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Capital Asset Pricing Compatible with Observed Market Value Weights

MICHAEL J. BEST and ROBERT R. GRAUER*

ABSTRACT

We show that the set of expected return vectors, for which an observed portfolio is mean variance (MV) efficient, is a two-parameter family. We identify ten ways to specify the time series behavior of the two parameters; the result highlights a number of inconsistencies involved in MV modelling. For each of the cases, it permits the inference of the time series of expected return vectors, as well as all the other Capital Asset Pricing Model (CAPM) variables, compatible with a known covariance matrix and the observed time series of market value weights. The empirical work shows that there are substantial case-to-case differences in the time series of mean vectors and many of them are quite different from the constant mean vector envisioned in tests of the CAPM.

MEAN VARIANCE (MV) portfolio selection, Markowitz [28], and the Capital Asset Pricing Model (CAPM) of Sharpe [39], Lintner [27], and Mossin [32], have formed the basis for much of the theory and empirical work in finance. Although the CAPM is a single-period model, it seems clear that both the theory and empirical tests of it have envisioned a multiperiod setting where, period-by-period, investors choose MV-efficient portfolios.¹ In this multiperiod scenario, there are no compelling theoretical reasons for assuming that the opportunity set is constant. However, the empirical literature has more or less been forced to assume a constant mean vector and covariance matrix.²

Tests of the model focused almost exclusively on whether assets plot on the Security Market Line (SML) and the evidence (Black, Jensen, and Scholes [6], Fama and MacBeth [16], and Blume and Friend [7]) seemed to support the CAPM, at least in its extended zero-beta form (Black [4]). However, there are at

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¹ This paper is concerned exclusively with discrete-time models. We assume, as does the majority of the literature, that the simple CAPM holds period-by-period. However, in continuous-time the simple CAPM relationships may not hold if the opportunity set shifts over time. See Merton [30]. Moreover, a number of papers have suggested that there are other difficulties with this approach. For example, see Rosenberg and Ohlson [38] and Cheng and Grauer [12].

² For clear statements of this assumption, see Fama and MacBeth [16], Fama [15], Roll [35], or Cheng and Grauer [11].

least two reasons for feeling uncomfortable with this evidence. First, if we generate MV-efficient frontiers from data sets comparable to those used in the empirical studies, or for individual securities, we find that the tangency portfolios, efficient in mean-standard deviation space and consistent with realistic values of either riskless or zero-beta interest rates, contain negative weights.³ Moreover, the “market” portfolio plots inside the frontier. These results are difficult to reconcile with the results from the SML tests, given Roll’s [35] reminder that securities plot on the SML if and only if the market portfolio is MV efficient. Second, the tests of the CAPM focusing on the SML assumed that the vector of betas is constant. This assumption, coupled with the assumption of a constant mean vector and covariance matrix, implies that the vector of market value weights does not change either. But market weights (and prices) change over time. Therefore, in this paper we assume that the covariance matrix is constant and infer the time series of expected return vectors and other CAPM variables that would be compatible with the time series of observed market value weights being MV efficient. This allows us to focus on two issues. The first is whether the single-period MV model is internally consistent in a multiperiod setting. The second is how different these time-varying-means scenarios are from the constant-means scenario envisioned in the testing literature.

Let μ , $\mathbf{x}_m$, and $\mathbf{1}$ be n -dimensional vectors containing one plus the expected rates of return, market value weights, and ones, respectively, and Σ be the variance-covariance matrix of asset returns, where throughout any vector $\mathbf{y}$ is assumed to be a column vector unless indicated to the contrary by transposition (e.g., $\mathbf{y}'$). Assume that, for an agreed upon universe of assets, Σ is known and $\mathbf{x}_m$ is observed. We define μ to be $(\Sigma, \mathbf{x}_m)$ -compatible if $\mathbf{x}_m$ is an efficient portfolio with respect to μ and Σ and show that, at a point in time t , the set of all $(\Sigma, \mathbf{x}_m)$ -compatible mean return vectors is a two-parameter family such that

$$\mu_t = \theta_{1t}\mathbf{1} + \theta_{2t}\Sigma\mathbf{x}_{mt}. \quad (1)$$

(1) provides an integrative framework in which we identify ten cases that focus on combinations of the two parameters that either: (i) restrict attention to a Sharpe-Lintner (SL) model; (ii) specify investor tastes (risk aversion); (iii) restrict measures of Reward-to-Risk; or (iv) minimize the distance of a $(\Sigma, \mathbf{x}_m)$ -compatible mean vector from some given vector, e.g., the vector of average historic returns. This allows us to explore the internal consistency of the model and infer the time series of expected returns for different cases.

Restrictions of the form (i)–(iii) have been discussed in the context of predicting the expected return on the market and form the basis for suggesting how firms might estimate required rates of return for capital budgeting purposes. And the assumption employed in the empirical literature that μ_t is constant is a stronger restriction than (iv). However, it is clear from (1) that it is impossible to satisfy simultaneously all four ways of restricting θ_{1t} and θ_{2t} . This implies that, if observed

³ For example, we found about half the weights were negative in the two data sets examined in this paper. Similarly, if nonnegativity constraints are imposed on either MV or expected utility portfolio selection models, the empirical evidence shows only about half the assets are held. See, for example, Grauer [20], Kallberg and Ziemba [25], Kroll, Levy, and Markowitz [26], and Pulley [34].

market value weights are to be MV efficient, they cannot be consistent simultaneously with each of the scenarios developed in the forecasting, capital budgeting, performance evaluation, and testing literature.

Additionally, the framework permits the inference of the time series of expected returns, and all the other CAPM variables, compatible with a known covariance matrix and the observed time series of market value weights. Each case may imply a different mean vector, a different frontier of risky assets, and a different SML, at a point in time. Moreover, each case implies that the means, the frontier of risky assets, the SML, and the betas must change at every point in time to insure that the observed and changing market value weights remain MV efficient. The key empirical issue is to identify how large these case-to-case differences are and, more specifically, to compare how different they are from the constant (μ, Σ) -scenario assumed in tests of the CAPM.

With the exception of Sharpe [41], Grauer [19], and Merton [31] there has been little academic research on estimating the expected returns of either individual securities or the market. Our empirical work most closely parallels Merton's. In estimating the expected return on the market portfolio, he assumed that the variance on the market is known and he restricted equilibrium expected excess returns on the market to be positive. We assume that the covariance matrix is known and infer the time series of expected return vectors and other CAPM variables. Merton predicated estimates on the assumption that the instantaneous return on the market follows an Itô process. We do not specify the process that the expected return vectors follow. We only require that the μ_t 's be $(\Sigma, \mathbf{x}_m)$ -compatible at each point in time. Merton's estimation procedures for Reward-to-Risk ratios are more sophisticated than ours, which suggests that future research might consider integrating his estimation procedures with our Reward-to-Risk cases. Finally, our empirical work is exploratory by design, very much in the spirit of Merton's. We are not concerned with "optimal" estimates of the covariance matrix. Without doubt, the estimates could be improved. The empirical relevance of the models rests on the proposition, developed theoretically by Merton and empirically by Rosenberg [37] and Black [5], that the estimates of variances and covariances calculated from time series data are more accurate than the estimates of means.⁴

The paper proceeds as follows. Section I develops the concept of $(\Sigma, \mathbf{x}_m)$ -compatible means which provides the basis for our analysis. Section II shows how to make a period-by-period CAPM tractable by specifying the time series behavior of θ_{1t} and θ_{2t} in (1). Section III examines some of the difficulties associated with MV modelling that are highlighted by our framework. Section IV describes the data and experimental design. Section V examines the time series of $(\Sigma, \mathbf{x}_m)$ -compatible means and other CAPM variables. Section VI contains

⁴ We note that the fully-specified intertemporal asset pricing model of Constantinides [14] comes reasonably close to satisfying the conditions that the market portfolio is MV efficient at each point in time and that the covariance matrix is stationary. Indeed, in such an economy, the market portfolio is MV efficient and "the covariance matrix of the rates of return of the subset of firms which are into production is also stationary" (p. 85). However, because of leverage and the nonstationarity of the interest rate, the covariance matrix of common stock returns is no longer stationary. On that subject, see Christie [13].

plots of MV-efficient frontiers and SMLs for three cases at two points in time which highlight the differences among cases in dramatic fashion. Section VII summarizes the paper.

