution, and it is more consistent with the spirit underlying utility-based asset pricing models since the distribution of asset returns remains unspecified even in the empirical work. The nonparametric approach turns out to be easy to implement with precision nearly indistinguishable from its parametric counterpart in this particular application. The application shows that log utility is consistent with the data over the period 1926–1981.

THE DISTRIBUTION OF ASSET returns is a fundamental quantity to be explained by financial economics. Consequently, utility-based models of asset pricing are of special interest since they allow the distributions of returns to be explained rather than assumed as in distribution-based models.

Despite this attractive feature, preference-based theories have received little empirical attention in financial economics. Perhaps there is a belief that a rigorous econometric investigation of these models requires distributional assumptions about asset returns. As a result, empiricists may view utility-based paradigms as unattractive, for the appropriate econometric models are burdened not only by assumptions about distributions but also preferences.

In the hope of stimulating empirical interest in utility-based models of asset pricing, this paper presents a statistical methodology that is appropriate for this class of theories. Since the suggested approach does not rely on strong distributional assumptions, it preserves the inherent attractiveness of preference-based models. To develop the intuition for the suggested methodology, the paper focuses on a particular equilibrium model for asset pricing that was developed in the seminal paper by Rubinstein [34]. This model provides a simple economic environment that is characterized by a power (isoelastic) aggregate utility function and a constant investment opportunity set. Once this situation is analyzed,

* School of Business, Indiana University and the Graduate School of Business, Stanford University, respectively. This research has benefited from the helpful comments made by our friends at Dartmouth, Harvard, Stanford, UBC, Wharton, and Yale where this paper was presented. We also thank Wayne Ferson, Allan Kleidon, Paul Pfleiderer, Peter Reiss, Mark Rubinstein, and Ken Singleton for many useful discussions. Of course, all remaining errors are our responsibility. Financial support was provided in part by the Stanford Program in Finance. This work was completed while the second author was a Batterymarch Fellow. Both sources of financial support are gratefully acknowledged.

extensions to more general, and admittedly more interesting, frameworks are reasonably straightforward.

However, the specific case under study has importance beyond its pedagogical value. Since the unknown parameter of isoelastic preferences is relative risk aversion (RRA), analyzing this utility function provides valuable information for several reasons. First, some theoretical results rely on log utility (e.g., Cox, Ingersoll, and Ross [4], Hakansson [14], Kraus and Litzenberger [23], Merton [27], Rubinstein [35], and Samuelson [36]); by estimating the difference between RRA and one, the appropriateness of these results can be judged. Second, the effects of changes in risk on the demand for risky assets and the savings decision depend on the magnitude of RRA (e.g., see Rothschild and Stiglitz [31]). Third, some of the current debate as to whether or not stock prices have excessive volatility relates to the degree of aggregate risk aversion (e.g., see Grossman and Shiller [12]). Indeed, the importance of RRA in financial economics can be judged by the number of papers which have also attempted to determine its value. A sample of such papers includes Ferson [7], Friend and Blume [8], Grossman and Shiller [12], and Hansen and Singleton [18, 19].

In fact, Hansen and Singleton [19] develop a methodology which also relies on very weak assumptions about the distribution of returns. Their paper considers a general economic environment which includes the economic model of this paper as a special case.¹ The generality that they achieve is important, but the intuition underlying their methods is more easily seen by focusing on a simpler setting as in this paper. Thus, one of the contributions of this work is that it may serve as a useful pedagogical introduction to Hansen and Singleton [19].

The next section provides the theoretical underpinnings of the econometric analysis, and a structural relation between RRA and certain moments of asset returns is established. Two alternative estimators of RRA are developed in the second and third sections. The first estimator is a parametric one, and it has the advantage of tractability. Unfortunately, this estimator depends critically on the distributional assumption (made in earlier studies of RRA) which is neither an implication of the theory nor consistent with the data. The second estimator is nonparametric. While it requires nonlinear estimation, the approach is intuitive and easy to implement. In Section IV, these two alternative approaches are applied to estimate RRA using monthly stock return data from 1926–1981. The relation between the nonparametric estimator and the approach developed by Hansen and Singleton [19] is discussed in Section V. Section VI concludes the paper.

I. The Relation between Preferences and the Moments of Security Returns

The well-known Euler condition for the dynamic consumption-portfolio problem faced by a representative individual under uncertainty is used to derive a relationship between relative risk aversion and the moments of security returns.

¹ However, this application was derived independently of Hansen and Singleton [19].

This first-order necessary condition for optimality with a time additive, von Neumann-Morgenstern utility function is

$$E\left[\frac{1}{1+r} \frac{U'(\tilde{C}_t)}{U'(C_{t-1})} (1 + \tilde{R}_{it}) | Z_{t-1}^*\right] = 1 \quad \forall i = 1, \dots, N \quad \forall t = 1, \dots, T \quad (1)$$

where

- $U'(\tilde{C}_t)$ = marginal utility in period t from consumption, $\tilde{C}_t$;
- r = rate of pure time discount;
- $\tilde{R}_{it}$ = return on asset i in period t ; and
- Z_{t-1}^* = information set available to the market in period $t - 1$.

This first-order condition reflects the loss of marginal utility of consumption today if one additional share of a security is purchased versus the gain in expected marginal utility tomorrow when the share is sold and the return is consumed. Among others, Lucas [25] discusses this equation in the discrete time case, and Grossman and Shiller [13] provide a derivation in the continuous time setting. Hansen, Richard, and Singleton [20] have also emphasized the importance of Equation (1) for econometric analyses of asset pricing models. One feature of (1) is that the equality holds for all time horizons. If agents make decisions daily, (1) is still relevant to the econometrician who has data sampled at monthly intervals; thus, by working with (1) temporal aggregation bias may be avoided.²

Without additional assumptions, Equation (1) provides little guidance for empirical research. In the work that follows, power (or isoelastic) utility is specified; that is,

$$U(C_t) = \frac{C_t^{1-B} - 1}{1-B}, \quad (2)$$

where $B = -U''(C_t)C_t/U'(C_t)$, relative risk aversion. By assuming isoelastic utility, Equation (2) can be restated as

$$E\left[\frac{1}{1+r} \left(\frac{\tilde{C}_t}{C_{t-1}}\right)^{-B} (1 + \tilde{R}_{it}) | Z_{t-1}^*\right] = 1. \quad (3)$$

Power utility is a natural choice because of its desirable theoretical properties. As a member of the HARA class of utility functions, Rubinstein [33] has established its aggregation properties over individuals in the economy. Further, log utility is a special case of isoelastic utility as B approaches one. Given the theoretical attractiveness of log utility (e.g., see Rubinstein [35]) as well as its use in theoretical work (see introductory section for examples), its empirical relevance needs investigation. Since B is estimated, its distance from one can be determined. Finally, Arrow [2], among others, has emphasized that absolute risk aversion should be decreasing, and power utility displays this characteristic.

² As Ken Singleton has emphasized to us, temporal aggregation bias is avoided as long as the econometrician has access to instantaneous consumption sampled at discrete intervals. Since Equation (1) is transformed into a statement involving the return on the market index, rather than consumption, temporal aggregation bias is avoided in this paper even without data on instantaneous consumption.

If appropriate measures of consumption were available,³ Equation (3) could be transformed to yield empirical implications. However, much of the early research on utility-based asset pricing models replaced aggregate consumption with the return on some proxy for the market portfolio; this substitution is also employed here to avoid measurement problems with consumption as well as to relate to these earlier studies. With additional assumptions Equation (3) can be rewritten as:

$$E\left[\frac{1}{1+r} (1-k)^{-B}(1+\tilde{R}_{mt})^{-B}(1+\tilde{R}_{it}) | Z_{t-1}^*\right] = 1, \quad (4)$$

where

R_{mt} = return on the market portfolio and

k = proportion of wealth consumed in every period (i.e., if C is consumption and W is wealth, then $C = kW$).

