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Divergence of Opinion in Complete Markets: A Note

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ABSTRACT

We consider an Arrow-Debreu model with agents who have different subjective probabilities. In general, asset prices will depend only on aggregate consumption and the distribution of subjective probabilities in each state of nature. If all agents have identical preferences then an asset with "more dispersed" subjective probabilities will have a lower price than an asset with less dispersed subjective probabilities if risk aversion does not decline too rapidly. It seems that this condition is likely to be met in practice, so that increased dispersion of beliefs will generally be associated with reduced asset prices in a given Arrow-Debreu equilibrium.

THERE HAVE BEEN SEVERAL recent investigations concerning the effect of heterogeneous probability beliefs on asset prices. Most of these investigations have taken place in the context of CAPM-like mean-variance models; see, for example, Lintner [5], Miller [7], Williams [10], Jarrow [4], and Mayshar [6].

Comparatively little has been done in the analysis of differences of opinion in an Arrow-Debreu contingent claims context. Rubinstein [8], Breeden and Litzenberger [2], and Breeden [1] have presented nice analyses of asset price determination in an Arrow-Debreu model, but most of their results have assumed commonly held probability beliefs.

In this paper we analyze the impact of divergence of opinion on asset prices in an Arrow-Debreu economy. The results serve to generalize the findings of Rubinstein and Breeden-Litzenberger to the case of different probability beliefs. Among the results we establish are:

1. In equilibrium, asset prices depend only on aggregate consumption and the distribution of subjective probability beliefs.
2. Asset values are an increasing function of any one individual's probability beliefs.
3. An increase in the "spread" of the probability beliefs of investors may increase or decrease equilibrium asset values depending on the value of a parameter of the utility function. However, the most likely effect is to decrease the asset values.

These results are especially interesting in light of the recent empirical work of Cragg and Malkiel [3]. They studied the relationship between ex post return and various measures of risk for common stocks and found that the measure of risk that performed best in their analysis was a measure of divergence of opinion

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about the asset returns. This seems like an interesting empirical finding in search of a model. This paper is an attempt to provide such a model.

I. The Arrow-Debreu Model

Suppose that there are n investors indexed by $i = 1, \dots, n$. There are S states of nature indexed by $s = 1, \dots, S$. Investor i has a von Neumann-Morgenstern utility function for consumption in state s denoted by $u_i(c_{is})$. This function is assumed to be strictly increasing and strictly concave in consumption. We assume that there are given endowments of consumption in state s by consumer i which we denote by $\bar{c}_{is}$.

Each consumer has a subjective probability distribution over the states of nature. We let π_{is} denote consumer i 's probability that state s will occur. We assume that there are a set of Arrow-Debreu securities that pay off one unit of consumption if and only if a given state of nature occurs. We let p_s denote the price of an Arrow-Debreu security that pays off in state s .

Each consumer chooses a portfolio of Arrow-Debreu securities by solving the following maximization problem:

$$\begin{aligned} \max \quad & \sum_{s=1}^S \pi_{is} u_i(c_{is}) \\ \text{subject to} \quad & \sum_{s=1}^S p_s c_{is} = \sum_{s=1}^S p_s \bar{c}_{is} \end{aligned}$$

We assume that final consumption in each state of nature is nonnegative, but the net position in each Arrow-Debreu security for a given individual may be positive or negative. Thus, there are no short sales restrictions of any sort. We will suppose that the standard conditions are satisfied so that an Arrow-Debreu equilibrium will exist. Note that this implies that each agent behaves strictly as a price-taker. In particular, agents do not acquire any information from market prices about other agents' probability beliefs; they behave purely as passive expected utility maximizers. Thus, the standard first-order conditions apply, which we note here for reference.

$$\pi_{is} u'_i(c_{is}) = \lambda_i p_s \quad (1)$$

Under standard assumptions, the pattern of consumption across states of nature in an Arrow-Debreu equilibrium will be Pareto-efficient. Similarly, all Pareto-efficient allocations of consumption can be supported as Arrow-Debreu equilibria. The results below are phrased in terms of an Arrow-Debreu equilibrium, but given these standard theorems of welfare economics, they can also be viewed as statements about Pareto-efficient risk bearing.

II. Asset Valuation in the Arrow-Debreu Model

Any asset can be valued in terms of the Arrow-Debreu prices. For example, let an asset have a payoff of x_s , in state of nature s . Then its equilibrium value must be:

$$v_x = \sum_{s=1}^S p_s x_s$$

Following the lines of Rubinstein [8], we can write this sum as:

$$v_x = \sum_{s=1}^S \frac{p_s}{\pi_{is}} x_s \pi_{is} = E_i \frac{p}{\pi_i} x$$

where E_i is the expectation with respect to agent i 's subjective probability distribution. Using the covariance identity, this term can be rewritten as:

$$v_x = \text{cov}_i\left(\frac{p}{\pi_i}, x\right) + E_i x E_i \frac{p}{\pi_i}$$

where cov_i is the covariance with respect to agent i 's probability distribution and symbols without the s subscript are random variables. Using the definition of expectation,

$$E_i \frac{p}{\pi_i} = \sum_{s=1}^S \frac{p_s}{\pi_{is}} \pi_{is} = \sum_{s=1}^S p_s$$

This sum is simply the value today of a certain payoff of one dollar next period. That is, it is the reciprocal of the risk-free rate of return, R_F .