I. $(\Sigma, \mathbf{x}_m)$ -Compatible Mean Returns

Our analysis builds on the efficient set mathematics but departs from the traditional MV model where both μ and Σ are given. As the efficient set mathematics for the traditional model is well known (see Fama [15], Merton [29], Roll [35], or Sharpe [40]), we simply note some key results and proceed from the traditional model directly to the concept of $(\Sigma, \mathbf{x}_m)$ -compatible mean returns. We formulate the MV investment problem as

$$\max\{s\mu' \mathbf{x} - \frac{1}{2}\mathbf{x}' \Sigma \mathbf{x} \mid \iota' \mathbf{x} = 1\}. \quad (2)$$

(2) can be interpreted as a parametric quadratic program. The efficient set of all μ_p, σ_p^2 pairs is traced out as the parameter s is varied. But we choose to interpret (2) as the maximization of a representative investor's MV "utility function" subject to his or her budget constraint. Under this interpretation, s is a measure of the investor's risk aversion or tastes.⁵

The first-order conditions for (2) are

$$\Sigma \mathbf{x} + \lambda \iota = s\mu, \quad (3)$$

$$\iota' \mathbf{x} = 1, \quad (4)$$

where λ is a Lagrange multiplier. Solving for $\mathbf{x}$ yields formulas for the weights in the market portfolio, say, for example,

$$\mathbf{x}_m = \frac{1}{a - \bar{r}_z c} (\Sigma^{-1} \mu - \bar{r}_z \Sigma^{-1} \iota), \quad (5)$$

where $a = \iota' \Sigma^{-1} \mu$ and $c = \iota' \Sigma^{-1} \iota$ are efficient set constants, Σ^{-1} is the inverse of the variance-covariance matrix, and $\bar{r}_z$ is the expected return on the zero-beta portfolio. Evaluating (3) for the market weights, $\mathbf{x}_m$, yields the well-known result that all securities plot on the SML in either mean-covariance space

$$\mu = \bar{r}_z \iota + \frac{(\mu_m - \bar{r}_z)}{\sigma_m^2} \Sigma \mathbf{x}_m, \quad (6)$$

or equivalently in mean-beta space

$$\mu = \bar{r}_z \iota + (\mu_m - \bar{r}_z) \beta, \quad (7)$$

where $\beta = \Sigma \mathbf{x}_m / \mathbf{x}_m' \Sigma \mathbf{x}_m$. From (3) and (6) we see that

$$1/s = (\mu_m - \bar{r}_z) / \sigma_m^2, \quad (8)$$

which states that in equilibrium the reciprocal of the representative investor's

⁵ The terms "utility" function and "risk aversion" are used in an MV framework. See Sharpe [42]. They are not to be confused with von Neumann-Morgenstern utility or with the Pratt [33]-Arrow [1] measures of risk aversion. However, the MV "utility" function is consistent with the expected utility theorem if the investor has negative exponential utility and makes joint normal probability assessments.

tastes parameter equals the reward-to-variability ratio of the market portfolio. From (3)–(6) we note that

$$\bar{r}_z = \lambda/s \quad \text{and} \quad s = 1/(a - \bar{r}_z c), \quad (9)$$

which shows that, given μ and Σ , s cannot be specified independently of $\bar{r}_z$. We also note that in the traditional SL model r_f replaces $\bar{r}_z$ in (5)–(9).

It is clear from (5) and (9) that an arbitrary choice of μ , Σ , and r_f may imply that we do not reach an equilibrium. For example, the $\mathbf{x}_m$ predicted by the MV model may not match the observed $\mathbf{x}_m$ or may not be strictly positive. In fact, with μ and Σ estimated from historical data, finding $\mathbf{x}_m > \mathbf{0}$ appears to be the exception rather than the rule.⁶

Therefore, we start with (2), but instead of solving for $\mathbf{x}_m$, given both μ and Σ , we infer the μ that makes an observed value of $\mathbf{x}_m$ the optimal solution to (2). The key result used in our subsequent analysis is given by the following theorem.

THEOREM. *If $\mathbf{x}_m \neq \mathbf{x}_q$, then the complete set of $(\Sigma, \mathbf{x}_m)$ -compatible expected return vectors is given by*

$$\mu = \theta_1 \iota + \theta_2 \Sigma \mathbf{x}_m,$$

where θ_1 and θ_2 are arbitrary scalar parameters with $\theta_2 \neq 0$.⁷

Proof: Suppose first that μ is $(\Sigma, \mathbf{x}_m)$ -compatible. Then $\mathbf{x}_m$ is optimal for (2) for some value of the parameter s . Let this value of s be s_m . Σ is positive definite and the single constraint of (2) is linear, therefore, $\mathbf{x}_m$ must satisfy the first-order conditions

$$s_m \mu = \lambda \iota + \Sigma \mathbf{x}_m. \quad (10)$$

Now $\mathbf{x}_m \neq \mathbf{x}_q$ implies $s_m \neq 0$. Consequently, the assertion of the theorem is satisfied with $\theta_2 = 1/s_m$ and $\theta_1 = \lambda/s_m$.⁸

COROLLARY. *Suppose μ is $(\Sigma, \mathbf{x}_m)$ -compatible, Σ is positive definite, and $\mathbf{x}_m \neq \mathbf{x}_q$. Then the efficient portfolios for μ and Σ are independent of θ_1 and are given by*

$$\mathbf{x}(s) = (1 - s\theta_2)\mathbf{x}_q + s\theta_2\mathbf{x}_m, \quad 0 < s < \infty. \quad (11)$$

Proof: From (3) and (4) the efficient portfolios are given by

$$\mathbf{x}(s) = \frac{\Sigma^{-1}\iota}{c} + s \left(\Sigma^{-1}\mu - \frac{a}{c}\Sigma^{-1}\iota \right).$$

(11) follows directly upon substitution for $\mathbf{x}_q$, a , and μ .

⁶ See footnote 3.

⁷ For mathematical completeness, we note that the minimum variance portfolio, $\mathbf{x}_q = \Sigma^{-1}\iota/c$, is independent of μ , so that if $\mathbf{x}_m = \mathbf{x}_q$, no knowledge of μ can be inferred and any μ is thus $(\Sigma, \mathbf{x}_m)$ -compatible. We also note that $\mathbf{x}_m = \mathbf{x}_q$ is an economically uninteresting, if not meaningless, case.

⁸ Conversely, if $\mu = \theta_1 \iota + \theta_2 \Sigma \mathbf{x}_m$ for some θ_1 and θ_2 , then $\mathbf{x}_m$ satisfies the first-order conditions for (2) with $s_m = 1/\theta_2$ and $\lambda = x_m \theta_1$.