Hakansson [14] points out that consumption is a constant proportion of wealth if individuals with infinite horizons have isoelastic utility functions, and the distribution of real production opportunities is constant and characterized by constant stochastic returns to scale.⁴ Alternatively, in a pure exchange economy, Rubinstein [34] demonstrates that if aggregate consumption growth rates are independently and identically distributed over time or if B equals one then (4) results with the return on the market index being independently and identically distributed as well.⁵ In fact, (4) is consistent with an equilibrium in *any* production economy in which aggregate consumption growth rates are independently and identically distributed.

Substituting $\tilde{R}_{mt}$ and R_{ft} (the return on the one period riskless bond) for $\tilde{R}_{it}$ in (4) and equating the left-hand sides of the resulting relations, it is evident after

³ Breeden, Gibbons, and Litzenberger [3] discuss some of the problems with measuring aggregate consumption over discrete time intervals. These problems include not only pure measurement error which arises from the necessity to sample aggregate consumption but also problems with the distinction between expenditures on goods and services versus actual consumption as well as temporal aggregation issues (see footnote 2).

⁴ While a constant real investment opportunity set is an extreme and unrealistic assumption, the advantages of generalizing this assumption to estimate RRA are by no means clear. Indeed, there is limited empirical evidence to support constant investment opportunities. For example, Hall [15] found consumption growth rates to be uncorrelated over time, and Fama [5] found that a constant expected real rate of interest was a plausible assumption for some research objectives. Of course, the theoretical assumptions leading to Equation (5) have the unfortunate implication that aggregate consumption growth will be perfectly correlated with growth in aggregate wealth.

⁵ Rubinstein [34] derives the more general result that the proportion of wealth consumed, k , changes in a deterministic fashion if time preference, r , is not constant and/or if investors have a finite time horizon. Rubinstein's [34] terminology differs from that in the text, for he assumes the growth in consumption is a "stationary random walk" which is used to imply that the growth rate is serially uncorrelated with identically distributed innovations. Of course, Rubinstein [34] does not assume that all asset returns (e.g., options) have stationary distributions—just the two "basic" portfolios (the riskless asset and the market portfolio).

a simple rearrangement that

$$\begin{aligned} E[(1 + \tilde{R}_{mt})^{1-B} | Z_{t-1}^*] &= (1 + r)(1 - k)^B \\ &= E[(1 + \tilde{R}_{mt})^{-B}(1 + R_{ft}) | Z_{t-1}^*] \\ E[(1 + \tilde{R}_{mt})^{1-B} | Z_{t-1}^*] - E[(1 + \tilde{R}_{mt})^{-B}(1 + R_{ft}) | Z_{t-1}^*] &= 0. \end{aligned} \quad (5)$$

Since a riskless asset has a nonstochastic return conditional on Z_{t-1}^* , R_{ft} can be brought outside the expectation operator. With a little algebraic manipulation,

$$E[(\tilde{x}_t - 1)\tilde{x}_t^{-B} | Z_{t-1}^*] = 0, \quad (6)$$

where $\tilde{x}_t = (1 + \tilde{R}_{mt})/(1 + R_{ft})$, a discrete time “excess return.” Equation (6) has implications not only for moments conditional on complete information but also conditional on coarser information. Assuming the relevant moments exist, the law of iterated expectations applied to (6) implies

$$E[(\tilde{x}_t - 1)\tilde{x}_t^{-B}] = 0. \quad (7)$$

Even if individuals have changing conditional expectations through time, the econometrician, by relying on (7), can still construct a valid test of the theory based on unconditional and fixed moments. There is no need to specify a model for the conditional expectations or even the variables which affect these conditional expectations. The significance of the law of iterated expectations for asset pricing models has been emphasized by Gibbons and Ferson [9], Grossman and Shiller [13], and Hansen and Singleton [18, 19].

Equation (7), which will be used throughout the paper, follows from replacing consumption in Equation (3) with the market index and from assuming the existence of an observable return on a riskless asset. While the estimators which follow can easily be extended to situations not relying on these two assumptions, the cost of such an extension is the loss of intuition, for in that case one must analyze a relation involving two or more random variables whereas Equation (7) has but a single random variable.⁶ Of course, there is one interesting case where (7) applies even if the investment opportunity set changes through time. In the case of log utility (i.e., $B = 1$), the market index can be substituted for consumption without strong restrictions on the temporal characteristics of asset returns. This particular case deserves emphasis, for log utility is an important null hypothesis to examine given the attention that it has received in theoretical work. Thus, this null hypothesis is testable in this framework with only weak assumptions concerning the time series on returns.

The next section develops the relevant econometrics assuming a lognormal distribution for the excess return on the market; this analysis simplifies Equation (7). The third section then discusses the case of the method of moments solution which works directly with Equation (7).

⁶ For example, for any two assets (one asset may be the market index) an alternative to equation (7) is

$$E\left\{\left(\frac{\tilde{C}_t}{\tilde{C}_{t-1}}\right)^{-B} (\tilde{R}_{1t} - \tilde{R}_{2t})\right\} = 0$$

II. A Parametric Estimator of RRA

For a transformed version of Equation (7), Rubinstein [34] suggested a parametric estimator for B which assumes that the total return on the market portfolio has a lognormal distribution. Similarly, if one assumes that $\tilde{x}_t$, the excess return on the market, has a lognormal distribution, then

$$B = \frac{E(\ln \tilde{x}_t)}{\text{Var}(\ln \tilde{x}_t)} + 0.5. \quad (8)$$

The derivation of (8) is provided by Rubinstein [34].

Given that the excess return on the market portfolio has a lognormal distribution, Equation (8) provides a relationship between RRA and the first two moments of this log excess return. Since the $(\ln x_t)$ has a normal distribution, the maximum likelihood estimators for $E\{\ln x_t\}$ and $\text{Var}\{\ln x_t\}$ are their sample equivalents⁷ ($\overline{\ln x}$ and s^2 , respectively). Since a function of maximum likelihood estimators is also a maximum likelihood estimator (Zehna [39]), the following has all of the well-known properties of maximum likelihood (i.e., consistency, asymptotic efficiency, and asymptotic normality):

$$\hat{b} = \frac{\overline{\ln x}}{s^2} + 0.5. \quad (9)$$

Determining the variance of the asymptotic distribution of $\hat{b}$ is straightforward given the variances of the asymptotic distributions of $\overline{\ln x}$ and s^2 . The normality of $(\ln x_t)$ implies that these two sample moments are independent with small sample variances:

$$\text{Var}\{\sqrt{T}(\overline{\ln x_t})\} = \text{Var}\{\ln \tilde{x}_t\}$$

and

$$\text{Var}\{\sqrt{T}s^2\} = \frac{2T}{T-1} [\text{Var}\{\ln \tilde{x}_t\}]^2,$$

where T = the number of observations. Combining the above with standard asymptotic theory⁸ yields:

$$\text{Var}\{\sqrt{T}\hat{b}\} = \frac{2[E\{\ln \tilde{x}_t\}]^2 + \text{Var}\{\ln \tilde{x}_t\}}{[\text{Var}\{\ln \tilde{x}_t\}]^2}. \quad (10)$$

The parametric approach has a shortcoming; the estimator (9) can be inconsistent if the lognormal assumption is violated. This lack of robustness is not surprising, for the functional form of (9) depends on the simplifications achieved by this distributional assumption.

