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Risk Aversion and Arbitrage

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ABSTRACT

This paper characterizes conditions under which asset returns and consumption are consistent with risk-averse preferences. It is shown that risk aversion is equivalent to "zero arbitrage" on a transformation of the payoff space. The implicit state prices which are dual to this no-arbitrage condition can be interpreted as prices of "pure consumption hedges." This zero-arbitrage restriction implies the usual restrictions associated with nonsatiation. The analysis holds in both complete and incomplete market settings.

THROUGH A NUMBER OF recent papers in financial economics, we have come to understand that the absence of arbitrage opportunities is equivalent to the existence of positive, implicit state-claim prices. In addition, given a particular allocation of consumption quantities for different states of the world, absence of arbitrage is necessary and sufficient for the consumption plan to be a consumer optimum for some monotonic preference ordering.

The purpose of this paper is to inquire into the additional restrictions on asset returns imposed by requiring that an allocation be a consumer optimum when preferences are concave, monotonic, von Neumann-Morgenstern (vNM), state-independent, and defined over a single good. We find that a necessary and sufficient condition is that there is absence of arbitrage not only on the original payoff space but also on a simple transformation of the payoff space. This transformation has an intuitively appealing form and can be interpreted as a type of risk adjustment.

Perhaps the classic development of the duality between absence of arbitrage and positive implicit state prices is contained in Ross [9]. Working with a finite number of assets and states, Ross shows that when positive state prices exist and price returns in a manner consistent with the observed market values of the assets, arbitrage is precluded. When markets are complete, the implicit state prices are uniquely determined and it is easy to check if they are positive. More generally, there will be many vectors of state prices which value assets in a manner consistent with their observed market prices. Absence of arbitrage opportunities implies that at least one of these is positive. Analogues of this result have been shown to hold in more general environments by Garman [3], Ross [10], Harrison and Kreps [6], and Hansen and Richard [5] among others. Several authors have also noted that zero-arbitrage restrictions are necessary and sufficient for an arbitrary positive consumption bundle to be a consumer optimum for some set of monotonic preferences. Kreps [7] provides, perhaps, the most general discussion of this point.

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Thus, lack of arbitrage both follows from nonsatiation and exhausts all the restrictions associated with nonsatiation. Risk aversion, as commonly defined, requires further the state-independence, concavity, and differentiability of the vNM utility index, and this provides the additional structure needed to restrict the joint behavior of prices and optimally chosen consumption allocations. In a complete market setting, Breeden and Litzenberger [1] show that for consumption to be optimal, it must be inversely related across states-of-the-world to the state-price/probability ratios. Dybvig and Ross [2] argue that in an incomplete market there must exist a state-price vector which, when normalized by the probabilities, is inversely related to an agent's consumption across states. Further, they show that this condition is sufficient for the consumption allocation to be a consumer optimum for some member of the class of preferences being considered. Various methods of actually constructing the preferences which rationalize the allocation from the implicit prices and probabilities are provided in Dybvig and Ross [2], Varian [11], and Green and Srivastava [4].

In this paper, we ask if there are other ways to characterize the relationship between consumption allocations and asset returns beyond the existence of implicit state prices which exhibit this inverse ordering property. Specifically, given the duality between the existence of positive state prices and zero arbitrage, might there also be a dual to this more restrictive existence condition? We describe this duality and show that it takes the form of ruling out arbitrage on a simple transformation of the set of asset payoffs at the original market values of the assets.

Thus, if preferences are required to be concave, differentiable, and state-independent as well as monotonic, a second zero-arbitrage condition must be met. While the second zero-arbitrage condition implies the first, the converse is not true. The transformed set of returns which must exhibit this zero-arbitrage condition are consumption measurable, and their relative magnitudes are determined by their usefulness in hedging consumption risks. The state prices which the dual system yields can be interpreted as Arrow-Debreu prices for pure "consumption hedges."

In the next section, we describe our notation and assumptions and review the consequence of zero arbitrage. We provide a proof of the duality between zero arbitrage and positive implicit state prices which is simpler than that in Ross [9]. Section II presents the major results. Section III outlines a generalization of the results to state spaces which are not discrete. Section IV concludes the paper.