Substituting from the first-order conditions for individual optimization, we now have:

$$v_x = \text{cov}_i(u'_i(c_{is}), x_s)/\lambda_i + \frac{E_i x}{R_F}$$

This formula states that in equilibrium an asset's value is the sum of its discounted expected return and its risk premium—its covariance with individual consumption. Denoting the risk measure by γ_i , we have the final form for the asset valuation formula:

$$v_x = \gamma_i + \frac{E_i x}{R_F}$$

In this expression v_x , x_s , and R_F are publicly observable while γ_i and $E_i x$ differ from person to person. Thus, the subjective beliefs of the agents about an asset's "risk" and "return" can not vary across individuals in an arbitrary way, but must satisfy the equilibrium pricing relation given above. This relation places restrictions on how the individual perceptions of risk and return relate to each other; if an agent tells us his or her beliefs about the expected return on an asset, then we can use the equilibrium pricing relationship to solve for the risk premium as a function of the publicly observable variables v_x and R_F . Note that the risk premium depends on the covariance between marginal utility and the asset payoffs. Since marginal utility is a decreasing function of consumption, assets whose payoffs are positively correlated with individual i 's consumption will have a negative risk premium for that individual, while assets that are negatively correlated will have a positive risk premium for that individual.

When the economic agents have homogeneous beliefs, we can go further. Following Rubinstein [8] and Breeden and Litzenberger [2], we proceed to

examine the individual first-order conditions in more detail. We write these conditions as:

$$\frac{u'_i(c_{is})}{\lambda_i} = \frac{p_s}{\pi_s}$$

The left-hand side of this expression is a decreasing function of consumption, so it has an inverse, $f_i(\cdot)$. Applying this inverse to each side of the expression and summing over the consumers, we have:

$$c_s = \sum_{i=1}^n c_{is} = \sum_{i=1}^n f_i\left(\frac{p_s}{\pi_s}\right)$$

The right-hand side of this expression is decreasing in p_s/π_s , so it has an inverse, $F(\cdot)$. Applying this inverse, we have:

$$\frac{p_s}{\pi_s} = F(c_s)$$

Substituting into the asset valuation formula, we have:

$$v_x = \text{cov}(F(c), x) + \frac{Ex}{R_F}$$

Thus, when all agents have common probability beliefs, the risk premium depends on the covariance of the asset payoffs with a decreasing function of *aggregate* consumption. This insight can be further generalized in two directions. Rubinstein [8] shows that if consumption and asset payoffs are bivariate normally distributed then the risk premium can be expressed in terms of the covariance with aggregate consumption alone, not just a nonlinear function of aggregate consumption. Breeden [1] shows that the same thing happens in a continuous time model. (Both of these simplifications can be applied to the γ_i term in the earlier valuation formula as well.)

However, when probability beliefs are heterogeneous such simplifications do not appear to be possible. Breeden and Litzenberger [2] observe that agents need only agree on the probability of occurrence of various levels of aggregate consumption; but other than this result, there is not much that is known.

Referring to the original expression for asset values, $v_x = \sum p_s x_s$, we see that if we can isolate the effect of divergence of probability beliefs on Arrow-Debreu prices, we will have determined the effect on all asset values. In the next section, we will see how this might be done.

III. Heterogeneous Probability Beliefs

In equilibrium each consumer maximizes expected utility as given above. Hence, optimal consumption must satisfy the first-order conditions given in Equation (1).

Since $u'_i(c_{is})$ is a strictly decreasing function, it has an inverse $f_i(\cdot)$. Thus, we can write:

$$c_{is} = f_i(\lambda_i p_s / \pi_{is}) \quad (2)$$

(Note that the definition of $f_i(\cdot)$ is slightly different than before.) Summing over all investors, we have:

$$\sum_{i=1}^n c_{is} = c_s = \sum_{i=1}^n f_i(\lambda_i p_s / \pi_{is}) \quad (3)$$

For fixed values of (π_{is}) and (λ_i) , the right-hand side of this expression is a strictly decreasing function of p_s . Hence, it has an inverse $F(\cdot, \pi_{1s}, \dots, \pi_{ns})$. Applying this function to each side of this expression, we have:

$$F(c_s, \pi_{1s}, \dots, \pi_{ns}) = p_s \quad (4)$$

In any given Arrow-Debreu equilibrium, the values of the λ_i terms are determined. Hence, the above equation implies that the equilibrium-contingent commodity prices are solely a function of *aggregate* consumption in each state and the distribution of beliefs about that state. Recording this fact for future reference, we note the following:

FACT 1. In equilibrium the Arrow-Debreu price for consumption in state s depends only on aggregate consumption in that state and the distributions of subjective probabilities that the state will occur. The Arrow-Debreu price is a decreasing function of consumption in state s and an increasing function of π_{is} for each $i = 1, \dots, n$.