⁷ Throughout this paper, the sample variance refers to the unbiased estimator using a degrees-of-freedom adjustment, not the true maximum likelihood estimator. This has no effect on the asymptotic properties.

⁸ The interested reader should consult Serfling [37; pp. 118–125] for theorems which provide the asymptotic distribution of a random variable given the asymptotic distribution of the random variables of which it is a function.

Table I
Inconsistency of the Parametric Estimator When Lognormal Distribution Violated

Parameters of the Underlying Distribution				
Probability of First Outcome	Value of First Outcome ($\ln x_1$)	Value of Second Outcome ($\ln x_2$)	$E\{[\ln x_t - E(\ln x_t)]^3\}$	True RRA
0.000	-5.311	0.005	-0.01771	0.706
0.331	-0.078	0.045	-0.00014	1.779
0.661	-0.037	0.085	0.00013	1.852
0.814	-0.023	0.125	0.00031	1.907
0.886	-0.016	0.165	0.00047	1.959
0.924	-0.012	0.205	0.00061	2.013
0.946	-0.009	0.245	0.00076	2.071
0.959	-0.008	0.285	0.00090	2.134
0.969	-0.006	0.325	0.00103	2.203
0.975	-0.005	0.365	0.00117	2.280
0.980	-0.004	0.405	0.00131	2.367
0.983	-0.003	0.445	0.00144	2.466
0.986	-0.003	0.485	0.00158	2.581
0.988	-0.002	0.525	0.00171	2.718
0.990	-0.002	0.565	0.00185	2.885
0.991	-0.001	0.605	0.00198	3.096
0.992	-0.001	0.645	0.00212	3.380
0.993	-0.001	0.685	0.00225	3.798
0.994	-0.000	0.725	0.00239	4.550
0.994	-0.000	0.765	0.00252	8.633

Note: The above values are based on a discrete distribution with only two outcomes for the excess return on the market index. In all cases, the outcomes and the probabilities are changed so that the first two moments are constant across all rows. These moments were set so that $E\{\ln x_t\} = 0.0044$ and $SD\{\ln x_t\} = 0.0577$ which corresponds to the CRSP value-weighted index for the period 1926–1981. Since the first two moments are constant, the lognormal estimator for RRA would converge to 1.81 for all rows even though this is not the true value of RRA.

A simple example demonstrates that the parametric estimator is inconsistent if the normality assumption is violated. If $(\ln x_t)$ is actually generated from a discrete distribution with only two outcomes, then this distribution can be characterized by three parameters—the value of each outcome and the probability of any one outcome.⁹ These parameters may be varied in such a way so that the mean and the variance of the $(\ln x_t)$ are constant yet the skewness of the distribution changes. If the mean and the variance remain constant, then the estimator given in (9) will always converge to the same value. However, true RRA as given by (7) is not a function of just the first two moments, so RRA may change as the skewness changes. Table I provides a range of values for RRA when the mean and the variance equal typical values for the log excess return on

⁹ Obviously, this distribution was not selected for its realism. Rather, a two-outcome distribution allows for an easy analytic solution to Equation (7). Alternative distributions may be used, but these could require extensive numerical integration. Since the purpose here is only to illustrate the concept, more realistic distributions were not analyzed.

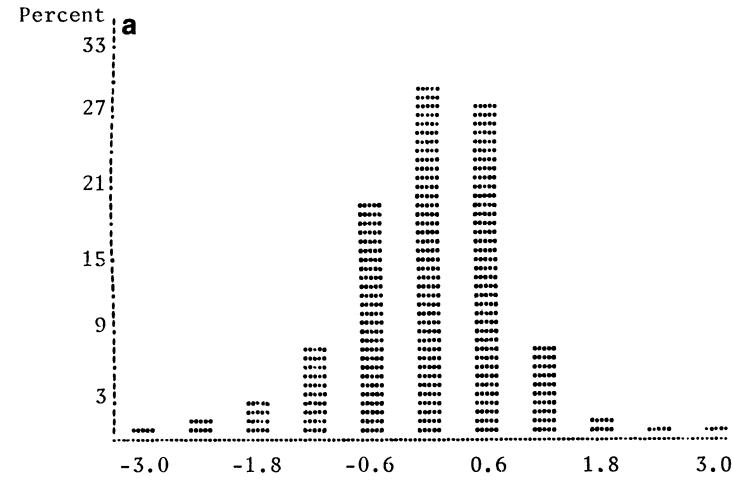


Figure 1a. CRSP Value-Weighted Index, 1/1926-12/1981

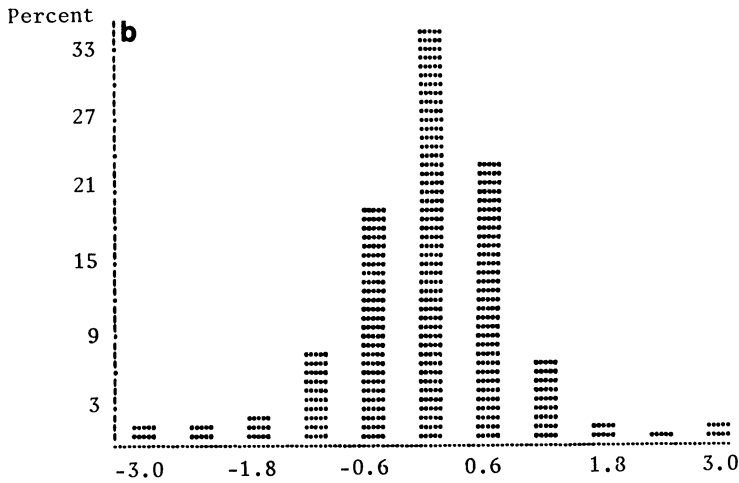


Figure 1b. CRSP Equal-Weighted Index, 1/1926-12/1981

Figure 1. Histograms of Standardized Monthly “Excess Returns” on Stock Market Indexes (The left-tail of each histogram has been truncated)

the value-weighted NYSE index. Clearly, the parametric estimator is not robust to this departure from normality of the log of the excess returns.¹⁰

Since the distributional assumption is critical in deriving (9), Figures 1a and 1b provide histograms of the log excess returns on the value-weighted NYSE index and the equal-weighted NYSE index for monthly data from 1926–1981. (A more detailed data description is provided in Section IV.) Table II quantifies the graphical presentations by providing summary statistics for the overall period

¹⁰ Interestingly, this example suggests that negative sample skewness results in the parametric estimator overstating RRA and conversely. The empirical evidence in Table III is consistent with the relationship observed in this illustration.