I. The Consequences of Zero Arbitrage

We begin with a market with a finite set of elementary states, $S = \{s_i\}_{i=1}^{m}$, and a finite number of assets. Asset returns in state s_i are given by the vector $x(s_i) = \{x_j(s_i)\}_{j=1}^{n}$. The $n \times m$ matrix X formed by the vectors $x(s_i)$ is called the "payoff matrix." Without loss of generality, we ignore any redundant assets and assume that X has rank $n \leq m$, and omit any states with zero probability. Markets are said to be complete when $n = m$. We denote the transpose of X by X'.

The market prices of the n assets are given by the vector $P \equiv \{P_j\}_{j=1}^{n}$. A portfolio is a vector $w \equiv \{w_j\}_{j=1}^{n}$, where w_j is the number of units of the j^{th} security purchased. The cost of a particular portfolio, then, will be $\sum_{j=1}^{n} w_j P_j = w'P$. The vector of state-dependent payoffs on the portfolio is $X'w$. Finally, we denote vectors of state prices as $q \equiv \{q(s_i)\}_{i=1}^{n}$. A vector of state prices is given implicitly by the market if it solves the system $Xq = P$.

Definition 1. The asset returns described by X and P exhibit zero arbitrage if $X'w \geq 0$, $X'w \neq 0$ implies $w'P > 0$.

To demonstrate the duality between zero arbitrage and positive state prices, Ross [9] assumes the matrix X is semipositive. We make the following weaker assumption.

Assumption 1. One of the n assets has the characteristic that $x_j(s_i) > 0$ for all i. This asset has a strictly positive price.

Clearly, a sufficient condition for Assumption 1 to hold is the existence of a riskless asset. Ross [10] employs Assumption 1, and also includes a discussion of the discrete state-space case. The result there, however, yields only semipositive state prices. The proof presented below yields strictly positive state prices and simplifies the proof in Ross [9].

THEOREM 1 (Ross [9]). *There exists a positive vector of state prices, q, $q(s_i) > 0$ for all i, if and only if X and P exhibit zero arbitrage.*

Proof: Necessity is easy to demonstrate. Let $w \in R^n$ be a vector of portfolio weights, and let $q \gg 0$ satisfy $Xq = P$. Then $w'Xq = w'P$ and $w'X \geq 0$, $w'X \neq 0$ implies $w'P = w'Xq > 0$. Thus, all portfolios with semipositive payoffs have positive prices.

Now let $K = \{y \in R^n : y = Xq,\ q \gg 0\}$. Suppose $P \notin K$. We must show this implies an arbitrage opportunity. K and P are disjoint convex sets and K is an open cone. By Assumption 1, K does not contain the origin. Theorems 11.3 and 11.7 of Rockafeller [8] then ensure that we can properly separate K and P with a hyperplane through the origin, and there exists $w \in R^n$, $w \neq 0$, such that $w'y > 0$ for all $y \in K$ and $w'P \leq 0$. Since $w'y > 0$ for all $y \in K$, $w'Xq > 0$ for all $q \gg 0$. This implies $w'X \geq 0$, since if $(w'X)_k < 0$ for some k and $(w'X)_h > 0$ for some h, then we can choose q_k to be large and q_h infinitesimal to make $w'Xq < 0$, which is a contradiction. Thus, $w'X \geq 0$, $w'X \neq 0$, and since $w'P \leq 0$ there is arbitrage. Q.E.D.

Suppose we are given a vector of state-contingent consumptions $\{c^*(s_i)\}_{i=1}^{m}$, with the characteristic $c^*(s_i) \geq 0$ for all i. We pose the following question: Can we find a set of monotonic preferences and an endowment pair $\{\bar{c}, W\}$ for which c^* solves the following consumption-portfolio problem?

Problem 1.

$$\max_{w} U(c(s_1), \cdots, c(s_m))$$

$$\text{subject to} \begin{cases} w'P \leq W \\ c(s_i) = w'x(s_i) + \bar{c}(s_i) \qquad i = 1, \cdots, m \end{cases}$$

Here W is initial wealth, $U(\cdot)$ is required to be strictly monotonic, and $\bar{c}(s_i)$ is the agent's endowment in state i. The answer to this question is yes, when X and P exhibit zero arbitrage. A proof in a more general setting is provided in Harrison and Kreps [6, p. 385]. If the market allows for arbitrage, then Problem 1 will not have a solution as long as $U(\cdot)$ displays nonsatiation. Thus, as discussed in Ross [9, pp. 203–204] there is an important sense in which positive implicit state prices exhaust all restrictions on consumption choices and asset returns implied by strict monotonicity of preferences. The next section exhaustively characterizes the restrictions associated with concave state-independent vNM preferences.