Proof: Only the last sentence remains to be proved. That the prices are decreasing in consumption is obvious from the definition of F . In order to prove the second part, we consider Equation (3). Hold aggregate consumption and the λ_i terms fixed in this equation and increase π_{is} . Then p_s must increase to maintain the equality. Q.E.D.

The first part of this fact is a generalization of Theorem 1 in Breeden and Litzenberger [2], alluded to earlier. Extending their discussion to this model, we note that two states of nature that have the same aggregate consumption and the same distribution of probability beliefs have the same Arrow-Debreu prices; thus, a set of Arrow-Debreu securities need only distinguish states with different values of aggregate consumption and different probability beliefs in order to support a given efficient pattern of consumption.

We now consider the impact of a change in the “spread” of the probability beliefs on asset prices. Consider two states of nature with the same value of aggregate consumption but with different probability beliefs. Suppose that the “average” probability over the investors is constant across the two states, but the “divergence of opinion” is higher in one state than the other. Which Arrow-Debreu price will be larger?

Intuition provides conflicting answers. One might argue that divergence of opinion about an asset’s payoffs makes the asset seem more risky. Hence, divergence of opinion will decrease the value of an asset. On the other hand, one can argue that the market price is determined by the optimists, so that increasing the divergence of opinion is likely to increase an asset’s price. Several of the CAPM-type models mentioned in the introduction support this latter view.

In order to develop some intuition, we might consider a simple reservation price model. Suppose that each consumer is limited to purchasing at most one unit of a given asset in fixed supply, and let the reservation prices differ across

consumers. Then the simple supply-demand diagram depicted in Figure 1 gives us the equilibrium price.

Now suppose that the reservation prices become more dispersed; i.e., the pessimists think that the asset is worth less and the optimists think that it is worth more. This will tend to rotate the demand curve clockwise about the average reservation price. Hence, the equilibrium price of the asset will increase or decrease as the supply of the asset is to the right or left of the pivot point.

Even in this simple model, an increase in the diversity of opinion has an ambiguous effect on asset prices. Thus, it seems unlikely that a definitive result is available in more general cases. However, the additional structure provided by the state-independent vonNeumann-Morgenstern utility functions does allow us to isolate the relevant parameter of the utility function that determines the effect of diversity on asset prices in a given equilibrium.

IV. The Effect of Diversity of Opinion

Let us consider a fixed equilibrium and a particular state. We will assume that all consumers have the same von Neumann-Morgenstern utility function $u(c_{is})$ with associated Arrow-Pratt measure of absolute risk aversion $r(c)$. We also introduce the *weighted* probabilities q_{is} for $i = 1, \dots, n$ defined by:

$$q_{is} = \pi_{is}/\lambda_i$$

In general, wealthier consumers will have lower marginal utilities of income so that their subjective probability beliefs will have a higher weight in the above expression. Using this notation we can rewrite Equation (3) as:

$$c_s = \sum_{i=1}^n f(p_s/q_{is}) \quad (5)$$

We now note the following:

FACT 2. The function $f(p_s/q_{is})$ is an increasing function of q_{is} . It is a concave or convex function of q_{is} as $r'(c)$ is greater than or less than $-r^2$.

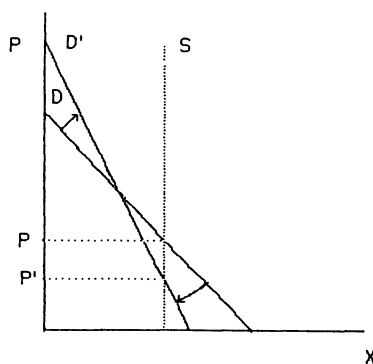


Figure 1. Effect of Increased Dispersion

Proof: Suppose that $r'(c) > -r^2$. Then it is a straightforward calculation to show that:

$$\frac{u'u'''}{u''u''} < 2$$

Using the fact that $f(u'(c)) \equiv c$, we can derive expressions for the derivatives of f in terms of the derivatives of u :

$$\begin{aligned} f' &= 1/u'' \\ f'' &= -u'''/(u'')^3 \end{aligned}$$

These expressions in turn can be used to calculate the following derivatives:

$$\begin{aligned} \frac{\partial f}{\partial q_{is}} &= -f'(p_s/q_{is})p_s/q_{is}^2 > 0 \\ \frac{\partial^2 f}{\partial q_{is}^2} &= \frac{p_s}{u''q_{is}^3} \left[2 - \frac{u'u'''}{u''u''} \right] \end{aligned}$$

Combining this last expression with the first inequality, we have the result. Q.E.D.

