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A Rational Expectations Model of Term Premia with Some Implications for Empirical Asset Demand Equations

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ABSTRACT

This paper derives the equilibrium time series processes characterizing the prices of bonds which differ by maturity using the CAPM relationship between expected returns. The assumption of rational expectations requires that asset demand behavior, which determines bond prices in equilibrium, be based on the covariances among returns that are implied by the assumption of market clearing. This requirement imposes nonlinear restrictions on the parameters in the solution for bond prices. Some implications for the types of comparative static exercises for which it is legitimate to assume invariant demand functions are discussed, and some numerical solutions for bond prices are derived.

THE PURE EXPECTATIONS MODEL of the term structure of interest rates has a long history in economics, but several authors (Shiller [29], Shiller, Campbell, and Schoenholtz [30], Jones and Roley [13], Campbell and Shiller [3], LeRoy and Porter [16], Mankiw and Summers [18]) have recently shown that U.S. interest rate data fairly decisively reject the joint hypothesis that the pure expectations model holds and expectations are rational. Such evidence is consistent with other empirical results suggesting financial assets differing by maturity or by issuer are not perfect substitutes. For example, Roley [25], Frankel [7, 8], Friedman [9], and Saito [27] all estimate systems of asset demand equations in which assets are imperfect substitutes because of differing risk characteristics. These empirical asset demand equations imply that expected excess returns are functions of relative asset supplies. However, the manner in which asset stocks affect returns is not restricted by any underlying model in the same way the pure expectations model of the term structure restricts the manner in which expectations of future short-term interest rates can affect current long-term rates. In addition, it is generally not possible to determine from these empirical studies what types of experiments can legitimately be analyzed under the assumption that the parameters of the asset demand equations will remain unchanged.

This paper develops a model of the term structure based on the joint hypothesis of rational expectations and risk aversion on the part of investors. Assets of different maturity can be imperfect substitutes, and the specific framework, based

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on the capital asset pricing model (CAPM), allows the role of both price expectational factors and quantity (i.e., asset stock) expectational factors to be modeled explicitly so that their possible contribution to deviations from the predictions of the pure expectations model can be investigated. It is also possible to show how the parameters of asset demand functions depend on the stochastic behavior of asset stocks and the short-term, riskless rate of interest.

Most models of the term structure when agents are risk averse (see Stiglitz [32], Roll [26]) are generally limited in their implications by two factors. First, unless some assumption is made about average holding periods, little can be said *a priori* about the sign of term premia. Second, earlier analyses have been of a partial equilibrium nature in which the uncertainty in holding period returns is treated as exogenous. In fact, this uncertainty is a reflection of the stochastic behavior of asset prices which, in turn, is a function of investor behavior. As discussed by LeRoy [14] and Lucas [17], the stochastic behavior of asset prices upon which risk averse agents base their demands for assets must, in a stationary rational expectations equilibrium, agree with the behavior of asset prices implied by that demand behavior and the assumption of market equilibrium. Asset demand models which take the stochastic processes describing all interest rates or asset prices as given will have only a limited ability to carry out comparative static exercises.

The model developed in this paper attempts to address this partial equilibrium nature of previous work. Asset demand equations are derived which are conditional on the stochastic properties of asset prices. These stochastic properties are then determined by the assumption of market equilibrium together with an assumption about the time series behavior of asset stocks and the riskless rate of interest. A general equilibrium rational expectations solution to the model is then found in which investors' assumptions about the stochastic behavior of asset prices is borne out by equilibrium prices.

Because of recent interest in estimating asset demand equations derived from a CAPM framework (Friedman [9], Frankel [7, 8]), a mean-variance framework is adopted in this paper. This is a severe restriction on the generality of the resulting model, but such a restriction seems necessary if explicit results are to be obtained. The advantage of the approach is that it allows one to derive the role of expectations of future asset stock movements in affecting current prices, just as the pure expectations model is able to derive the role of expectations of future short-term interest rate movements. The model shows that the assumption of a rational expectations equilibrium imposes a nonlinear restriction which links the variance-covariance matrix of asset prices and the parameters of the asset demand equations. To evaluate the role of a variety of factors on the term structure of interest rates, numerical solutions to the model are calculated under alternative assumptions about the variables exogenous to the model. The numerical examples suggest that changes in the volatility of short-term interest rates have large effects on the term structure.