II. Risk-Averse Preferences and Arbitrage

To restrict asset returns and consumption allocations beyond consistency with positive state prices, more structure must be put on preferences. In this section, we show that if $U(c(s_1), \cdots, c(s_m))$ is required to be vNM, state-independent, differentiable, and strictly concave, the added restrictions on asset returns and consumption can be summarized by a "zero-arbitrage" condition on a transformation of the payoff matrix.

Consider Problem 1 more carefully. If U is differentiable and a vector $c^* = \{c^*(s_i)\}_{i=1}^m$ is to be an optimal consumption choice for Problem 1, it must satisfy the first-order conditions:

$$\sum_{i=1}^m x_j(s_i) \frac{\partial U(c^*)}{\partial c(s_i)} = \lambda P_j \qquad j = 1, \cdots, n \tag{1}$$

For a given set of implicit state prices, q, we have

$$\sum_{i=1}^n x_j(s_i) q(s_i) = P_j \qquad j = 1, \cdots, n \tag{2}$$

It is thus a simple matter to identify $q(s_i)$ with the normalized marginal utility $[\partial U(c^*)/\partial c(s_i)]/\lambda$ and to observe that monotonicity merely restricts these quantities to be positive.

Suppose, however, that we require U to be of the form

$$U(c(s_1), \cdots, c(s_m)) = \sum_{i=1}^m \pi(s_i) u[c(s_i)] \tag{3}$$

where $\pi(s_i)$ is the (assumed known) probability of state i occurring. We assume that $u(\cdot)$ is monotonic, differentiable, and strictly concave. If c^* is to be an optimal allocation, we must again be able to find positive implicit state prices which we can identify with the marginal utilities of consumption in the various states. If U has the form in (3) then these marginal utilities are $\pi(s_i)u'(c(s_i))$, where $u'(\cdot)$ is the derivative of $u(\cdot)$. This in turn implies that:

$$u'[c^*(s_i)] = \lambda \frac{q(s_i)}{\pi(s_i)} \tag{4}$$

If $u(\cdot)$ is to be strictly concave, $u'(\cdot)$ must be a decreasing function. As noted in Breeden and Litzenberger [1] and Dybvig and Ross [2] this implies an inverse ordering across states between the consumptions and the price/probability ratios, $r(s_i) \equiv q(s_i)/\pi(s_i)$. The graphs of all possible closed, proper concave functions on the real line are precisely the nonincreasing curves in R^2 (Theorem 24.3, Rockafeller [8]). Green and Srivastava [4] provide a simple construction of a concave utility function which rationalizes a given c^* for a given set of $\{r(s_i)\}_{i=1}^m$ which has the property that $[c^*(s_i) - c^*(s_j)][r(s_i) - r(s_j)] \leq 0$ for all i, j. Dybvig and Ross [2] first noted that the existence of state-price probability ratios having this property was both necessary and sufficient for c^* to be "efficient."

Given the duality between the existence of positive state prices and zero arbitrage, it is perhaps not surprising that there is also a zero-arbitrage dual to the existence of price/probability ratios inversely ordered with consumption. Without loss of generality, let us order the states inversely with the consumption vector c^*, so that $i \geq j$ implies $c(s_i) \leq c(s_j)$. Zero arbitrage implies the following system has a positive solution, q,

$$Xq = P \tag{5}$$

or, alternatively, that there be a positive solution to the system

$$X\Pi r = P \tag{6}$$

The $m \times m$ matrix Π has $\pi(s_1), \cdots, \pi(s_m)$ as diagonal elements and zeroes off the diagonal, and r is the vector of price/probability ratios.