⁹ Suppose we believe in the simple additive form of the Fisher equation: nominal rates are real rates plus the rate of inflation. It follows directly from (11) that if the rate of inflation increases and all other interest rates, including the riskless rate, increase by the same amount, then there will be no change in the market value weights. Ironically, this is almost the same “no-change-in-prices” problem that arises when μ , Σ , r_f , and β are assumed to be constant in many of the tests of the MV model.

II. Specification of the Time Series Behavior of θ_{1t} and θ_{2t}

These results provide a framework in which we can make a period-by-period CAPM tractable by giving structure to the time series behavior of θ_{1t} and θ_{2t} . We do so by choosing combinations of the two parameters that restrict $\bar{r}_{zt}$, tastes, Reward-to-Risk measures, or minimize the distance of μ_t from a given mean vector.

A. Sharpe-Lintner Cases

The most obvious restriction to place on a period-by-period CAPM is to assume that the riskless return equals the zero-beta return, i.e., $\theta_{1t} = \bar{r}_{zt} = r_{ft}$. As an empirical proposition, r_{ft} is estimated as the return on a government security of the appropriate maturity.

B. Tastes Cases

In the absence of a formal model of time-dependent tastes, the common argument is that tastes remain constant. In light of (8) and because $\bar{r}_z$ is unobservable, a natural way to specify a constant-tastes model is to set the reciprocal of the tastes parameter equal to the historical SL reward-to-variability ratio, i.e.,

$$\theta_{2t} = \frac{1}{s_t} = \frac{\bar{r}_m - \bar{r}_f}{\hat{\sigma}_m^2}. \quad (12)$$

C. Reward-to-Risk Cases

Restricting the time series of Reward-to-Risk measures has been explored in the framework of predicting the expected return on the market. We explore four ways of restricting the time series of Reward-to-Risk measures in the more general framework of predicting μ_t and the rest of the CAPM variables.

First, we might assume that the expected return on the market is constant and equal to its historic average, i.e., $\mu_{mt} = \bar{r}_m$. However, there are serious reasons for feeling uncomfortable with the constant-market-mean hypothesis: (a) It is reasonable to expect that, in an equilibrium characterized by risk-averse investors, μ_{mt} will exceed r_{ft} and that μ_{mt} will depend on r_{ft} . However, in the early 1980s the riskless return exceeded the historic market mean. (b) r_{mt} is measured in nominal terms and it is reasonable to expect that μ_{mt} will depend on the rate of inflation.

A second way to restrict Reward-to-Risk measures attempts to alleviate both these concerns by assuming that the risk premium is constant, i.e., $\mu_{mt} - r_{ft} = \bar{r}_m - \bar{r}_f$. It explicitly recognizes the dependence of μ_{mt} on r_{ft} and implicitly takes into account the level of inflation. With the exception of Merton's [31] approach, this is the usual way of predicting expected returns on the market. (See Ibbotson and Sinquefeld [23, 24], Van Horne [43], or Brealey and Myers [8].) Note, however, that specifying a constant risk premium assumes that the level of market risk is constant.

The third and fourth ways to restrict Reward-to-Risk measures take account of changing market risk by assuming that either the slope of the Capital Market Line (CML) is constant, i.e., $(\mu_{mt} - r_{ft})/\sigma_{mt} = (\bar{r}_m - \bar{r}_f)/\hat{\sigma}_m$, or that the reward-to-variability ratio is equal to a constant, i.e., $(\mu_{mt} - r_{ft})/\sigma_{mt}^2 = (\bar{r}_m - \bar{r}_f)/\hat{\sigma}_m^2$. The appeal of the latter formulation is in its tie to restrictions on investor tastes. For example, if we estimate tastes from (12), a case with a constant reward-to-variability ratio given an SL model is equivalent to a case with a constant reward-to-variability ratio given constant tastes and to a case with an SL model and constant tastes.

D. Cases That Minimize the Change in the Mean Vector

Although the other sets of cases have received some attention in the literature, the motivation for this paper came from the observations that when μ and Σ are estimated from historic data the $\mathbf{x}_m$ predicted by the MV model may not match the observed $\mathbf{x}_m$ or the $\mathbf{x}_m$ predicted by the MV model may not be strictly positive. The observations are difficult to reconcile with the testing assumption that μ and Σ are constant. But we wondered whether small changes in μ might alleviate this problem. Therefore, we examine a series of minimum shifts in the mean vector required to make the MV weights correspond to the observed market value weights. It seems natural to consider the smallest change defined in terms of the Euclidian norm given by¹⁰

$$\|\mathbf{y}\|^2 = \sum_i y_i^2,$$

where $\mathbf{y}$ is any n -dimensional vector and the y_i are its components. In the cases considered here, the objective is

$$\min \|\mu_t - \mu_0\|^2, \quad (13)$$

where μ_t is specified in (1) and in our empirical work μ_0 is the vector of average returns.

Minimizing (13) with respect to θ_{1t} and θ_{2t} provides the μ_t closest to μ_0 . If estimates of the time series of μ_t differ very little from μ_0 , we may feel more comfortable in adopting the assumption that μ_t is, at least approximately, constant over time. However, the μ_t in this case has an implied zero-beta return (that will not match the observed r_{ft}) and an implied tastes parameter. If we adhere to the SL model, it makes more sense to minimize (13) with respect to θ_{2t} and to set $\theta_{1t} = r_{ft}$. On the other hand, we may wish to specify investor tastes by minimizing (13) with respect to θ_{1t} , setting $\theta_{2t} = 1/s_t$ from (12).

Explicit formulas for θ_{1t} and θ_{2t} , in each of ten cases, are presented in Table I. Basically, SL restrictions set $\theta_{1t} = r_{ft}$ in cases 1–4 and 9. Tastes set θ_{2t} in cases 5–7 and 10. Reward-to-Risk sets θ_{2t} in cases 1–4 and θ_{1t} in cases 5–7. To derive the formulas for θ_{1t} and θ_{2t} in the Reward-to-Risk cases multiply (1) by $\mathbf{x}'_{mt}$ to find $\mu_{mt} = \theta_{1t} + \theta_{2t}\sigma_{mt}^2$. Then substitute μ_{mt} in the case description and solve. Restrictions that minimize the change in the mean vector set θ_{1t} and θ_{2t} in case

¹⁰ Defining the smallest change in terms of the Euclidean norm is both intuitive and tractable. But a natural alternative might be to weight the deviations by the market value weights.

Table I
A Summary of Different Cases That Specify θ_{1t} and θ_{2t} Where
 $\mu_t = \theta_{1t}\iota + \theta_{2t}\Sigma\mathbf{x}_{mt}$

Case Number	Case Description	Formulas for $\theta_{1t} \equiv \bar{r}_{zt}$	Formulas for $\theta_{2t} \equiv \frac{\mu_{mt} - \bar{r}_{zt}}{\sigma_{mt}^2}$
Sharpe-Lintner (SL) and Reward-to-Risk Cases			
1	Expected return on market constant given SL	r_{ft}	$(k_m - r_{ft})/\sigma_{mt}^2$
2	Excess return on market constant given SL	r_{ft}	k_p/σ_{mt}^2
3	Slope of CML constant given SL	r_{ft}	k_{sd}/σ_{mt}
4	Reward-to-variability ratio constant given SL	r_{ft}	k_v
Tastes (Risk Aversion) and Reward-to-Risk Cases			
5	Expected return on market constant given tastes.	$k_m - 1/s_t \sigma_{mt}^2$	$1/s_t$
6	Excess return on market constant given tastes	$r_{ft} + k_p - 1/s_t \sigma_{mt}^2$	$1/s_t$
7	Slope of CML constant given tastes	$r_{ft} + k_{sd}\sigma_{mt} - 1/s_t \sigma_{mt}^2$	$1/s_t$
Cases That Minimize the Change in the Mean Vector			
8	Minimize change in mean vector	$\frac{-\mathbf{h}'_t \mathbf{h}'_t \mu_0 + \mathbf{h}'_t \mathbf{h}_t \iota' \mu_0}{n \mathbf{h}'_t \mathbf{h}_t - (\mathbf{h}'_t \iota)^2}$	$\frac{n \mathbf{h}'_t \mu_0 - \mathbf{h}'_t \iota' \mu_0}{n \mathbf{h}'_t \mathbf{h}_t - (\mathbf{h}'_t \iota)^2}$
9	Minimize change in mean vector given SL	r_{ft}	$\frac{\mathbf{h}'_t (\mu_0 - r_{ft} \iota)}{\mathbf{h}'_t \mathbf{h}_t}$
10	Minimize change in mean vector given tastes	$1/n(\iota' \mu_0 - 1/s_t \mathbf{h}'_t \iota)$	$1/s_t$