Roll [26] uses a CAPM framework in a similar fashion to relate term premia to the betas of bonds of differing maturity. He treats the betas as exogenous, however. The present paper determines each bond's risk factor endogenously as a function of the stochastic processes characterizing the underlying asset stocks and the one-period rate of interest.

The approach developed here is less ambitious than recent work building on the consumption CAPM of Breeden [2]. Cox, Ingersoll, and Ross [5], for example, show how instantaneous interest rates and expected marginal rates of substitution are related in a continuous time model in which preferences are time additive. The uncertainty in the instantaneous rate, and therefore in the term structure, is then shown to depend on the economy's stochastic technology. The resulting expressions derived for bond prices are consistent with utility maximization and market equilibrium (see also Sundaresan [33] and Cox, Ingersoll, and Ross [4]). However, most empirical work on the term structure and on asset demand functions has utilized a discrete time approach, and a discrete time framework is assumed in the model developed here.

The first-order Euler conditions relating marginal rates of substitution to expected asset returns have also formed the basis for recent empirical investigations of asset demands (Grossman and Shiller [10], Hansen and Singleton [11]). This empirical work is in the partial equilibrium tradition of earlier research in that asset return uncertainty is treated as exogenous.

The next section reviews some of the recent empirical evidence that has cast doubt on the pure expectations model of the term structure. Section II develops the model of asset demand and Section III characterizes the general equilibrium solution under rational expectations. Section IV discusses the numerical simulations, while the paper's conclusions are summarized in Section V.

I. The Pure Expectations Model: Recent Evidence

Several recent papers have provided evidence suggesting that the pure expectations model of the term structure is not supported by the behavior of interest rates in the U.S. This evidence has taken two forms. First, researchers have found variables which seem to aid in the prediction of excess holding period returns, and second, the volatility of long-term interest rates seems to be too great to be consistent with the pure expectations model. Each of these two forms of evidence will be briefly surveyed in this section.

To provide a framework for discussion, consider the price $P(t, n)$ at time t of a zero coupon bond which pays \$1 at time $t + n$. The n-period rate of interest at time t is given by $R(t, n) = -(\ln P(t, n))/n = -p(t, n)/n$, where $p = \ln P$. The one-period return obtained from purchasing an n-period bond at time t and selling it at time $t + 1$ is given by $H(t, n) = p(t + 1, n - 1) - p(t, n)$, since an n-period bond at time t is an $(n - 1)$-period bond at $t + 1$. $H(t, n)$ is unknown at time t, but we can consider the expectation of H conditional on the information set known at time t. Defining $1(t, n)$ as the liquidity premium, we can write the identity

$$E_t H(t, n) = R(t, 1) + 1(t, n) \tag{1}$$

where $R(t, 1)$ is the riskless one-period interest rate at time t, and E_t denotes the mathematical expectation, conditional on the time t information set.

The pure expectations model is based on the assumption that bonds of all maturities are perfect substitutes. This implies that $1(t, n) = 0$ for all t and n. However, most formulations of the model allow for term-specific, but constant, risk premia. This amounts to assuming that we can write $1(t, n) = 1(n)$ for all t. Under this additional assumption, the excess holding return on an n-period bond, equal to $H(t, n) - R(t, 1)$, has expected value, conditional on information available at time t, equal to $1(n)$. Since $1(n)$ is a constant, $H(t, n) - R(t, 1) - 1(n)$ should be uncorrelated with any variable in the information set.