The requirement that the price vector P and payoffs X be consistent with a strictly risk-averse consumer who chooses c^* is equivalent to requiring that among the solutions to (6) there exists at least one with the property that $r(s_i) < r(s_{i+1})$ if $c^*(s_i) > c^*(s_{i+1})$ and $r(s_i) = r(s_{i+1})$ if $c^*(s_i) = c^*(s_{i+1})$. Let $c_1^*, \cdots, c_K^*$ denote the K distinct values of the consumption variable, and $r(c_k^*)$, $k = 1, \cdots, K$, the corresponding implicit price/probability ratios. Concavity is a cardinal property relating to differences in utility. Thus, there is a fundamental sense in which risk aversion cannot restrict the behavior of prices and returns over states where the consumption variable is the same. Breeden and Litzenberger [1] noted in a complete markets context that assets whose payoffs differ only on these states, and yet have the same expectation conditional on consumption, must have the same value. This allows us to reduce the dimensionality of the payoff matrix as follows. Let $X(c^*)$ be the $n \times K$ matrix with rows $E(\tilde{x}_j \mid c^* = c_k^*)$, $k = 1, \cdots, K$, and let $\pi(c_k^*) \equiv \Pr\{c^* = c_k^*\}$. We denote by $\Pi(c^*)$ the $K \times K$ matrix with $\pi(c_k^*)$ on the k^{th} diagonal and zeroes off the diagonal. Then we have:

LEMMA. *If r solves (6) and is consistent with risk aversion and consumption allocation, c^*, then $r(c^*) \equiv \{r(c_k^*)\}_{k=1}^K$, where $r(c_k^*) < r(c_{k+1}^*)$, solves:*

$$X(c^*)\Pi(c^*)r(c^*) = P \tag{7}$$

Proof: Suppose $c_k^* = c^*(s_l) = c^*(s_{l+1}) = \cdots = c^*(s_{l+h})$. If $u(\cdot)$ is differentiable, the superdifferential is uniquely valued so that $r(s_l) = r(s_{l+1}) = \cdots = r(s_{l+h}) \equiv r(c_k^*)$. The contribution of the payoffs in these states to the market value of asset j is:

$$\textstyle\sum_{i=l}^{l+h} r(s_i)\pi(s_i)x_j(s_i) = r(c_k^*)\pi(c_k^*)E(\tilde{x}_j \mid c^* = c_k^*)$$

Therefore, if q correctly prices the marketed assets over the whole state space, $q(c_k^*) = r(c_k^*)\pi(c_k^*)$ will "price" the payoffs in $X(c^*)$ which is $n \times K$. Strict concavity requires strictly decreasing marginal utility so that, with the c_k^*'s ordered inversely with k, $r(c_k^*)$ is strictly increasing with k. Q.E.D.

A strictly increasing solution $r(c^*)$ to (7) is equivalent to a strictly positive solution ε to the system:

$$X(c^*)\pi(c^*)L\varepsilon = P \tag{8}$$

or

$$\hat{X}\varepsilon = P \tag{9}$$

where L is a $K \times K$ lower triangular matrix of ones, and $\hat{X} \equiv X(c^*)\Pi(c^*)L$ is an $n \times K$ matrix of linearly transformed payoffs. Structuring the problem in this way simply involves taking note of the fact that requiring $r(c^*)$ to be positive and increasing is equivalent to requiring that

$$r(c_k^*) = \textstyle\sum_{i=1}^{k} \varepsilon_i \tag{10}$$

for some set of positive numbers $\{\varepsilon_1, \cdots, \varepsilon_K\}$. Alternatively, we can view this as a requirement that there exist positive state-claim prices which correctly price the payoffs in $\hat{X}$; i.e., which solve $\sum_{k=1}^{K} \hat{x}_j(c_k^*)\varepsilon_k = P_j$ for each asset j, where consumption now describes the relevant states. This in turn implies that a second zero-arbitrage condition must be met, as is stated in Theorem 2.

THEOREM 2. *An allocation $c^*(s_i)$, $i = 1, \cdots, m$, will be a consumer optimum for some strictly concave, differentiable, state-independent, vNM, utility function in the market described by X and P if and only if the payoffs $\hat{X}$ and prices P exhibit zero arbitrage.*

Proof: The arguments presented above prove necessity if we can apply Theorem 1 to ensure that there is a duality between the existence of the positive state prices ε_k, $k = 1, \cdots, K$, and zero arbitrage on the payoffs $\hat{X}$ at prices P. The condition required for Theorem 1 to hold for $\hat{X}$ is that the existence of an asset with strictly positive payoffs in X implies its existence in $\hat{X}$. Let x_f denote the strictly positive row in X. The i^{th} entry in the corresponding row of $\hat{X}$ is $\hat{x}_f(c_k^*) = \sum_{k=i}^{K} \pi(c_k^*)E(\tilde{x}_f \mid c^* = c_k^*)$. These are all strictly positive since we have assumed that the states all have strictly positive probability.