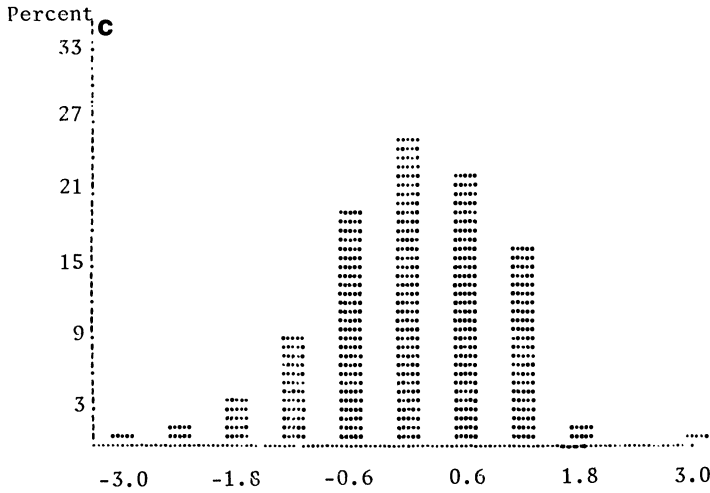


Figure 1c. CRSP Value-Weighted Index, 7/1939–12/1952

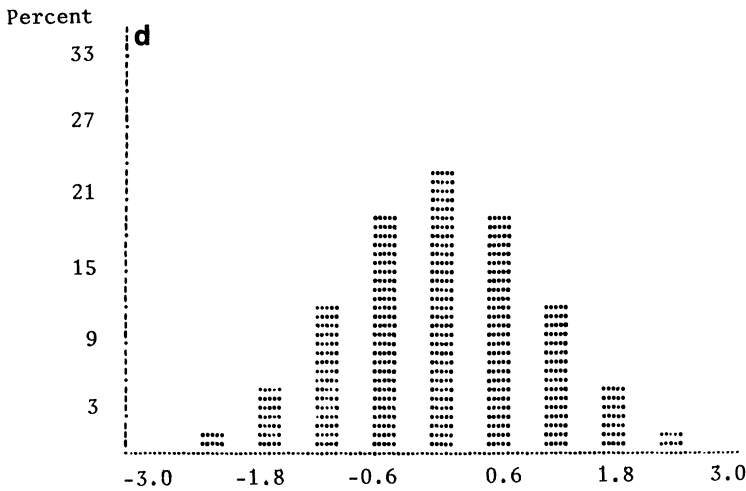


Figure 1d. Standardized Normal Distribution

Figure 1—continued

and several subperiods. The negative sample skewness of the value-weighted index, which is apparent in the histogram, is confirmed in Table II. Further, the Kolmogorov D -statistic, which is also reported in the second table, rejects a normality assumption for either index. The next section proposes an alternative estimation scheme.

III. A Method of Moments Estimator of RRA

An advantage of utility-based pricing theories is the lack of strong distributional assumptions regarding asset returns. If statistical inference concerning such theories necessitates additional distributional assumptions, then these theoretical

Table II
Summary Statistics on the Log of Monthly "Excess Returns" on the Value-Weighted and Equal-Weighted Indexes (1926-1981)

Subperiod (No. of Observations)	Value-Weighted Index				Equal-Weighted Index			
	Sample Mean	Sample Std. Dev.	Sample Skewness ^c	D-Statistic for Normality ^b		Sample Std. Dev.	Sample Skewness	D-Statistic for Normality (p-Value)
				Sample Mean	(p-Value)			
1/1926-12/1981 (672)	0.0044	0.0577	-0.360	0.0071	0.081 (<0.01)	0.0769	0.551	0.098 (<0.01)
1/1926-12/1952 (324)	0.0053	0.0716	-0.353	0.0086	0.099 (<0.01)	0.0975	0.551	0.111 (<0.01)
1/1953-12/1981 (348)	0.0035	0.0409	-0.339	0.0057	0.054 (0.01)	0.0506	-0.154	0.059 (0.01)
1/1926-6/1939 (162)	0.0010	0.0909	-0.085	0.0032	0.094 (<0.01)	0.1228	0.681	0.101 (<0.01)
7/1939-12/1952 (162)	0.0097	0.0446	-1.298	0.0140	0.070 (0.05)	0.0629	0.185	0.082 (<0.01)
1/1953-6/1967 (174)	0.0078	0.0339	-0.450	0.0090	0.067 (0.05)	0.0377	-0.264	0.075 (0.02)
7/1967-12/1981 (174)	-0.0009	0.0465	-0.157	0.0021	0.062 (0.10)	0.0608	-0.006	0.047 (>0.15)

^a The sample skewness is calculated as the third sample moment *after* the random variable has been standardized so that its sample mean equals zero and sample standard deviation equals one. Under normality, this measure of skewness should equal approximately zero.

^b The D-statistic is the usual Kolmogorov test for normality.

models are burdened by assumptions concerning both preferences as well as the probability distributions for returns. Thus, this paper attempts to estimate B directly from Equation (7) with a minimal amount of additional distributional assumptions; in this paper such a solution is referred to as a method of moments.

An obvious estimator of RRA follows directly from Equation (7). Replacing the population moment in (7) with its sample equivalent, RRA could be estimated by finding that value of b such that

$$f(b) = \frac{1}{T} \sum_t (x_t - 1)x_t^{-b} = 0 \quad (11)$$

where $b \in (0, +\infty)$. Under reasonable assumptions a unique value of b which satisfies (11) always exists.¹¹ Various search procedures are available to find the appropriate value of b ; Section IV employs a simple gradient search algorithm which was quite satisfactory in all the data analysis.

The only difficulty with this method of moments estimator is the determination of the appropriate measure of its precision. Under the assumption that $\hat{x}_t$ is independently and identically distributed,¹² the following equation for the variance of the asymptotic distribution of b is derived in Appendix A:

$$\text{Var}(\sqrt{T}b) = \frac{E\{[(\hat{x}_t - 1)\hat{x}_t^{-B}]^2\}}{[E\{(\hat{x}_t - 1)\hat{x}_t^{-B} \ln \hat{x}_t\}]^2} \quad (12)$$

Fortunately, Equation (12) has a nice intuitive interpretation. In solving (11) for b , one could draw a graph with $f(b)$ on the vertical axis and b on the horizontal axis. When the graph of $f(b)$ crosses the horizontal axis, this point is b . Figure 3 illustrates a typical graph of $f(b)$. The denominator of (12) is the square of the

¹¹ In solving (11) for b , one could draw a graph with $f(b)$ on the vertical axis and b on the horizontal axis. If investors are risk-averse (i.e., $B > 0$), then $E\{\hat{x}_t\} > 1$, and to solve (11) for b , it is assumed that $f(b = 0) = (1/T) \sum_t (x_t - 1) > 0$. Furthermore, as long as the sample contains at least one observation on x_t different from one, then $f'(b)$ is strictly negative. Thus, the graph of $f(b)$ starts above the horizontal axis at $b = 0$ and monotonically declines towards that axis. To guarantee a solution for (11) requires that the graph of $f(b)$ eventually crosses the horizontal axis. If the sample contains one observation on x_t less than one, then it is easy to verify that $\lim_{b \rightarrow \infty} f(b) = -\infty$. Figure 3 illustrates a typical graph of $f(b)$. Using a similar line of argument, one can establish the existence of a unique solution to (7), the population counterpart to (11), as long as: (i) $E[x_t^{-B}(x_t - 1) \ln x_t]$ and (ii) $E[x_t^{-B}(x_t - 1)(x_t^\lambda - 1)/\lambda]$, for some $\lambda > 0$, are finite. These conditions assure that the derivative of (7), the limit of (ii) as λ declines toward 0, exists and is equal to (i). See Theorems 6 and 15 in Royden [32; pp. 81 and 88].