This implication has been tested and rejected by Jones and Roley [13], Shiller, Campbell, and Schoenholtz [30], and Campbell and Shiller [3]. Campbell and Shiller [3], for example, show that $H(t, n) - R(t, 1) - 1(n)$ is correlated with the spread between long and short rates, $R(t, n) - R(t, 1)$. This correlation should be zero under the pure expectations model since both $R(t, n)$ and $R(t, 1)$ are observed at time t. Similarly, Jones and Roley [13] find a statistically significant coefficient on foreign holdings of U.S. Treasury securities and on $R(t, n)$ in a regression of $H(t, n)$ on a set of variables which included a constant and $R(t, 1)$.

These results cast doubt on the simple expectations model of the term structure, but they provide no real alternative framework. Researchers may find correlations between term premia and addition variables if they consider a long enough list of possible variables, but what is needed is a framework which would suggest which variables should help to explain premia and the exact manner in which they should affect such premia. One advantage of the pure expectations model was its implication that expectations of future bond prices should affect current long-term bond prices in a very specific, and testable, way.

The second body of work which has found evidence to reject the pure expectations model has focused on the volatility of long-term interest rates. Shiller [29], LeRoy and Porter [16], and Singleton [31] all find evidence that suggests long-term interest rates are too volatile to be consistent with the expectations model of asset pricing. Shiller [29], for example, develops a test along the following lines. Suppose $R(t, 2)$ is the yield on a two-period bond. Under the linear approximation to the pure expectations model, $R(t, 2) = (½)[R(t, 1) + E_tR(t+1, 1)]$. Define $R'(t, 2) = (½)[R(t, 1) + R(t+1, 1)]$ as the perfect foresight two-period rate. Under the assumption of rational expectations, $R'(t, 2) = R(t, 2) + e(t+1)$ with $e(t+1)$ and $R(t, 2)$ uncorrelated. It follows that $\text{var } R' > \text{var } R$. Shiller shows that this type of argument can be used to construct variance bounds for the long-term interest rate. Observed long rates violate this bound. Similar evidence is found by LeRoy and Porter [16] and Singleton [31].

These results on the volatility of long rates have been criticized along several lines (see LeRoy [15]). Flavin [6] derives the small sample distributions of the test statistics used in volatility tests and shows that they differ significantly from the asymptotic distributions used in the earlier papers. Using the appropriate small sample distributions, Flavin finds that U.S. interest rate data do not reject the pure expectations model.

Flavin, like the other authors, assumes that the short-term interest rate follows a stationary stochastic process. Nelson and Plosser [21] and Campbell and Shiller [3] report evidence suggesting that the short-term rate is nonstationary. As Nelson and Plosser point out, such nonstationarity would imply much greater volatility in long rates than would be implied by the stationary processes assumed in the tests of excess volatility.

Finally, Pesando [22] derives the variance of the term premium, relative to the variance of the short-term interest rate, which would be necessary to produce the high volatility found in the long-term interest rate data. He then argues, on *a priori* grounds, that the implied variance of the term premium is implausibly large. His results, however, are based on the assumption that term premia and short rates are uncorrelated.

While these criticisms suggest that the evidence on excess volatility is not conclusive, the apparent predictability of excess holding period returns does indicate that the assumption of constant term premia is inappropriate. This implies that a model of term premia is required.

II. The Model

In order to examine the relationship between term premia and expectations, a simple form of the CAPM model of equilibrium asset pricing is used. This has the advantage of imposing a fairly rigid structure on the manner in which asset stocks and expectations of future asset prices can affect current asset prices. In this section, the basic framework is developed. It will be assumed that there exist outstanding stocks of government bonds of maturity one through k periods.[1] These bonds are, for simplicity, taken to be pure discount bonds; an n-period bond pays the bearer \$1 in n periods. There is no default. Let $X(t, n)$ be the aggregate number, at time t, of bonds maturing at $t + n$. Assets of the same time to maturity are assumed to be perfect substitutes no matter when they were issued. Hence, in the absence of any debt operations by the government, $X(t, n) = X(t - 1, n + 1)$. As before, $P(t, n)$ is the time t price of an n-period bond and $p(t, n)$ is the natural log of $P(t, n)$. By construction, $P(t, 0) = 1$ or $p(t, 0) = 0$.