To prove sufficiency we note that zero arbitrage on $\hat{X}$ and P implies by Theorem 1 that a strictly positive K vector, ε, exists which solves (9). We can use these, as in (10), to construct the vector $r(c^*)$. From this point, we simply follow the argument in Dybvig and Ross ([2], Theorem 1) and connect the points $(c_k^*, r(c_k^*))$ with line segments in R^2, which will be downward sloping. By integrating over this function to an arbitrary $c \geq 0$, we obtain the required $u(c)$. Q.E.D.

Both the transformed payoff matrix, $\hat{X}$, and the vector of implicit state prices for this payoff matrix, ε, have interesting interpretations. The first entry in the j^{th} row of $\hat{X}$ is simply the unconditional expectation of the payoff on asset j,

$$\hat{x}_j(c_1^*) = \textstyle\sum_{k=1}^{K} E(\tilde{x}_j \mid c^* = c_k^*)\pi(c_k^*) = E(\tilde{x}_j) \tag{11}$$

As we move along the row, we sum over fewer and fewer consumption states.

Since $c^*_{k-1} > c^*_k > c^*_{k+1}$, the k^{th} entry of the j^{th} row of $\hat{X}$ is:

$$\begin{aligned} \hat{x}_j(c^*_k) &= \textstyle\sum_{i=k}^{K} E(\tilde{x}_j \mid c^* = c^*_i)\pi(c^*_i) \\ &= E[\tilde{x}_j \mid c^* \leq c^*(s_k)]\Pr[c^* \leq c^*(s_k)] \end{aligned} \tag{12}$$

Note that the factor $\Pr[c^* \leq c^*(s_k)]$ is the same for all assets. Therefore, the fundamental determinants of the relative values of the assets are the conditional expectations $E[\tilde{x}_j \mid c^* \leq c^*(s_k)]$. Consider two assets with the same unconditional expected payoff. Given an optimal consumption allocation, c^*, the asset which is highly correlated with consumption will have low expected payoffs conditional on $c^* \leq c^*(s_k)$. The price of any payoff, by Theorem 2, can be written as a positive linear combination of these conditional expectations. Accordingly, this asset will, *ceteris paribus*, command a lower price than one which is negatively correlated with the consumption variable. It is less valuable as a hedge against consumption risk. If it does not, then the consumption variable must not be an optimum for preferences in this class. The effect of requiring consumption to be an optimum is to reduce the set of admissable price vectors to those exhibiting this type of behavior. The Appendix illustrates graphically the nature of this reduction.

The implicit state prices associated with zero arbitrage on $\hat{X}$, ε_k, $k = 1, \cdots, K$, can be interpreted as the prices of pure "consumption hedges." What type of asset, or payoff stream $x_j(s_i)$, $i = 1, \cdots, m$, would have a price equal to ε_k? Since $P_j = \sum_i q(s_i)x_j(s_i) = \sum_k \varepsilon_k \hat{x}_j(c^*_k)$, this would be a payoff stream for which the k^{th} entry in the j^{th} row of $\hat{X}$ was a one, and the other entries were zeroes. The associated payoff in X, x_j, is described as follows:

$$x_j(s_i) = \begin{cases} 0 & \{i : c^*(s_i) < c^*_{k-1}\} \\ -\dfrac{1}{\pi(c^*_{k-1})} & \{i : c^*(s_i) = c^*_{k-1}\} \\ \dfrac{1}{\pi(c^*_k)} & \{i : c^*(s_i) = c^*_k\} \\ 0 & \{i : c^*(s_i) > c^*_k\} \end{cases} \tag{13}$$