Notes: $\mathbf{h}_t = \Sigma \mathbf{x}_{mt}$, while k_m , k_p , k_{sd} , k_v are constants with respect to the market mean, risk premium, risk premium-to-standard deviation ratio (or slope of the Capital Market Line (CML)), and risk premium-to-variance ratio, respectively. In the empirical section, we usually set them equal to their historic averages. Given that we establish tastes from the historic reward-to-variability ratio case 4 is equivalent to two other cases: SL given constant tastes and reward-to-variability constant given constant tastes.

8, θ_{2t} in case 9, and θ_{1t} in case 10. To find the formulas for θ_{1t} and θ_{2t} in these cases, substitute for μ_t from (1) in the case description and minimize.

III. Difficulties Associated with MV Modelling

A. Problems of Endogeneity and Consistency

Our framework highlights one of the central rules of modelling: only so many variables may be exogenous. Yet, we do not have to search far to find examples of overspecification. The testing literature, and Roll's critique of it, assumed that μ , Σ , r_f , and β are constant over time, which implies that virtually all the other variables including the market value weights are constant as well. Unfortunately this contradicts the empirical reality of changing market value weights. On the other hand, the exogenous specification of μ , Σ , and r_{ft} breaks no rules of modelling, but there are no guarantees that the MV weights will match the

observed market value weights, nor that the MV weights will be positive, nor that the MV weights will lie on the *efficient* frontier.

Our approach specifies Σ , $\mathbf{x}_{mt}$, θ_{1t} , and θ_{2t} exogenously by holding Σ constant and equating the time series of MV-efficient market value weights to their observed values. We choose θ_{1t} and θ_{2t} by placing restrictions on $\bar{r}_{zt}$, on tastes, on Reward-to-Risk measures, and by minimizing the distance of μ_t from μ_0 . But if these four variables are specified in any one of the cases, then all the other CAPM variables must be determined endogenously. Whatever is exogenous in one case may be endogenous in another, and vice versa. Simply stated, it is impossible to satisfy simultaneously all four ways of restricting θ_{1t} and θ_{2t} . Furthermore, when we specify the time series behavior of θ_{1t} and θ_{2t} , we also determine the other CAPM variables both at a point in time and through time. For example, if any one of the four Reward-to-Risk measures μ_{mt} , $\mu_{mt} - r_{ft}$, $(\mu_{mt} - r_{ft})/\sigma_{mt}$, or $(\mu_{mt} - r_{ft})/\sigma_{mt}^2$ is specified as *constant* over time, then the other three must *change* with changes in the riskless rate and market value weights. But at a point in time r_{ft} , $\mathbf{x}_{mt}$, $\mathbf{x}_q$, $\mathbf{x}_{zt}$, Σ , σ_{mt}^2 , σ_q^2 , σ_{zt}^2 , and β_t are the same across all cases.¹¹

B. Problems with Zero-Beta Returns

The return on the zero-beta portfolio has attracted considerable attention, both theoretical and empirical. The Fama-MacBeth estimates of zero-beta returns have even appeared in tests of the efficient markets hypothesis (see Charest [10]). While the theoretical importance of the concept in a single-period setting is beyond question, we see that seemingly reasonable sets of time series restrictions on θ_{1t} and θ_{2t} in a period-by-period CAPM setting may not yield results consistent with the single-period model. For example, it is well-known that in a single-period model, with different borrowing, r_b , and lending, r_l , returns, equilibrium implies $\mu_m > r_b > \bar{r}_z > r_l$. See Brennan [9] and Black [4]. But it is apparent from the formulas for θ_{1t} in Table I that it is possible for r_{ft} to exceed $\bar{r}_{zt}$. Moreover, there are serious problems associated with empirical estimates of zero-beta returns. For a given μ and Σ , $\bar{r}_z$ is unique, if the market is MV efficient. But Grauer [18] showed by example, and Roll [36] showed analytically, that if the market proxy is not MV efficient there are an infinite number of portfolios orthogonal to the market, each with a different composition, variance, and expected return. Thus, while the SML studies found a time series of *realized* zero-beta returns that yielded an average zero-beta return that exceeded the average riskless (lending) return, there is no reason to believe that the empirical results are anything but an artifact, because there is no guarantee that the market proxy was MV efficient.

In contrast, we infer a time series of *expected* zero-beta returns that is compatible with the changing weights in the market portfolio (proxy) being on

¹¹ Perhaps the only real surprise is that at a point in time the weights in the zero-beta portfolio, $\mathbf{x}_{zt}$, are the same in all cases. To see why, we note that, by definition, portfolios $\mathbf{x}_m$ and $\mathbf{x}_z$ are orthogonal if $\mathbf{x}_m' \Sigma \mathbf{x}_z = 0$. Roll [35] showed that if $\mathbf{x}_m$ is an efficient portfolio when $\mathbf{x}_z$ is a unique frontier (or minimum variance) portfolio. Now consider any two cases i and j , with $\mu^i \neq \mu^j$. The i^{th} case has a market portfolio $\mathbf{x}_m^i$ with a unique zero-beta portfolio $\mathbf{x}_z^i$. The same is true for the j^{th} case. However, $\mathbf{x}_m^i = \mathbf{x}_m^j = \mathbf{x}_m$, which implies that $\mathbf{x}_z^i = \mathbf{x}_z^j = \mathbf{x}_z$.

an MV frontier at each point in time, and that is unique given the time series specification of θ_{1t} and θ_{2t} . Moreover, at a point in time $\mathbf{x}_{zt}$ and σ_{zt}^2 are unique across cases. But, as θ_{1t} , θ_{2t} , and μ_t vary from case to case, so does $\bar{r}_{zt}$.

IV. The Data and the Experimental Design

A. The Data

We study two data bases. The first extends Cheng-Grauer's (CG) [11] 20 beta-ranked portfolios to cover the 1935–1979 period. It is patterned after Fama-MacBeth to provide a comparison with previous studies of the CAPM.¹² The second provides robustness by extending the data to include five major asset categories: common stock, corporate securities, real estate, U.S. government securities, and municipal bonds. This data base, covering the 1947–1978 period, was compiled by Ibbotson and Fall (IF) [22].