Each investor is assumed to choose bond holdings to maximize a concave, twice continuously differentiable function of the mean and variance of end-of-period wealth. Assuming transaction costs are zero and ignoring taxes, the CAPM equilibrium relation between expected one-period returns and the riskless rate of interest is given by

$$E_t H(t, n) = R(t, 1) + R \textstyle\sum_{j=2}^{k} \sigma_{n-1,j-1} A(t, j) \qquad (2)$$

where σ_{ij} is the covariance between $p(t, i)$ and $p(t, j)$, $A(t, j)$ is the proportion of j-period bonds in the aggregate portfolio, and R is a measure of aggregate relative risk aversion which will be assumed to be constant.[2] Since $H(t, n) = p(t + 1, n - 1) - p(t, n)$ and $R(t, 1) = -p(t, 1)$, Equation (2) can also be written as

$$E_t p(t + 1, n - 1) - p(t, n) = -p(t, 1) + R \textstyle\sum \sigma_{n-1,j-1} A(t, j) \qquad (3)$$

[1] Other types of assets could be incorporated into the model. The basic assumption, to be made below, is that the economy's aggregate portfolio of outside assets is not Granger-caused by the prices of any asset with maturity greater than one period.

[2] If r_i is the coefficient of relative risk aversion for the i^{th} individual, W_i is the wealth of individual i, and W is aggregate wealth, then $R = 1/\Sigma[W_i/r_i W]$.

Equation (3) can be solved for the portfolio shares and interpreted as a set of aggregate asset demand equations, or it can be interpreted as a set of equilibrium conditions determining current asset prices, given the existing aggregate portfolio of government bonds and expectations of future asset prices:

$$p(t, n) = E_t p(t + 1, n - 1) + p(t, 1) - R \sum \sigma_{n-1,j-1} A(t, j) \quad (4)$$

for $n = 2, \cdots, k$. Equivalently, $E_t H(t, n) - R(t, 1) = 1(t, n) = R \sum \sigma_{n-1,j-1} A(t, j)$. By recursive substitution, Equation (4) can be shown to imply

$$p(t, n) = \sum_{i=1}^{n-1} [E_t p(t + i, 1) - R \sum_{j=2}^{k} \sigma_{n-i,j-1} E_t A(t + i - 1, j)] \quad (5)$$

Equation (5) expresses the current price of an n-period bond as the sum of two factors. The first, $\sum E_t p(t + i, 1)$, captures the effects of the expected path of the one-period bond price over the life of the n-period bond. It corresponds, therefore, to the value of $p(t, n)$ predicted by the pure expectations model of the term structure. This parallel can be made more apparent by noting that the yield to maturity of an n-period bond is equal to $-(1/n)p(t, n) = R(t, n)$, while $-p(t, 1) = R(t, 1)$ is the riskless one-period rate of interest. Hence, if investors are risk neutral, $R = 0$, and Equation (5) implies $R(t, n) = (1/n) \sum E_t R(t + i, 1)$. This is just one form of the standard expectations model of the term structure.

If investors are not risk neutral, current asset prices will differ from the values predicted by the pure expectations model in order to compensate investors for the risk associated with holding bonds of maturity greater than one period. This risk correction factor, as can be seen from (5), depends on the covariances among the prices of different bonds and the expected composition of the aggregate portfolio of government debt over the life of the bond. In addition, the covariances, while treated as givens by each individual investor, will be determined in equilibrium by the assumption that asset prices fluctuate to equate asset demands and supplies. The advantage of (5) is that it restricts the manner in which expectations of future values of the $A(t, j)$'s can affect current asset prices in the same manner as it restricts the way expectations of future $p(t, 1)$'s can affect current prices.