¹² While a specific distributional assumption about returns has been avoided, the requirement of independent and identical draws for $\hat{x}_t$ is still quite strong. However, the approach can be extended using generalization of the Lindeberg-Levy Central Limit Theorem (Hansen [17] and Hansen and Singleton [19]), and the formula (12) is *exactly* the same. Since the theoretical justification in Section I for replacing consumption with the market return is more consistent with the simple version of the Central Limit Theorem, this generalization has not been emphasized. Of course, if the null hypothesis is that of log utility (i.e., $B = 1$), then allowing for dependent and nonidentical draws for $\hat{x}_t$ is an important extension since such time series properties are consistent with the theory used to derive (7). Fortunately, Equation (12) is general enough to handle such a situation. Equation (12) still obtains when the conditional investment opportunity set changes through time as long as $\hat{x}_t$ is a stationary and ergodic stochastic process. (For example, the economic model of Cox, Ingersoll, and Ross [4] implies a changing conditional investment opportunity set with returns which follow a stationary and ergodic stochastic process.)

slope of $f(b)$ at B while the numerator equals the variability of the line at B . As the variance of the line increases or as the absolute value of the slope of the line decreases, the asymptotic variance increases. Such behavior in the variance of b is to be expected. If the line in Figure 3 has more ex ante variability at the crossing point on the horizontal axis or if the line flattens out near the crossing point, then selecting a point estimate (or finding the crossing point) becomes more difficult.

Since the method of moments estimator does not rely on assumptions about the exact distribution of $\hat{x}_t$, (11) is robust to departures from lognormality of $\hat{x}_t$, unlike its parametric counterpart. The price of this robustness is the potential loss of statistical efficiency if the distributional assumption is appropriate. This disadvantage is analyzed by studying the relative asymptotic efficiency, the ratio of the asymptotic variances of the parametric estimator [given in (10)] divided by that of the method of moments approach [see (12)]. Assuming lognormality, Appendix B derives this relative efficiency, RE , as:

$$RE = \frac{\text{Var}(\hat{b})}{\text{Var}(b)} = \frac{2\mu^2 + \sigma^2}{\exp\left\{\frac{\mu^2}{\sigma^2} + \mu + \frac{\sigma^2}{4}\right\} \left[1 + \exp\{-2\mu\} - 2 \exp\left\{-\mu - \frac{1}{2}\sigma^2\right\}\right]}$$

where $\mu = E\{\ln \hat{x}_t\}$ and $\sigma^2 = \text{Var}\{\ln \hat{x}_t\}$.

Figure 2 summarizes the inefficiency of the nonparametric estimator versus its parametric counterpart. Each curve represents various combinations of $E\{\ln \hat{x}_t\}$ and $\text{Var}\{\ln \hat{x}_t\}$ which yield the same relative efficiency when x_t is lognormal. The points on the plot represent estimated means and variances for the market index over different subperiods from 1926–1981. As is clear from the figure, typical values for the mean and variance suggest that 99% of the precision obtained by the parametric estimator, $\hat{b}$, is obtained by the nonparametric alternative, b . Given the robustness of the latter estimator, this small drop in precision makes the method of moments approach very attractive.

IV. Some Empirical Results

Other studies have attempted to estimate aggregate RRA. One class of studies has utilized cross-sectional surveys of the portfolio holdings of individuals to measure RRA using wealth, not consumption.¹³ For example, Friend and Blume [8] estimate RRA to be 2 and roughly constant across different levels of wealth.

The other class of research uses time series data on asset returns and aggregate consumption. Grossman and Shiller [12] claim RRA is close to 4 based on a graphical examination of the data. However, there was no attempt to estimate this value precisely or to determine confidence intervals about the estimate.¹⁴

¹³ In these studies, RRA is derived through a Taylor series approximation of the utility function. However, in the case where RRA is a constant through time (i.e., isoelastic utility), this approximation is not necessarily accurate for power utility functions with discrete time periods (Loistl [24]). The approximation may also break down in continuous time models if the stochastic process for security returns contains jump components.

¹⁴ Grossman and Shiller [11] provide a more rigorous estimation scheme for RRA. Their main focus is on working with consumption data which has been time averaged.

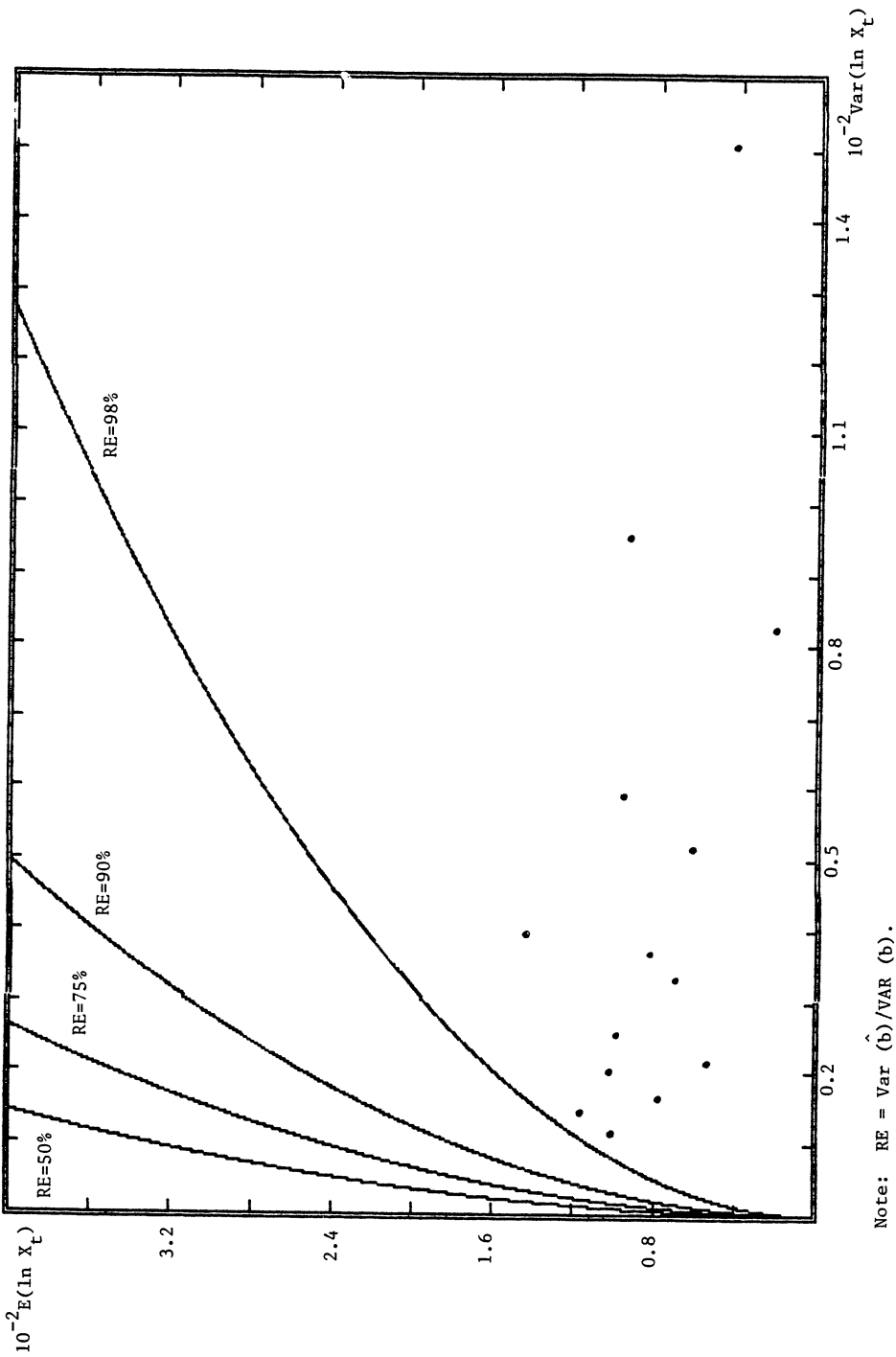


Figure 2. The Relative Efficiency (RE) of the Parametric versus Nonparametric Estimators of RRA for Different Pairs of $\{E(\ln X_t), \text{Var}(\ln X_t)\}$. Each Curve Represents all Pairs Which Yield a Particular Value of RE. The Points Represent Sample Means and Variances for CRSP Indexes over Different Subperiods

Hansen and Singleton [19] develop a clever approach involving instrumental variables. Using monthly consumption data and stock returns since 1959, they determine RRA to be somewhere between -1.6 and 1.6 .