This payoff has an expected value of zero, and so represents a "fair game" only if it has a price of zero. No risk-neutral investor would purchase it for a positive price. It can only have positive value if it serves as a hedge for some consumer who is risk-averse. For the consumption stream we have hypothesized, with $c^*_{k-1} > c^*_k$, it serves this function as a pure hedge. It has negative covariance with the consumption variable over these two "consumption states." The positive price this "consumption hedge" commands is precisely ε_k. If c^*_{k-1} were not larger than c^*_k, the risk-averse consumer would want to "arbitrage" with this asset by selling it short, simultaneously receiving the proceeds of ε_k for current consumption or riskless investment and smoothing consumption across the two states. It is just this sort of "arbitrage" that the second no-arbitrage condition we have described rules out. The set of admissable price vectors are those which rule out costless hedging of consumption risk.

III. Generalization to Continuous State Spaces

The purpose of this section is to briefly outline how Theorem 2 can be generalized to continuous state spaces. In more general spaces of contingent payoffs the duality between positive state prices and zero arbitrage is ensured by versions of the Hahn-Banach theorem. These generalizations have been considered and elaborated by Ross [10], Kreps [7], Harrison and Kreps [6], and Hansen and Richard [5]. Each of these authors makes slightly different assumptions and employs a slightly different notion of what "arbitrage" means. Kreps [7] considers these complications in depth. In the interests of brevity and consistency, we will rely on the results of Harrison and Kreps [6].

Let $\bar{X}$ denote a linear space of payoffs defined on a probability space $(S, \mathscr{S}, \pi)$, and let X denote the subspace of portfolios of marketed assets. Prices of the portfolios in X are given by a linear functional V which is continuous in the L^2-norm. Strict positivity of V on X and the exclusion of "free lunches" as defined in Kreps [7] imply lack of arbitrage in this context. Theorems 1 and 2 of Harrison and Kreps [6] then ensure that, for any $x \in X$, $V(x) = E(\tilde{r}\tilde{x})$ for some strictly positive random variable $\tilde{r}$ which is bounded in L^2. This random variable can be interpreted as the state prices implicit in the valuation operator V, normalized by the probabilities.

If $\tilde{c}^*$ is a positive random variable defined on $(S, \mathscr{S}, \pi)$ and is the optimal consumption choice for a consumer who can freely trade the payoffs in X, then the existence of the positive random variable $\tilde{r}$, and the associated lack of arbitrage opportunities, would characterize exhaustively the restrictions on asset prices implied by nonsatiation (see Harrison and Kreps [6, p. 385]). If the consumer is also an expected utility maximizer with some state-independent, differentiable, concave utility index, then a necessary and sufficient condition for $\tilde{c}^*$ to be an optimum for the consumer is that $\tilde{r}$ lie within the set of marginal utilities for such a utility index. This implies that $\tilde{r}$ must be measurable with respect to the σ-algebra generated by $\tilde{c}^*$, or that $\tilde{r} = r(\tilde{c}^*)$. Strict concavity then implies that $r[\cdot]$ be strictly decreasing in its argument and we can write $r[\tilde{c}^*] = \int_{c^*}^{+\infty} \varepsilon(t)\, dt$ for some positive function $\varepsilon(\cdot)$. Together these arguments justify the following steps:

$$V(\tilde{x}) = E(\tilde{r}\tilde{x}) = E[E((\tilde{r}\tilde{x}) \mid c^*)] = E[r(\tilde{c}^*)E(\tilde{x} \mid c^*)]$$

$$= E\left[\left(\int_{c^*}^{+\infty} \varepsilon(t)\, dt\right) E(\tilde{x} \mid c^*)\right] \qquad (14)$$

The following theorem shows that an integration by parts argument can be applied to transform this last expression into one of the form $E(e\hat{x}) = V(x)$, where both $\hat{x}$ and e are consumption measurable random variables.

THEOREM 2′. *A necessary and sufficient condition for $\tilde{c}^*$ to be an optimum for some consumer with a state-independent, strictly concave, and differentiable vNM utility index is zero arbitrage on the payoff space $\hat{X}$ with typical element $\hat{x} = E(\tilde{x} \mid c < c^*)\Pr\{c \leq c^*\}$.*

Proof: If the function $r(c)$ is strictly decreasing and continuous, its integral on $[0, c^*]$ will provide the vNM index required. Thus, Equation (14) is both necessary and sufficient and we need only demonstrate it is equivalent to the conditions in the theorem.