Models of the term structure generally take the behavior of the short-term rate as exogenous with respect to the term structure (see, for example, Nelson [20] or Sargent [28]). Given a description of the time series properties of the short rate, the stochastic processes governing the evolution of longer rates can be derived from the pure expectations hypothesis. In the present framework, suppose $p(t, 1)$ can be represented as a moving average process: $p(t, 1) = \sum b_i e(t - i) + p'$, where p' is any (constant) deterministic component of $p(t, 1)$ and $e(t)$ is a white noise process. The pure expectations hypothesis can be shown to imply that $p(t, n) = \sum c_i(n) e(t - i) + np'$, with $c_i(n) = [b_i + b_{i+1} + \cdots + b_{n+i-1}]$. This follows from (5) with $R = 0$, under the assumption that investors use the true process governing $p(t, 1)$ to forecast future values of the price of one-period bonds.

If $R \neq 0$, then some assumption about the behavior of the composition of the aggregate portfolio must be made before (5) can be used to derive the implied solution for the price of an n-period bond. In order to solve for the term structure of bond prices, the time series behavior of $p(t, 1)$ and $A(t, n)$, $n = 2, \cdots, k$, will be taken as exogenous. Further, it will be assumed that $(p(t, 1), A(t, 2), \cdots, A(t,$

$k)) = (p(t, 1), A'(t)) = z'(t)$ can be represented by a multivariate, mixed autoregressive, moving average process:

$$z(t) = B(L)z(t-1) + C(L)e(t) + z' \tag{6}$$

where z' is a constant deterministic component and $B(L)$ and $C(L)$ are $k \times k$ matrices of polynomials in the lag operator L.[3] Writing $B(L) = B(0) + B(1)L + \cdots$, the j^{th} row of $B(i)$ will be denoted $b_j(i)$. Similarly, let $c_j(i)$ be the j^{th} row of $C(i)$. $e(t)$ in (6) is a $k \times 1$ vector, $e'(t) = (e(t, 1)\ e(t, 2) \cdots e(t, k))$, of random disturbance terms. It is assumed that $e(t)$ is distributed independently over time with mean zero and variance-covariance matrix $\Omega = [\text{diag } \omega_{ii}]$. Thus, $e(t, 1)$ is the innovation in the price of the one-period bond, while $e(t, i)$ for $i > 1$ is the innovation in $A(t, i)$. Since $A(t, 1)$ does not appear in (6), the constraint on the sum of the $A(t, i)$'s places no restrictions on either $B(L)$ or $C(L)$; the further constraint that $0 \leq A(t, i) \leq 1$ for all t and i would.

The rational expectations equilibrium solution for $p(t, n)$ is much more complicated when investors are risk averse than when the pure expectations model holds. This is so for two reasons. First, as Equation (5) shows, $p(t, n)$ now depends on expectations of future asset stock proportions of all maturity. In addition, the assumption of rational expectations requires that investor perceptions of the risk characteristics of bonds of various maturity actually agree with the riskiness of the assets in the market equilibrium. Specifically, this requires that the covariances, σ_{ij}, which appear in (5) equal the actual covariances among asset prices implied by the assumption of market clearing. Thus, if equilibrium leads to a solution for $p'(t) = (p(t, 1)\ p(t, 2) \cdots p(t, k))$ of the form $p'(t) = p' + H(L)z(t-1) + Q(L)e(t)$ where H and Q are matrices of polynomials in L, then (5) must be satisfied by this solution with $\sigma_{ij} = Q_i(0)\Omega Q_j(0)'$ where $Q_m(0)$ is the m^{th} row of $Q(0)$. This requirement of covariance consistency will lead to nonlinear (quadratic) restrictions on the solution matrix $Q(L)$.

III. Equilibrium Bond Prices

The equilibrium solution for bond prices can be obtained by employing the method of undetermined coefficients, using (6) and a proposed solution for $p(t)$ to eliminate expectational variables from the equilibrium condition (4). Equilibrium will be characterized by a vector of constants, p', and two polynomials in L, $H(L)$ and $Q(L)$, such that

$$p(t, n) = p_n' + \sum h_n(i)L^i z(t-1) + \sum q_n(i)L^i e(t) \tag{7}$$

where $h_n(i)$ and $q_n(i)$ are $1 \times k$ vectors which satisfy, for $i = 0, \cdots$ and $n = 2, \cdots, k$

$$h_1(i) = b_1(i) \tag{8a}$$

$$h_n(i) = h_{n-1}(i+1) + h_1(i) - R \sum \sigma_{n-1,j-1} b_j(i) + h_{n-1}(0)B(i) \tag{8b}$$

$$q_1(i) = c_1(i) \tag{9a}$$

[3] At this stage, no restrictions need be placed on the roots of $I - B(L)L$ or the order of $C(L)$; $z(t)$ can be nonstationary.