B. The Experimental Design

We assumed that the agreed on universe of assets was either the five IF asset classes or the 20 beta portfolios and for each data set estimated a covariance matrix from historical return data. Then, for each of the cases, we inferred the time series of μ_t and the other CAPM variables. The exercise provides a wealth of information on heretofore unobservable variables. Counting variables that are the same in some or all cases, we estimated well over 500,000 items for the two data sets. However, space limitations preclude reporting any but a subset of the results. Cases 2, 4, 8, and 9 appear to be the most interesting. Cases 8 and 9 come closest to capturing the spirit of the constant opportunity set, measured in nominal terms, that is envisioned in the testing literature.¹³ Case 2 extends the usual way of predicting the expected return on the market and implicitly takes the level of inflation into account. Case 4 takes account of both the level of inflation and changing market risk and is consistent with constant tastes.¹⁴

¹² While essentially the same grouping procedure was applied in the CG and Fama-MacBeth studies, there was one exception: for a company to be included in the CG data base it had to have a full data complement in the test period. This meant that fewer companies appear in the CG data base (about 650 versus 850 per year in the 1960s). For details, see Cheng and Grauer [11, p. 665–66].

¹³ However, they do not explicitly guarantee that θ_{2t} will be positive. While we did not find an instance where θ_{2t} was negative, a negative θ_{2t} value would cast further doubt on the constant (μ, Σ) -scenario envisioned in the testing literature.

¹⁴ We originally specified 18 cases. For details, see Best and Grauer [2, 3]. There are no strong a priori reasons for assuming that investor risk aversion either varies or is constant over time. Therefore, in six cases we explicitly modelled varying tastes by setting

$$\begin{aligned}\theta_{2t} &= s_t^{-1} = a_0 - r_{ft}c_0, & \text{if } s_t > 0, \\ &= s_t^{-1} = (\bar{r}_m - \bar{r}_f)/\hat{\sigma}_m^2, & \text{otherwise.}\end{aligned}$$

However these cases yielded some highly unlikely estimates of expected returns. For example, in some cases the expected rates of return on U.S. Government securities and real estate were negative over 62 percent of the time. On the other hand, the constant-market-mean cases 1 and 5 yielded somewhat more realistic estimates of expected returns. But the time series characteristics of the estimates were less than satisfactory. The expected returns were high (low) when the riskless rate was low (high).

We report the minimum, average, maximum, and standard deviation for each variable. However, because these four descriptive statistics do not capture the time series character of the data, we also correlate each variable with the riskless rate of interest. This provides two convenient benchmarks. On the one hand, if we adhere to the constant (μ, Σ) -scenario used in tests of the SML we should expect to find that $\rho(r_f, \mu_j) = 0$, and that there is no variability in the μ_{jt} series, for all j . On the other hand, if we believe that real returns are constant over time and that nominal returns change to reflect changes in the rate of inflation, we should expect to find that $\rho(r_f, \mu_j) = +1$, and that the variability in each μ_{jt} series is the same as in the r_{ft} series.

V. Time Series Results

Table II describes the historical data. Rates of return are on the left side of the table and market value weights are on the right. The table provides evidence of the empirical reality of changing market value weights that provide the driving force for our model.

Table III describes variables that are constant across cases but not over time. In particular, note that the standard deviation of the return on the market proxy varies with time. This suggests that cases reflecting changing market risk may be preferred to those that simply assume the market risk premium is constant.

Table IV shows the results for IF cases 8, 2, and 4. The means are increasingly variable, as we move from case 8 to 4, and exhibit different time series behavior. The differences in the Reward-to-Risk ratios are even more pronounced. Perhaps more important, r_{ft} exceeds $\bar{r}_{zt}$ 43.75 percent of the time in case 8. This is clearly inconsistent with single-period theory but is part of the price we pay in attempting to embed a single-period model in a multiperiod framework.¹⁵

Table V shows the results for CG cases 8, 9, and 2. In case 8 there is little variability in the expected return estimates and the correlation coefficients are near zero. These results seem to support the constant (μ, Σ) -scenario adopted in empirical studies of the SML. However, r_{ft} exceeds $\bar{r}_{zt}$ 16.67 percent of the time. Case 9, which also captures the essence of a constant opportunity set subject to the restriction that $\bar{r}_{zt} = r_{ft}$, exhibits increased variability in the expected rate of return vector and pronounced differences in the time series behavior of the means.

On the other hand, the average values of the case 2 expected returns are lower than the case 8 and case 9 expected returns and the historic means. But they change much more, with the variability matching that of the riskless rate.

¹⁵ From a broader point of view, one of the striking results in all the cases is that the average predicted expected returns on real estate (municipals) is consistently below (above) the average historic return. We attribute this result to two main sources. First, real estate indices may not reflect the true variability in real estate returns (see Hoag [21]). Second, the CAPM assumes that there are no transactions costs or taxes. These assumptions may be acceptable when analyzing a homogeneous group of assets. However, they appear less acceptable for analyzing this particular data set where real estate carries higher transactions costs than the other asset categories and municipals are the only tax-free asset.

Table II
Descriptive Statistics for the Historical Data

Panel A: Ibbotson-Fall Data											
Rates of Return (percent)						Market Value Weights (percent)					
Com	FIC	Real	USG	Mun	Mkt	Com	FIC	Real	USG	Mun	
Min	-28.33	-7.11	2.65	-2.52	-14.34	-1.22	11.27	5.85	40.01	11.25	2.77
Ave	11.79	3.03	8.19	3.23	2.02	6.97	24.47	7.60	45.60	17.00	5.34
Max	50.52	14.65	15.04	11.84	21.07	17.40	36.58	11.19	53.05	33.66	7.33
Std	18.02	5.53	3.53	3.78	7.62	4.66	7.90	1.60	3.75	5.78	1.17
$\rho(\cdot, r_f)$	-0.45	0.07	0.60	0.42	-0.04	-0.11	0.66	-0.63	-0.15	-0.79	0.79

Panel B: Cheng-Grauer Data											
P_1	P_5	P_{10}	P_{15}	P_{20}	Mkt	P_1	P_5	P_{10}	P_{15}	P_{20}	
Min	-37.20	-35.67	-31.99	-25.94	-23.56	0.43	0.89	1.35	2.18	1.53	
Ave	1.63	1.36	1.20	1.12	0.90	0.99	2.60	4.64	7.01	8.63	
Max	59.98	40.11	33.12	25.51	23.48	2.00	11.58	16.57	16.42	17.66	
Std	9.00	6.94	6.14	4.96	3.74	.27	1.76	2.98	3.33	4.32	
$\rho(\cdot, r_f)$	0.02	0.05	0.05	0.05	0.02	-0.91	0.92	0.99	0.99	1.00	

Definition of symbols: Com, Common Stock; FIC, Fixed Income Corporate Debt; Real, Real Estate; USG, U.S. Government Securities; Mun, Municipal Securities; Mkt, Market where the market return is a value-weighted average of either the five-asset category returns or the 20 beta-ranked portfolio returns; Min, minimum value; Ave, arithmetic average; Max, maximum value; Std, standard deviation; $\rho(\cdot, r_f)$, correlation of r_t , the risk-free rate of return, with the variable in column j ; and P_i , Portfolio i , $i = 1, 5, 10, 15, 20$, where P_1 is the high beta portfolio.

Note: The Ibbotson-Fall returns are annual (sample size = 32) and the Cheng-Grauer returns are monthly (sample size = 540).