The fact that $f(p_s/q_{is})$ has a definite curvature allows us to use the standard techniques of Rothschild-Stiglitz [9] to determine the effect of diversity of opinion on asset prices.

FACT 3. If $f(p_s/q_{is})$ is an increasing concave (convex) function of q_{is} then a mean-preserving spread across the population in q_{is} must decrease (increase) the equilibrium value of p_s .

Proof: Refer to Equation (5). Since $f(p_s/q_{is})$ is a concave function of q_{is} , a mean-preserving spread in the distribution of q_{is} will decrease the value of the sum. If c is to remain fixed, this means that p_s must decrease. Q.E.D.

Note that this fact holds only for a “cross-sectional” comparison of asset prices in a *fixed* equilibrium. Thus, if we have two states s and t with the same aggregate consumption but more dispersed beliefs in s than in t in the sense that the distribution of weighted beliefs in s is a mean-preserving spread of the one in t , then the Arrow-Debreu price for consumption in s will be less than the Arrow-Debreu price for consumption in t .

Facts 2 and 3 taken together indicate that the crucial determinant effect of dispersion of opinion on asset prices is whether risk aversion declines “too rapidly.” However, this condition in itself is not terribly transparent. There is an equivalent expression for the condition in terms of the Arrow-Pratt measure of *relative* risk aversion.

FACT 4. Let $\rho(c) = r(c)c$ be the coefficient of relative risk aversion. Then $r'(c) > -r^2$ if and only if the consumption elasticity of relative risk aversion is greater than $1 - \rho$. That is, $\rho'c/\rho > 1 - \rho$.

Proof: Differentiating $\rho = rc$, we have:

$$\rho' = r + r'c$$

Thus, $r' > -r^2$ can be expressed as:

$$\frac{\rho' - r}{c} > -\frac{\rho^2}{c^2}$$

which reduces to:

$$rc - \rho'c < \rho^2$$

or

$$\rho - \rho^2 < \rho'c$$

$$1 - \rho < \frac{\rho'c}{\rho} \quad \text{Q.E.D.}$$

Thus, if we consider the family of constant relative risk averse utility functions, we see that the condition will be satisfied when $\rho > 1$. Since the empirical evidence indicates that ρ is at least 2, it seems that equilibrium asset prices should generally decrease with an increase in diversity of opinion.

It is an easy calculation to check other commonly used functional forms. For example, quadratic utility and constant absolute risk aversion each imply $f(p_s/q_{is})$ is a concave function of q_{is} . Thus, asset prices will decrease with an increase in diversity of opinion in both of these cases.

The most convenient general class of expected utility functions is the HARA, or linear risk tolerance class defined by:

$$-\frac{u'(c)}{u''(c)} = a + bc$$

FACT 5. If the representative utility function is of the HARA class, then increasing the dispersion of opinion will decrease asset prices if and only if $b < 1$.

Proof: By Fact 2 and direct computation. Q.E.D.

V. An Example with Constant Relative Risk Aversion

The family of constant relative risk averse utility functions provides a nice example. Here the first-order conditions for utility maximization have the form:

$$\pi_{is}c_{is}^{-\rho} = \lambda_i p_s$$

Solving for c_{is} , we have:

$$c_{is} = \left(\frac{\lambda_i}{\pi_{is}} \right)^{-1/\rho} p_s^{-1/\rho}$$

Summing this over $i = 1, \dots, n$ and rearranging, we have:

$$p_s^{1/\rho} c_s = \sum_{i=1}^n \left(\frac{\pi_{is}}{\lambda_i} \right)^{1/\rho}$$

For $\rho > 1$ the right-hand side of this expression is a concave function of π_{is}/λ_i , so that a mean-preserving spread of $\pi_{is}/\lambda_i = q_{is}$ across individuals will decrease the right-hand side. Hence, p_s must decrease to maintain the equality.

Finally, let us consider the general condition given in Fact 2. We might well ask when this condition is met with equality. That is, for what von Neumann-Morgenstern utility function is $r'(c) = r(c)^2$? It is not difficult to verify that $u(c) = \ln c$ is the essentially unique solution to this differential equation. Thus, logarithmic utility is the borderline case not only in the class of constant relative risk averse utility functions, but also in the class of *all* utility functions. As long as risk aversion declines less rapidly than it does in the case of logarithmic utility, an increase in the diversity of opinion will be associated with decreased asset prices.

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