$$q_n(i) = q_{n-1}(i + 1) + q_1(i) - R \sum \sigma_{n-1,j-1}c_j(i) + h_{n-1}(0)C(i) \tag{9b}$$

$$\sigma_{nj} = q_n(0)\Omega q_j'(0) \tag{10}$$

The vector of constants, p', must satisfy

$$p_n' = p_{n-1}' + p_1' - R \sum \sigma_{n-1,j-1}A_j' + h_{n-1}(0)z' \tag{11}$$

where A_j' is the j^{th} element in z', and all the summations in (8)–(11) run from $j = 2$ to $j = k$.

If p', $H(L)$, and $Q(L)$ solve (8)–(11), then $p(t) = p' + H(L)z(t-1) + Q(L)e(t)$ is consistent with market equilibrium, given the behavior of $p(t, 1)$ and $A(t)$ assumed in (7), and the covariances on which investors base their asset demands will be equal to the actual covariances observed in the market equilibrium.

For a given $\Sigma = [\sigma_{ij}]$, Equations (8), (9), and (11) yield unique solutions for $H(L)$, $Q(L)$, and p_n'. For $n = 2$, $b_1(i + 1) + b_1(i) - R \sum \sigma_{1,j-1}b_j(i) + h_1(0)B(i)$ is the unique solution for $h_2(i)$, for all i. If $h_s'(i)$ is the solution for $h_s(i)$ for all i, then $h_s'(i + 1) - b_1(i) - R \sum \sigma_{s,j-1}b_j(i) + h_s'(0)B(i)$ is the unique solution for $h_{s+1}(i)$. A similar argument applies to the solution for $q_j(i)$. Bubble solutions cannot arise here because bond prices are anchored by the requirement that $p(t, 0) = 0$. Equations (8), (9), and (11), however, only impose expectational consistency on the means. Any solution to these equations is still consistent with an equilibrium in which investors hold systematically incorrect expectations about the riskiness of assets of different maturity. Equation (10) imposes consistency on the covariances by requiring investors' subjective estimates of Σ to equal the objective values implied by the assumption of market clearing. Since (10) is quadratic in the parameters of $Q(0)$, there will generally be multiple solutions.

Suppose $B(L)$ is of order r and $C(L)$ is of order s; $b_j(i) = 0$ for $i > r$ and $c_j(i) = 0$ for $i > s$. Then, for all n and for $i > r$, (8b) implies $h_n(i) = h_{n-1}(i + 1) = 0$. There exists then, a unique $k \times k$ matrix $H(L) = H(0) + H(1)L + \cdots + H(r)L^r$ of order less than or equal to r which satisfies (8) for a given Σ. Similarly, the maximum order of $Q(L)$ is s and there exists a unique solution to (9) for a given Σ.

Using (10), Equations (8b) and (9b) can be written for $n \geq 2$ as

$$h_n(i) = h_{n-1}(i + 1) + h_1(i) - R \sum q_{n-1}(0)\Omega q_{j-1}'(0)b_j(i) + h_{n-1}(0)B(i) \tag{12a}$$

$$q_n(i) = q_{n-1}(i + 1) + q_1(i) - R \sum q_{n-1}(0)\Omega q_{j-1}'(0)c_j(i) + h_{n-1}(0)C(i) \tag{12b}$$

It does not seem possible to develop a simple analytic expression for the solution values of $h_n(i)$ and $q_n(i)$ using (12a) and (12b). However, some general conclusions about the outcome of comparative static experiments can be drawn.