Table III
Descriptive Statistics for Variables Constant over All Cases

Panel A: Ibbotson-Fall Data								
Betas								
	Com	FIC	Real	USG	Mun	σ_m	σ_z	r_f
Min	2.63	0.36	-0.00	-0.04	0.38	2.95	2.45	0.50
Ave	3.30	0.73	0.18	0.14	0.79	4.69	2.78	3.53
Max	3.83	1.42	0.56	0.63	1.61	6.58	3.64	8.00
Std	0.38	0.33	0.17	0.21	0.38	1.16	0.37	2.11
$\rho(\cdot, r_f)$	-0.49	-0.68	-0.64	-0.67	-0.68	0.66	-0.66	1.0

Panel B: Cheng-Grauer Data								
Betas								
	P1	P5	P10	P15	P20	σ_m	σ_z	r_f
Min	1.51	1.20	1.06	0.83	0.58	4.68	3.59	-0.06
Ave	1.62	1.29	1.15	0.92	0.65	5.13	3.74	0.24
Max	1.73	1.38	1.24	1.00	0.74	5.60	3.95	1.15
Std	0.06	0.05	0.05	0.05	0.04	0.25	0.09	0.23
$\rho(\cdot, r_f)$	-0.19	-0.19	-0.18	-0.17	-0.19	0.19	-0.19	1.00

Definition of symbols: σ_m , standard deviation of the return on the market portfolio; σ_z , standard deviation of the return on the zero-beta portfolio. Other symbols are defined in Table II.

Table IV
Descriptive Statistics for Selected Cases: Ibbotson-Fall Data

	Com	FIC	Real	USG	Mun	μ_m	Pf	Pf/σ_m^2	s
Case 8 Minimize the change in the mean vector ^a									
Min	9.40	4.81	3.55	3.53	4.85	4.59	-2.08	-10.77	0.04
Ave	10.82	4.97	3.73	3.65	5.09	5.65	2.13	14.00	0.10
Max	11.14	5.50	3.78	3.94	5.87	6.59	4.18	45.09	0.16
Std	0.48	0.21	0.06	0.08	0.30	0.62	1.75	15.27	0.04
$\rho(\cdot, r_f)$	0.63	-0.65	0.70	-0.19	-0.65	0.67	-0.96	-0.92	0.65
Case 2 Excess return on the market constant given SL									
Min	12.25	4.15	1.97	1.95	4.28	3.95		7.95	0.03
Ave	14.89	6.03	4.13	4.01	6.24	6.97	3.45	18.98	0.07
Max	19.88	9.87	8.27	8.06	10.00	11.45		39.64	0.13
Std	1.85	1.57	1.78	1.71	1.56	2.11		10.13	0.03
$\rho(\cdot, r_f)$	0.79	0.84	0.97	0.95	0.79	1.00		-0.67	0.64
Case 4 Reward-to-variability ratio constant given SL									
Min	5.52	2.59	1.13	1.42	2.87	1.97	1.38		
Ave	15.18	5.72	3.91	3.76	5.87	7.23	3.71	15.91	0.06
Max	24.71	10.25	8.32	8.07	10.40	13.47	6.89		
Std	5.86	2.22	1.97	1.89	2.20	3.48	1.73		
$\rho(\cdot, r_f)$	0.84	1.00	0.99	0.99	1.00	0.92	0.64		

Definition of symbols: s, tastes; μ_m , expected rate of return on the market portfolio; $Pf = \mu_m - r_f$. Other symbols are defined in Tables II & III. Symbols that denote percentage rates of return in the tables denote unity plus rates of return in the text.

^a In the same row order, the $\bar{r}_z$ statistics are 2.39, 3.36, 3.79, 0.42, and 0.66. In cases 2 and 4, $\bar{r}_z = r_f$.

Table V
Descriptive Statistics for Selected Cases: Cheng-Grauer Data

	P1	P5	P10	P15	P20	μ_m	Pf	Pf/σ_m^2	s
Case 8 Minimize the change in the mean vector ^a									
Min	1.62	1.38	1.28	1.11	0.92	1.11	-0.00	-0.02	0.31
Ave	1.62	1.38	1.28	1.12	0.92	1.17	0.94	3.59	0.37
Max	1.63	1.39	1.29	1.12	0.93	1.24	1.24	4.90	0.42
Std	0.00	0.00	0.00	0.00	0.00	0.03	0.22	0.96	0.03
$\rho(\cdot, r_f)$	-0.15	0.06	0.16	0.24	-0.03	0.18	-0.99	-0.98	0.18
Case 9 Minimize the change in the mean vector given SL									
Min	1.36	1.32	1.26	1.00	0.69	1.04	0.13	0.53	0.22
Ave	1.70	1.40	1.28	1.07	0.83	1.14	0.90	3.45	0.31
Max	1.81	1.43	1.30	1.27	1.24	1.28	1.17	4.52	1.90
Std	0.08	0.02	0.01	0.05	0.10	0.06	0.19	0.80	0.13
$\rho(\cdot, r_f)$	-1.00	-0.99	0.94	0.99	0.99	0.77	-0.98	-0.97	0.86
Case 2 Excess return on the market is constant given SL									
Min	1.06	0.84	0.74	0.58	0.39	0.64		2.23	0.31
Ave	1.37	1.14	1.04	0.88	0.69	0.93	0.70	2.66	0.38
Max	2.31	2.08	1.98	1.82	1.63	1.85		3.18	0.45
Std	0.22	0.22	0.22	0.22	0.22	0.23		0.25	0.04
$\rho(\cdot, r_f)$	0.98	0.99	0.99	0.99	0.99	1.00		-0.19	0.19

^a In the same row order, the $\bar{r}_z$ statistics are 0.40, 0.45, 0.50, 0.03, and 0.16. In cases 2 and 9, $\bar{r}_z = r_{ft}$.

Moreover, the means are correlated almost perfectly with the riskless rate. We note that, for the CG data only, the expected returns in cases 2, 3, 4, 6, and 7 are quite similar (except case 2 predicts lower average expected returns). Taken together these results seem to support the view that the distribution of excess returns may be approximately constant. However, caution is in order when drawing conclusions based on any subset of the data. For example, the tastes and Reward-to-Risk measures are quite different. In particular, the excess returns on the market vary in cases 3, 4, and 7. Moreover, if carried to the extreme, a constant excess return opportunity set and a changing riskless return imply that the market value weights never change.

In a broader sense, cases 5–8 and 10 imply that $\bar{r}_{zt} \neq r_{ft}$. This is a perfectly acceptable result provided $\bar{r}_{zt} > r_{ft}$. But over 59 percent of our $\bar{r}_{zt}$ estimates were less than the corresponding r_{ft} values.¹⁶ This is in sharp contrast to other estimates of zero-beta returns. And while the market proxies in our study are clearly imperfect and the estimates of $\bar{r}_{zt}$ may also reflect a flaw in our admittedly simple estimation procedures, it suggests that conclusions drawn from zero-beta-based CAPM tests should be viewed with suspicion. In fact, tests of CAPM-like models might require other data and new techniques that account for changing means, betas, and market weights. Frankel and Dickens [17] provide a first attempt to move in this very direction.

¹⁶ For the full 18 cases examined in our preliminary work, see footnote 14. $\bar{r}_{zt}$ was greater than r_{ft} 74 percent of the time.

VI. Graphical Results

Figures 1–4 show minimum variance frontiers of risky assets, parabolas in MV space, and the lines connecting the market and minimum variance portfolios. These lines have a slope of $\theta_{2t} = 1/s_t$ and a y-intercept of r_{ft} , in an SL model, or $\bar{r}_{zt}$, otherwise. Figures 1–3 show the 1948-frontier, drawn with solid lines, and the 1969-frontier, drawn with broken lines.

Figure 1 shows the frontiers for case 8, the case that is closest to the constant (μ, Σ) -scenario envisioned in tests of the CAPM. The frontiers are in fact fairly close together. We also note three other points. First, the variance on the market increases from 1948 to 1969. This may be attributed directly to the change in the market value weights between the two dates. Second, case 8 implies that investors are more risk averse in 1969 because the slope of the line connecting $\bar{r}_{zt}$, the market and the minimum variance portfolio, is less steep. Third, in 1948 the zero-beta rate, 2.53 percent, exceeds the riskless rate, 0.81 percent, as it should.