RRA has been estimated by three groups of researchers using a dynamic model for expectations and a lognormal distribution assumption. With stock return data, Hansen and Singleton [18] estimate a range of values for RRA from 0.07 to 0.62 while Ferson [7], using bond data, reports a range from -1.4 to 5.4 . Both studies involve nonlinear techniques on postwar consumption data. With a similar methodology, but in a Bayesian framework, Hall [16] concludes that RRA is in a range from -26.3 to 25.6 , depending on whether stock returns or interest rates are employed with annual sampling intervals.

The empirical results in this section are based on monthly data from 1926–1981. Two proxies for the market portfolio are examined—the value-weighted and equal-weighted indexes of the New York Stock Exchange from the Center for Research in Security Prices (CRSP) at the University of Chicago. Interest rates are needed to form the “excess returns” on these market proxies. Prior to 1953, monthly returns¹⁵ on U.S. treasury bills are from Ibbotson and Sinquefeld [22]. From 1953–1981, yields on 30-day (approximately) bills are from the CRSP Government Bond File. While all returns are nominal, the transformation to “excess returns” may still satisfy the theoretical underpinnings given in Section I; a sufficient, but not necessary, condition is that inflation over the sampling interval is known with certainty.

Using the value-weighted index, Figure 3 illustrates the nonlinear search for the method of moments estimate. In all subperiods and for both indexes, convergence to the estimate occurred within three iterations.¹⁶ Using both the method of moments and the parametric alternative, Table III summarizes the estimates of RRA for both indexes across various subperiods.

In all but two subperiods (i.e., July 1939–December 1952 and January 1953–June 1967) estimated RRA is within 2 standard errors of 1; i.e., the data cannot reject log utility as the appropriate specification for the aggregate utility function. As was noted above, the formula for the standard error remains valid even if returns are not independent and identically distributed, so the acceptance of the null hypothesis that $B = 1$ holds for economic environments where the conditional investment opportunity set is not constant. Furthermore, the results for the subperiod 1939–1952 are not clearly rejecting log utility, for the method of moments estimate is within the usual confidence intervals. Since the parametric estimator is not robust, the method of moments estimate for 1939–1952 deserves the primary attention. Figure 1c visually confirms the departure from normality for the value-weighted index for this subperiod, and Table II quantifies the inadequacy of this distributional assumption.

¹⁵ Economic theory underlying (7) calls for interest rates on one-period bonds, not an uncertain return series which includes capital gains and losses. However, a reliable monthly interest rate prior to 1953 is difficult to construct. This deficiency in the data motivated the subperiod breakdown in the work that follows.

¹⁶ The starting point for the nonlinear solution to (11) was the parametric estimator given in (8). The convergence criterion was to continue to search for b until $[(1/T) \sum_i (x_i - 1)x_i^{-b}] < 0.0000001$.

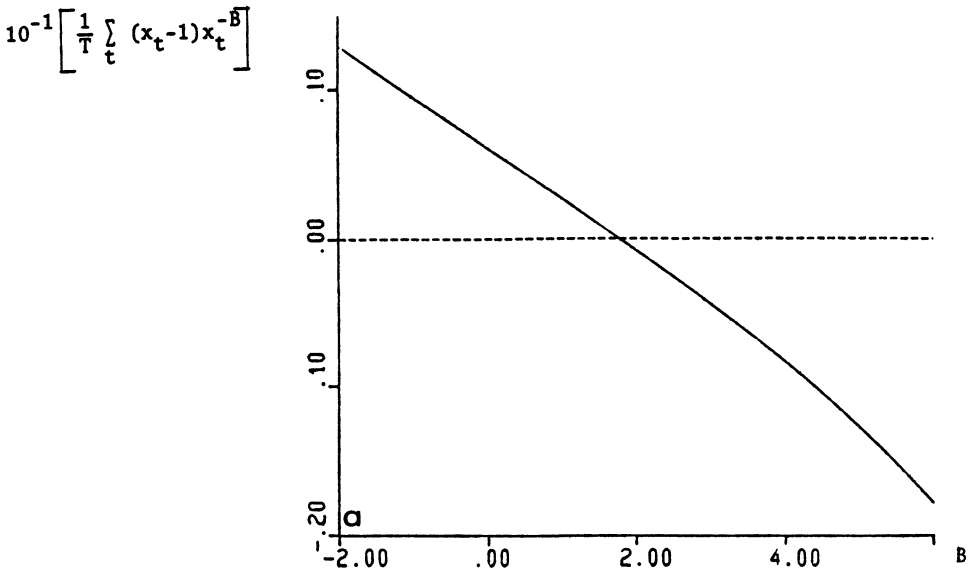


Figure 3a. A Fine Grid for the Axes

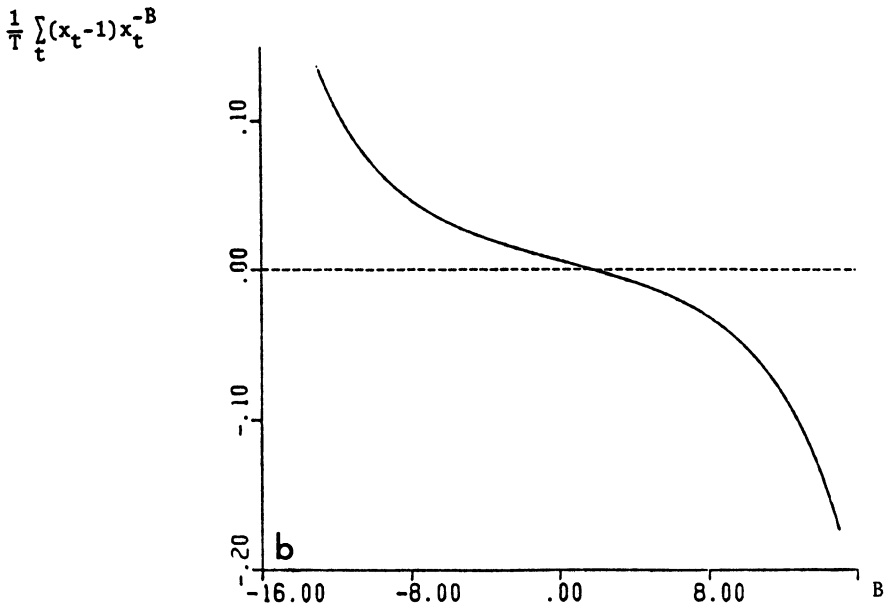


Figure 3b. A Coarse Grid for the Axes

Figure 3. Graphical Determination of the Value of RRA, B , for Which the Equilibrium Condition Is Satisfied, i.e., the Left-Hand Side of Equation (7) Must Equal Zero. (x_t is the excess return on the CRSP Value-Weighted Index, and the monthly observations are from 1/1926–12/1981).