Let $F(c^*)$ denote the cumulative density of the consumption variable. Let $dH(c^*) = E(\tilde{x} \mid c^*)dF(c^*)$ and $G(c^*) = \int_{c^*}^{\infty} \varepsilon(c)\, dc$. Then, we can write (14) as:

$$\begin{aligned} V(x) &= E\{r(c^*)E(\tilde{x} \mid c^*)\} \\ &= \int_0^{\infty} G(c^*)\, dH(c^*) \\ &= G(c^*)H(c^*)\Big|_0^{\infty} - \int_0^{\infty} H(c^*)\, dG(c^*) \end{aligned} \tag{15}$$

The last step is simply an integration by parts where $H(c^*) = \int_0^{c^*} dH(c) = E(\tilde{x} \mid c < c^*)F(c^*)$ and $dG(c^*) = -\varepsilon(c^*)$. Note that $H(0) = 0$ and $G(\infty) = 0$, so we can write the value of $x \in X$ as:

$$\begin{aligned} V(x) &= -\int_0^{\infty} H(c^*)\, dG(c^*) \\ &= \int_0^{\infty} E(\tilde{x} \mid c \leq c^*)F(c^*)\varepsilon(c^*)\, dc^* \\ &= \int_0^{\infty} \hat{x}(c^*)\varepsilon(c^*)\, dc^* \end{aligned} \tag{16}$$

Dividing and multiplying $\varepsilon(c^*)$ by $dF(c^*)$ gives $V(x) = E(\hat{x}e)$, where $e(c^*)$ is a positive normalization of $\varepsilon(c^*)$. The existence of this positive random variable is equivalent to no arbitrage. Q.E.D.

By the same arguments as given in Section II, the payoff space $\hat{X}$ may be viewed as the original payoff space adjusted for "consumption risk" and reduced to "consumption events." The random variable $\tilde{e}$ can be interpreted as a normalization of the implicit prices of "consumption hedges." It corresponds to the vector $\varepsilon(c_j^*)/\pi(c_j^*)$. The functional $V(\circ)$ corresponds to the vector P.

IV. Conclusion

Conventional wisdom suggests that behavioral consequences of risk aversion are that, while more of everything is better, consumers will sometimes take less in order to smooth consumption across contingencies. In terms of asset prices, the condition that more is better translates into zero arbitrage or the existence of positive implicit state prices. We have shown that the second condition, willingness to hedge, translates into positive implicit prices for all pure consumption hedges and zero arbitrage on a returns space which summarizes the relationship between consumption and asset payoffs.

Appendix

In this appendix, we discuss and illustrate graphically the added restrictions on the set of admissable price vectors which are imposed by concavity of the utility index.

Figure 1, adapted from Garman [3], illustrates the requirements associated with zero arbitrage for the case of two states and two assets. The vector z_1 represents the payoffs on the assets in the first state, $[x_1(s_1), x_2(s_1)]$, and similarly $z_2 = [x_1(s_2), x_2(s_2)]$. To form an arbitrage portfolio, we must choose a vector of portfolio weights which has a negative inner product with the price vector P. Such vectors form angles of more than 90° with the price vector, and thus must lie to the southwest of the line orthogonal to P, $P^{\perp}$. The other requirement for arbitrage, that the portfolio have positive payoffs in each state, requires vectors of weights which form acute angles with the vectors of payoffs z_1 and z_2. Such vectors lie to the northeast of the orthogonal subspaces $z_1^{\perp}$ and $z_2^{\perp}$. To preclude arbitrage there must be no "room" to the left of $P^{\perp}$ and to the right of $z_1^{\perp}$ and $z_2^{\perp}$ in either the southeast or northwest quadrant of the figure. This implies a price vector, such as P', that falls within the cone which has z_1 and z_2 as boundaries. Any price vector in this cone can be written as a positive linear combination of the payoffs in the two states. The weights in Figure 1, q_1 and q_2,