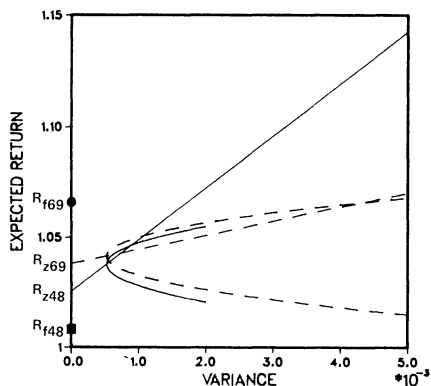


Figure 1. Case 8 Frontiers in 1948 (solid line) and 1969 (broken line)

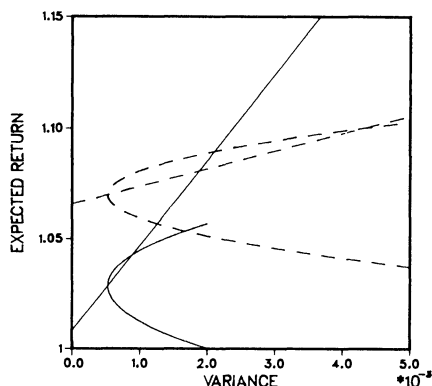


Figure 2. Case 2 Frontiers in 1948 (solid line) and 1969 (broken line)

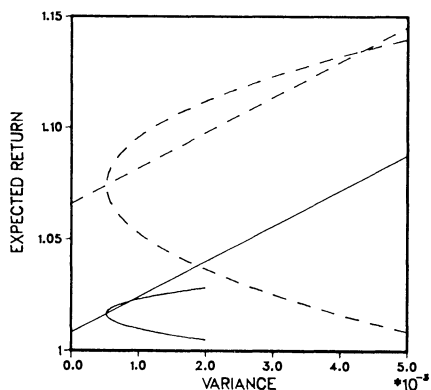


Figure 3. Case 4 Frontiers in 1948 (solid line) and 1969 (broken line)

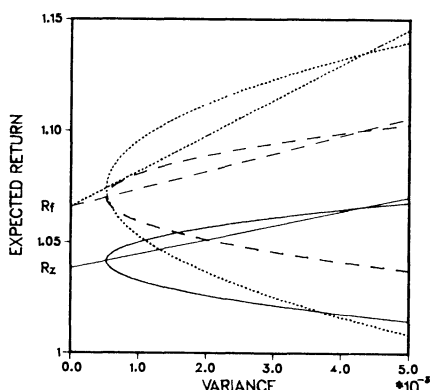


Figure 4. Case 8 (solid line), Case 2 (broken line), and Case 4 (dotted line) Frontiers in 1969

But in 1969 the riskless rate, 6.58 percent, exceeds the zero-beta rate, 3.79 percent.

Figure 2 shows the frontiers for case 2, the case with a constant market risk premium and $\theta_{1t} = r_{ft}$. Moving from 1948 to 1969, the excess return on the market is constant at 3.45 percent but the shape of the frontier of risky assets changes and investors are less risk averse in 1969.

Figure 3 shows the frontiers for case 4, a case with constant tastes, a constant reward-to-variability ratio, and $\theta_{1t} = \bar{r}_{zt} = r_{ft}$. Compare the sets of frontiers in Figures 1, 2, and 3, particularly in Figures 2 and 3. In 1948, the case 4 frontier is more compact than the case 2 frontier, i.e., the excess return on the market is 1.42 percent for case 4 versus 3.45 percent for case 2. In 1969, the situation is just the reverse, i.e., the excess return on the market is 6.89 percent for case 4 versus 3.45 percent for case 2. Again this may be attributed directly to the change in the market value weights. As the weights changed the variance on the market increased, and for the constant reward-to-variability case so did the excess return on the market.

Figure 4 shows the frontiers for case 8, with solid lines, case 2, with broken lines, and case 4, with dotted lines, in 1969. Note the differences in the shape and positions of the frontiers.

Figure 5 shows the SMLs for these same three cases in 1948 and 1969. At a point in time the betas of each asset are the same across cases. However, when the market value weights change so do the betas. Interestingly, the betas are lower in 1969 even though the case 2 and 4 expected returns are higher. Moreover, the relative degree of risk changes for real estate, the least risky in 1948, and U.S. Government securities, the least risky in 1969.¹⁷ Finally, we note the positions of the SMLs at different points in time. In 1948, in the positive quadrant, the case 4 SML plots below the case 2 and 8 SMLs. But in 1969 the higher case 4 expected excess return on the market causes its SML to plot above the case 2 and 8 SMLs.

VII. Summary

This paper has shown that the set of expected return vectors for which an observed portfolio is MV efficient is a two-parameter family, $\mu_t = \theta_{1t}t + \theta_{2t}\Sigma\mathbf{x}_{mt}$. The time series behavior of θ_{1t} and θ_{2t} was specified by placing restrictions on $\bar{r}_{zt}$, on tastes, on Reward-to-Risk measures, and by minimizing the distance of μ_t from μ_0 . This permitted the inference of the time series of expected return vectors, and all the other CAPM variables, compatible with a known covariance matrix and the observed time series of market value weights.

The estimates of the expected returns and other CAPM variables differed from case to case. The expected returns from one case, 8, appeared to be consistent with the constant (μ, Σ) -scenario envisioned in tests of the CAPM. But case 8, as well as cases 5, 6, 7, and 10, predicted expected zero-beta returns that were less than concurrent riskless returns. More specifically, over 59 percent of our $\bar{r}_{zt}$

¹⁷ In 1969 the betas for real estate and U.S. Government securities were -0.00 and -0.04 , respectively. This explains the profusion of points plotting near the y-axis in 1969.

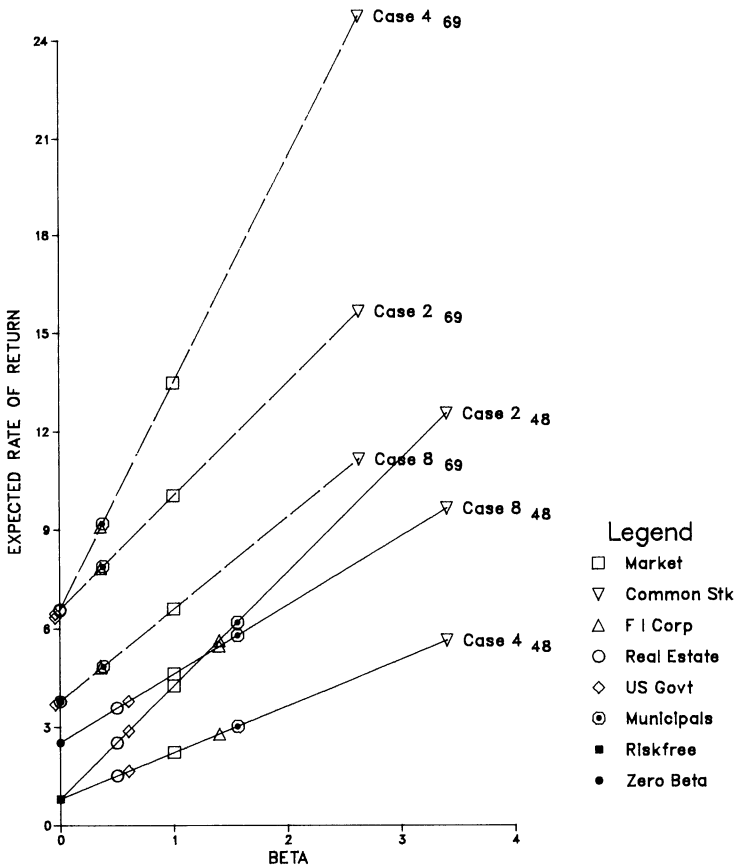


Figure 5. Case 2, 4, and 8 Security Market Lines in 1948 and 1969

estimates were less than the corresponding r_{ft} values. The expected returns from a number of other cases, 2, 3, 4, 6, and 7, appeared to be more consistent with a constant opportunity set measured in real returns. But these cases involved quite different time series behavior for the expected excess return on the market and for measures of investor risk aversion. Our framework makes it clear that these differences and inconsistencies are the price we pay in attempting to embed a single-period model in a multiperiod framework. Simply stated, it is impossible to satisfy simultaneously all four ways of restricting θ_{1t} and θ_{2t} .