Table III
Estimates of Relative Risk Aversion (1926-1981)

Subperiod (No. of Observations)	RRA Using Value-Weighted Index			RRA Using Equal-Weighted Index		
	Method of Moments	Parametric with Lognormal Distribution	Hausman's Specification Test ^a (p-Value)	Method of Moments	Parametric with Lognormal Distribution	Hausman's Specification Test (p-Value)
	(Std. Error)	(Std. Error)		(Std. Error)	(Std. Error)	
1/1926-12/1981 (672)	1.79 (0.68)	1.81 (0.67)	26.57 (<0.0001)	1.71 (0.52)	1.70 (0.51)	14.23 (0.0002)
1/1926-12/1952 (324)	1.52 (0.79)	1.54 (0.78)	5.43 (0.0198)	1.42 (0.59)	1.40 (0.57)	8.42 (0.0037)
1/1953-12/1981 (348)	2.55 (1.33)	2.58 (1.32)	18.46 (<0.0001)	2.68 (1.09)	2.70 (1.07)	6.05 (0.0139)
1/1926-6/1939 (162)	0.62 (0.87)	0.62 (0.86)	0.00 (1.00)	0.72 (0.64)	0.71 (0.64)	1.86 (0.1726)
7/1939-12/1952 (162)	4.69 (2.02)	5.35 (1.84)	103.92 (<0.0001)	3.81 (1.49)	4.03 (1.31)	15.66 (<0.0001)
1/1953-6/1967 (174)	7.00 (2.35)	7.29 (2.35)	-1256.88 ^b (0.00)	6.77 (2.17)	6.99 (2.13)	51.05 (<0.0001)
7/1967-12/1981 (174)	0.09 (1.64)	0.09 (1.63)	0.11 (0.7401)	1.06 (1.25)	1.06 (1.25)	0.11 (0.7401)

^a Hausman's [21] specification test is distributed asymptotically as chi square with one degree of freedom in this application.
^b The chi-square statistic is negative because the *estimated* asymptotic variance of the nonparametric estimator is less than that of its parametric counterpart. Under the null hypothesis, the precision of the two approaches should be reversed since the parametric estimator is a maximum likelihood estimator. Thus, the *p*-value is set equal to zero.

The results from the overall period suggest no *statistically* significant departure from log utility. The *economic* distinction between RRA equal to one versus (say) two may not be very important given the behavior of an individual to a timeless gamble (see Ferson [7] and Pratt [29]). An individual with log utility will pay a half basis point of wealth to avoid a fair coin toss which risks 1% of current wealth. When RRA is two, the individual will pay one basis point of wealth to avoid the same gamble.

Three final observations are based on Table III. First, the results are not sensitive to the index except for the last subperiod (July 1967–December 1981). Given the arbitrariness in selecting an index, such results are comforting.¹⁷ Second, the standard errors confirm the analysis in Section III on the relative asymptotic efficiency of the two alternative estimation schemes. While the precision of the lognormal estimator is greater than that of the method of moments, the difference is insignificant. Finally, Table III provides a test of the distributional assumption underlying the parametric estimator. Rather than compare the fit of log excess returns to a normal distribution (as was done in Table II with the Kolmogorov D -statistic), Hausman's [21] specification test can be used.¹⁸ This statistic compares the parametric and nonparametric estimates which should be equal in large samples if log excess returns are normally distributed. However, under the alternative hypothesis of nonnormality, the parametric estimator is not necessarily consistent, and the difference between the estimates may be nonzero. The advantage of the Hausman test is that it measures departures from normality with an interesting metric. Nonnormality is only important here to the extent that the parametric estimator is inconsistent, and the asymptotic chi-squared test reported in Table III directly examines this issue. Table III confirms the results in Table II, for log excess returns do violate normality in all but two subperiods for each index. However, despite the rejection of lognormality by the Hausman test, the most striking feature of Table III is the strong similarity between the method of moments and the parametric approach. For most subperiods, the difference between the two point estimates is economically insignificant. Thus, the potential problems with the parametric approach are not important in this application.

By working with the market index and not consumption, time preference cannot be estimated. Since portfolio allocation is not affected by time preference with isoelastic utility, consumption as well as wealth must be known to identify the time preference parameter. However, given the proportion of wealth that is consumed, time preference can be uncovered from the data. From (5) when RRA is not equal to one, pure time preference, r , is given by

$$r = \frac{E\{(1 + R_{mt})^{1-B}\}}{(1 - k)^B} - 1 \quad (13)$$

where k = proportion of wealth consumed.

¹⁷ For the period February 1953 to December 1977, RRA was estimated using an even broader index constructed by Stambaugh [38]. Both the nonparametric estimate of 2.76 as well as the parametric estimate of 2.75 correspond closely to the values that we obtained for the CRSP Indexes over this same subperiod. We thank Rob Stambaugh for providing us with this index.

¹⁸ We thank Richard Startz for suggesting this test to us.

If $B = 1.79$, then the numerator in (13) is 0.9974 for the value-weighted index from 1926–1981. Ando and Modigliani [1] suggest that the proportion of wealth consumed, k , is between 4 and 8% (annualized). This suggests a range of 4 to 11% for time preference. In contrast with Mehra and Prescott [26] and Nordhaus and Dulant [28], these parameter estimates for both time preference as well as RRA are within the range of economically acceptable bounds.

V. Applying the Hansen and Singleton Methodology to the Case of Many Assets

Until now, this research has focused on estimating RRA using two assets—the market portfolio and the riskless security. The econometric analysis for the case of many assets is outlined in this section.

In most of the empirical work on utility-based asset pricing models, a specific utility function (usually logarithmic) was assumed, and then the appropriate risk measures and the risk premium were estimated. Fama and MacBeth [6], Kraus and Litzenberger [23], and Roll [30] use log utility and then test the implied risk-return relation across assets. Grauer [10] examines the specification for a range of specific aggregate utility functions. Few¹⁹ have estimated the appropriate specification of RRA from the data and then tested the cross-sectional implications of such a value.

Equation (5) implies:

$$E\{(1 + \tilde{R}_{mt})^{-B}(1 + \tilde{R}_{it}) - (1 + \tilde{R}_{mt})^{-B}(1 + R_{ft})\} = 0 \quad \forall i = 1, 2, \dots, N.$$

As in the derivation of Equation (7), the above result can be transformed as

$$E\{\hat{x}_t^{-B}(\hat{x}_{it} - 1)\} = 0 \quad \forall i = 1, 2, \dots, N \quad (14)$$

where $\hat{x}_t = (1 + \tilde{R}_{mt})/(1 + R_{ft})$ and $\hat{x}_{it} = (1 + \tilde{R}_{it})/(1 + R_{ft})$. A nonparametric estimator²⁰ for B using individual securities would replace (14) with its sample equivalent and search for the appropriate value of RRA. Unlike the case where only the excess return on the market is utilized, a single estimate of B which simultaneously satisfies (14) for all N nonlinear equations is not expected. In other words, when many assets are observed, RRA is overidentified.

Fortunately, Hansen and Singleton [19] have developed a very general econometric approach which could be adapted to this problem and which would select a single estimate for B . Their procedure begins by replacing the population expectations in (14) with their sample counterparts for all N equations. B would then be estimated by choosing that value which minimizes the weighted-squared

¹⁹ Hansen and Singleton [18] [19] are two exceptions to this statement, and both papers rely on consumption data not a market index. Hansen and Singleton [18] use a lognormal distribution and conditional expectations while Hansen and Singleton [19] use unconditional expectations, as in this paper, but emphasized instrumental variables for the estimation.