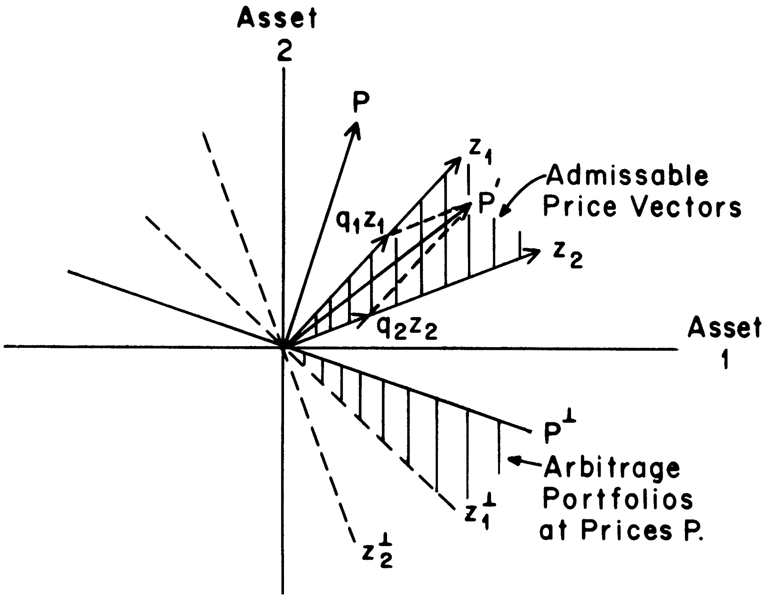

Figure 1. Arbitrage portfolios form an acute angle with the price vector and obtuse angles with asset payoffs in each state, z_1 and z_2. Price vector P' does not allow arbitrage and yields positive state prices q_1 and q_2.

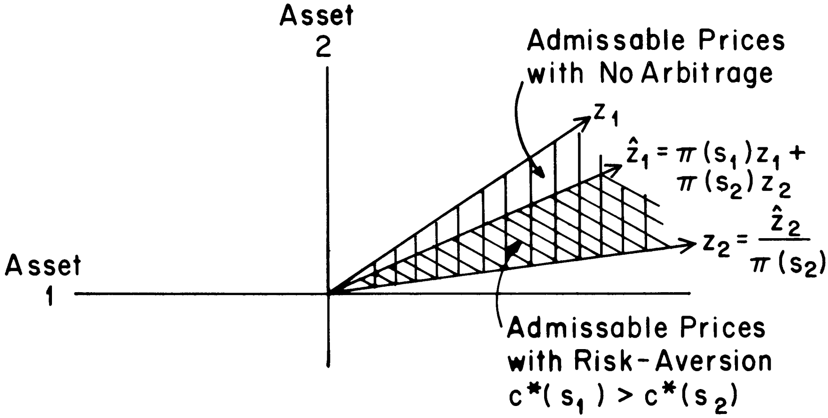

Figure 2. Requiring preferences to exhibit risk aversion shrinks the set of admissable price vectors from the zero-arbitrage cone to the smaller cone associated with no arbitrage for payoffs $\hat{z}_1$ and $\hat{z}_2$.

serve as the implicit state prices. If we added another state, but there were still only two assets, the nature of the restrictions on admissable price vectors would not change. If, for instance, the third vector, z_3, fell within the cone generated by z_1 and z_2, the set of admissable price vectors would remain exactly the same. The implicit state prices, however, would no longer be uniquely determined by P'.

Figure 2 illustrates how the set of admissable price vectors shrinks when the two assets are priced in a manner consistent with vNM preferences and consumption with $c^*(s_1) > c^*(s_2)$. The price vector must be consistent not only with zero arbitrage on X, but also on $\hat{X}$. The first column of $\hat{X}$, the vector of unconditional expected payoffs, is denoted $\hat{z}_1 \equiv \pi(s_1)z_1 + \pi(s_2)z_2$. It is a convex combination of z_1 and z_2 and so must fall within the interior of the cone. On the other hand, $\hat{z}_2 = \pi(s_2)z_2$ is simply a scalar multiple of z_2. With zero arbitrage on $\hat{X}$, the admissable price vectors must lie in the cone generated by $\hat{z}_1$ and $\hat{z}_2$, the area which is cross-hatched in Figure 2. Thus, the set of admissable price vectors collapses down on the vector z_2 as $\pi(s_2)$ approaches unity. As long as both probabilities are positive, it will be strictly smaller than the set of no-arbitrage price vectors.

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