REFERENCES

1. Kenneth J. Arrow. "The Theory of Risk-Bearing." *Essays in the Theory of Risk-Bearing*. Chicago: Markham Publishing Company, 1971.
2. Michael J. Best and Robert R. Grauer. "Capital Asset Pricing Compatible with Observed Market Value Weights: The Theory." Working Paper, Department of Economics, Simon Fraser University (September 1982, January 1983).
3. ——— and ———. "Capital Asset Pricing Compatible with Observed Market Value Weights: The Empirical Evidence." Working Paper, Department of Economics, Simon Fraser University (October 1982, January 1983).

4. Fischer Black. "Capital Market Equilibrium with Restricted Borrowing." *Journal of Business* 45 (July 1972), 444-55.
5. ———. "Studies of Stock Price Volatility Changes." Proceedings of the 1976 Meetings of Business and Economic Statistics Section, American Statistical Association (1976), 177-81.
6. ———, Michael C. Jensen, and Myron Scholes. "The Capital Asset Pricing Model: Some Empirical Tests." In Michael C. Jensen (ed.), *Studies in the Theory of Capital Markets*. New York: Praeger Publishing Co., 1972.
7. Marshall E. Blume and Irwin Friend. "A New Look at the Capital Asset Pricing Model." *Journal of Finance* 28 (March 1973), 19-34.
8. Richard Brealey and Stewart Myers. *Principles of Corporate Finance*. New York: McGraw-Hill, 1981.
9. Michael J. Brennan. "Capital Market Equilibrium with Divergent Borrowing and Lending Rates." *Journal of Financial and Quantitative Analysis* 6 (December 1971), 1197-1208.
10. Guy Charest. "Split Information, Stock Returns and Market Efficiency—I." *Journal of Financial Economics* 6 (June/September 1980), 265-96.
11. Pao L. Cheng and Robert R. Grauer. "An Alternative Test of the Capital Asset Pricing Model." *American Economic Review* 70 (September 1980), 660-71.
12. ——— and ———. "An Alternative Test of the Capital Asset Pricing Model: Reply." *American Economic Review* 72 (December 1982), 1201-7.
13. Andrew A. Christie. "The Stochastic Behavior of Common Stock Variances." *Journal of Financial Economics* 10 (December 1982), 407-32.
14. George M. Constantinides. "Admissible Uncertainty in the Intertemporal Asset Pricing Model." *Journal of Financial Economics* 8 (March 1980), 71-86.
15. Eugene F. Fama. *Foundations of Finance*. New York: Basic Books, 1976.
16. ——— and James D. MacBeth. "Risk, Return, and Equilibrium: Empirical Tests." *Journal of Political Economy* 81 (May 1973), 607-36.
17. Jeffrey A. Frankel and William T. Dickens. "Are Asset-Demand Functions Determined by CAPM?" IBER Working Paper No. 140, University of California (June 1983).
18. Robert R. Grauer. "Generalized Two Parameter Asset Pricing Models: Some Empirical Evidence." *Journal of Financial Economics* 6 (March 1978), 11-32.
19. ———. "The Inference of Tastes and Beliefs from Bond and Stock Market Data." *Journal of Financial and Quantitative Analysis* 13 (June 1978), 273-97.
20. ———. "Investment Policy Implications of the Capital Asset Pricing Model." *Journal of Finance* 36 (March 1981), 127-41.
21. James W. Hoag. "Towards Indices of Real Estate Value and Return." *Journal of Finance* 35 (May 1980), 569-80.
22. Roger G. Ibbotson and Carol G. Fall. "The United States Market Wealth Portfolio." *Journal of Portfolio Management* (Fall 1979).
23. ——— and Rex A. Siquefield. "Stocks, Bonds, and Inflation: Year-by-Year Historical Returns (1926-1974)." *Journal of Business* 49 (January 1976), 11-47.
24. ——— and ———. *Stocks, Bonds, Bills and Inflation: The Past (1926-1976) and the Future (1977-2000)*. Charlottesville, Virginia: The Financial Analysts Research Foundation, 1977.
25. J. G. Kallberg and W. T. Ziemba. "Comparison of Alternative Utility Functions in Portfolio Selection Problems." *Management Science* 29 (November 1983), 1257-76.
26. Yoram Kroll, Haim Levy, and Harry M. Markowitz. "Mean-Variance Versus Direct Utility Maximization." *Journal of Finance* 39 (March 1984), 47-61.
27. John Lintner. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economics and Statistics* 47 (February 1965), 13-47.
28. Harry Markowitz. "Portfolio Selection." *Journal of Finance* 7 (March 1952), 77-91.
29. Robert C. Merton. "An Analytic Derivation of the Efficient Portfolio Frontier." *Journal of Financial and Quantitative Analysis* 7 (September 1972), 1851-72.
30. ———. "An Intertemporal Capital Asset Pricing Model." *Econometrica* 41 (September 1973), 867-87.
31. ———. "On Estimating the Expected Return on the Market: An Exploratory Investigation." *Journal of Financial Economics* 8 (December 1980), 323-61.

32. Jan Mossin. "Equilibrium in a Capital Asset Market." *Econometrica* 32 (October 1966), 768-83.
33. John Pratt. "Risk-Aversion in the Small and in the Large." *Econometrica* 32 (January-April 1964), 122-36.
34. Lawrence B. Pulley. "A General Mean-Variance Approximation to Expected Utility for Short Holding Periods." *Journal of Financial and Quantitative Analysis* 16 (September 1981), 361-73.
35. Richard Roll. "A Critique of the Asset Pricing Theory's Tests; Part 1: On Past and Potential Testability of the Theory." *Journal of Financial Economics* 4 (March 1977), 129-76.
36. ———. "Orthogonal Portfolios." *Journal of Financial and Quantitative Analysis* 15 (December 1980), 1005-23.
37. Barr Rosenberg. "The Behavior of Random Variables with Non-stationary Variance and the Distribution of Security Prices." Working Paper 11, Graduate School of Business, University of California, Berkeley, 1972.
38. ——— and James Ohlson. "The Stationary Distribution of Returns and Portfolio Separation in Capital Markets: A Fundamental Contradiction." *Journal of Financial and Quantitative Analysis* 11 (September 1976), 393-402.
39. William F. Sharpe. "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk." *Journal of Finance* 19 (September 1964), 425-42.
40. ———. *Portfolio Theory and Capital Markets*. New York: McGraw-Hill, 1970.
41. ———. "Imputing Expected Security Returns from Portfolio Composition." *Journal of Financial and Quantitative Analysis* 9 (June 1974), 463-72.
42. ———. *Investments*. Second Ed., Englewood Cliffs, N.J.: Prentice-Hall, 1981.
43. James C. Van Horne. *Financial Management and Policy*. Fifth Ed., Englewood Cliffs, New Jersey: Prentice Hall, 1980.