²⁰ Naturally, a distributional assumption can be added to simplify the estimation of RRA. A bivariate lognormal distribution for $\hat{x}_t$ and $\hat{x}_{it}$ provides a very tractable estimator for B .

deviations of these N sample moments from zero. In other words,²¹

$$\text{Min}_b \left[\frac{1}{T} \sum_t x_{mt}^{-b} (\underline{x}_t - \underline{1}) \right]' S^{-1} \left[\frac{1}{T} \sum_t x_{mt}^{-b} (\underline{x}_t - \underline{1}) \right] \quad (15)$$

where

$$\begin{aligned} \underline{x}_t' &= (x_{1t} x_{2t} \cdots x_{Nt}), \\ \underline{1}' &= (1, 1, \dots, 1), \text{ and} \\ S^{-1} &= N \times N \text{ weighting matrix which reflects the asymptotic variance-covariance matrix of the sample counterparts to (14).} \end{aligned}$$

If the theory underlying (14) is correct, then the sample counterparts to (14) should be close to zero in large samples when evaluated at true RRA. This observation provides the basis for a test of the theory by measuring the distance of the objective function (15) from zero. The asymptotic distribution of such a test statistic is intuitive. The objective function (15) is just sum of squared, asymptotically normal random variables which have means equal to zero and which are weighted by S^{-1} so as to transform them into uncorrelated random variables. Not surprisingly, Hansen and Singleton [19] have demonstrated that the objective function evaluated at the optimal value b does have an asymptotic chi-squared distribution.

The estimator developed in Section III can be viewed as a special case of the Hansen and Singleton [19] approach when there are no overidentifying restrictions to test. In this particular case, the squared deviation of the left-hand side of (11) from zero is zero when b is selected in the appropriate fashion.

VI. Conclusions

Utility-based models of asset pricing can be estimated and tested without strong distributional assumptions concerning asset returns. By selecting a general parameterization for an aggregate utility function, the appropriateness of certain specializations can be determined from available data. Here an isoelastic utility function was selected, and log utility, which is a special case of the isoelastic class, is consistent with the data from 1926–1981.

Although distributional assumptions can simplify the estimation of utility-based asset pricing models, such an approach may not be robust to departures from the assumed distribution. The suggested nonparametric estimator, although nonlinear, is easy to compute and provides reasonably precise estimates—at least for the data analyzed here.

²¹ In terms of the notation of Hansen and Singleton [19], $h(b) = f(b) = x_{mt}^{-b}(x_t - 1)$. Hansen and Singleton [19] distinguish between $h(b)$ and $f(b)$ because their procedure also incorporates the use of instrumental variables. Since (7) holds for both unconditional expectations and conditional expectations given any predetermined variables (or instruments), it must be the case that $\hat{x}_t^{-b}(\hat{x}_t - 1)$ is uncorrelated with these instruments. That is, the unconditional covariance between $\hat{x}_t^{-b}(\hat{x}_t - 1)$ and any instrument must be zero. The knowledge that these unconditional covariances must equal zero provide additional equations from which to estimate b . Throughout this paper, we have relied only on an instrument equal to one for all t .

Given the limited amount of statistical analyses of utility-based paradigms of asset pricing, the empirical usefulness of such models was open to question. While this paper has relied on a simple and special case to highlight the conceptual issues, it is clear that preference-based theories can yield tractable econometric models. Future empirical work should further investigate the implications of such theories.

Appendix A

A Derivation of the Asymptotic Standard Error for the Method of Moments Estimator

The following provides a derivation of the asymptotic standard error for the methods of moments estimator. An exact first-order Taylor series expansion of (11) around the population value of B yields

$$f(B) - f'(b^*)[b - B] = 0$$

where $b^* \in (b, B)$, and $f'(b^*)$ is the first derivative of $f(b)$, evaluated at b^* . Or,

$$b - B = [f'(b^*)]^{-1}f(B).$$

If $\tilde{x}_t$ is independently and identically distributed through time, then $(\tilde{x}_t - 1)\tilde{x}_t^{-b}$ is also. Thus, Equation (7) and the Lindeberg-Levy central limit theorem imply that $\sqrt{T}f(b)$ is distributed asymptotically as normal with mean zero and variance equal to the $\text{Var}\{(\tilde{x}_t - 1)\tilde{x}_t^{-B}\}$, which can be estimated by

$$\frac{1}{T} \sum_t [(x_t - 1)x_t^{-b}]^2.$$

Since $\sqrt{T}f(B)$ converges to a random variable with a known distribution and $[f'(b^*)]^{-1}$ converges in probability to a constant, the product, $[f'(b^*)]^{-1}\sqrt{T}f(B)$, converges in distribution to the distribution of $\sqrt{T}f(b)$ scaled up by this constant (Serfling [37, p. 19]). Thus, for large samples, b is normally distributed about B with variance given by Equation (12).

Appendix B

The Relative Asymptotic Efficiency of the Parametric Estimator Versus the Method of Moments Approach Assuming a Lognormal Distribution

From Equation (12),

$$\text{Var}\{\sqrt{T}b\} = \frac{E\{[(x_t - 1)x_t^{-B}]^2\}}{[E\{(x_t - 1)x_t^{-B} \ln x_t\}]^2}.$$

Based on footnote 11, the above can be written as

$$\text{Var}\{\sqrt{T}b\} = \frac{E\{[(x_t - 1)x_t^{-B}]^2\}}{\lim_{\lambda \rightarrow 0} \left[E \left\{ (x_t - 1)x_t^{-B} \left(\frac{x_t^\lambda - 1}{\lambda} \right) \right\} \right]^2}.$$

Under the assumption that x_t has a lognormal distribution with $E\{\ln x_t\} = \mu$ and $\text{Var}\{\ln x_t\} = \sigma^2$, the above equation can be rewritten as:

$$\text{Var}\{\sqrt{T}b\} = \frac{\exp\{-2B\mu + 2B^2\sigma^2\} A_1 + A_2}{\left[\exp\left\{-B\mu + \frac{1}{2} B^2\sigma^2\right\}\right]^2 \left\{\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [A_3(\lambda) + A_4(\lambda)]^2\right\}}$$

where

$$A_1 = \exp\{2\mu + 2(1 - 2B)\sigma^2\}$$

$$A_2 = 1 - 2 \exp\{\mu + \frac{1}{2}(1 - 4B)\sigma^2\}$$

$$A_3(\lambda) = \exp\{(1 + \lambda)\mu + \frac{1}{2}[(1 + \lambda)^2 - 2B(1 + \lambda)]\sigma^2\}$$

$$A_4(\lambda) = 1 - \exp\{\lambda\mu + \frac{1}{2}[\lambda^2 - 2\lambda B]\sigma^2\} - \exp\{\mu + \frac{1}{2}(1 - 2B)\sigma^2\}.$$

Taking the limit using l'Hôpital's rule:

$$\text{Var}\{\sqrt{T}b\} = \exp\{B^2\sigma^2\}$$

$$\frac{\exp\{2\mu + 2(1 - 2B)\sigma^2\} - 2 \exp\{\mu + \frac{1}{2}(1 - 4B)\sigma^2\} + 1}{[\exp\{\mu + \frac{1}{2}(1 - 2B)\sigma^2\}(\mu + (1 - B)\sigma^2) - (\mu - B\sigma^2)]^2}.$$

Under the lognormal assumption, $B = (\mu/\sigma^2) + 0.5$, so the above can be rewritten as

$$\text{Var}\{\sqrt{T}b\} = \frac{1}{\sigma^4} \exp\left\{\frac{\mu^2}{\sigma^2} + \mu + \frac{\sigma^2}{4}\right\} \left[\exp\{-2\mu\} - 2 \exp\left\{-\mu - \frac{1}{2}\sigma^2\right\} + 1\right].$$

Recalling from the text that relative efficiency, RE , is defined as the $\text{Var}\{\sqrt{T}\hat{b}\}/\text{Var}\{\sqrt{T}b\}$, the above equation can be combined with (10) to yield the equation for RE at the end of Section III.

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