Since the $q_j(i)$'s are functions of the $c_j(i)$'s and Σ is a function of the $q_j(i)$'s, empirically estimated asset demand or term structure relationships implied by the model will not remain invariant to changes in the stochastic behavior of either $p(t, 1)$ or the $A(t, n)$'s. From (3), if $\Sigma^{-1} = [\sigma^{ij}]$,

$$A(t, i) = (1/R) \sum \sigma_{i-1,n-1}\{E_t p(t + 1, n - 1) - p(t, n) + p(t, 1)\} \tag{13}$$

Frankel [8] has recently estimated demand functions which take the form of (13), and he has imposed the restriction that Σ be equal to the covariance matrix of actual returns. However, Equations (8)–(11) show that Σ is determined by the time series behavior of $p(t, 1)$ and the $A(t, n)$'s. Changes in the elements of $B(L)$, $C(L)$, or Ω will produce shifts in Σ. Estimating (13) over a sample period during which monetary policy operating procedures changed, for example, would be inappropriate if the policy change affected the manner in which the short-term interest rate responded to financial market disturbances.

Empirically estimated functions of the form of (13) or (3) cannot generally be used to estimate the impact of changes in asset stocks, even if, as in Frankel's work, the parameter restrictions implied by the presence of Σ in (13) are imposed. For example, Frankel [8] tests the hypothesis that a rise in the supply of long-term government debt increases the expected excess return on long-term debt by testing whether the variance of the excess return on such debt is significantly positive. From (3), the coefficient on $A(t, n)$ in the equation for $E_tH(t, n) + p(t, 1)$ is $R\sigma_{n-1,n-1}$ so that Frankel tests, in the present notation, $H_0: \sigma_{n-1,n-1} = 0$.

This, however, is not a test of the hypothesis that an innovation in $A(t, n)$ increases $E_tH(t, n) + p(t, 1)$. To see this, (7) can be used to show that

$$\begin{aligned} E_tH(t, n) + p(t, 1) = {} & \textstyle\sum_{i=0}^{r} [h_{n-1}(i + 1) - h_n(i) + b_1(i)]z(t - i - 1) \\ & + h_{n-1}(0)z(t) + \textstyle\sum_{i=0}^{s} [q_{n-1}(i + 1) - q_n(i) \\ & + c_1(i)]e(t - i) + p'_{n-1} - p'_n + p'_1 \end{aligned} \tag{14}$$

Using (12),

$$\begin{aligned} E_tH(t, n) + p(t, 1) = {} & \textstyle\sum_{i=0}^{r} R \sum_{j=2}^{k} \sigma_{n-1,j-1} b_j(i) z(t - i - 1) \\ & + \textstyle\sum_{i=0}^{s} R \sum_{j=2}^{k} \sigma_{n-1,j-1} c_j(i) e(t - i) + R \sum_{j=2}^{k} \sigma_{n-1,j-1} A'_j \end{aligned} \tag{15}$$

From (15), it follows that $R\sigma_{n-1,n-1}$ measures the effect on $E_tH(t, n) + p(t, 1)$ of a change only in A'_n, the *deterministic* component of the stock of asset n.[4] The effect on the excess expected return to asset n of an innovation in asset n is equal to $R \sum \sigma_{n-1,j-1} c_{jn}(0)$ where $c_{jn}(0)$ is the n^{th} element in the $1 \times k$ row vector $c_j(0)$. Notice that this effect cannot be estimated solely from an estimate of the asset demand functions in (13) since an estimate of $c_{jn}(0)$ is also required.

Equation (15) shows that $R\sigma_{n-1,n-1}$ correctly measures the effect on the expected excess return on n-period bonds due to an innovation in the proportion of n-period bonds in the aggregate portfolio if and only if the n^{th} column of $C(0)$ has a 1 in the n^{th} row and zeros elsewhere. This is just the condition that innovations in asset n do not instantaneously cause either the price of one-period bonds or the proportion in the portfolio of any assets except 1 and n. It is unlikely, however, than an innovation in $A(t, n)$ which is balanced by an equal change in $A(t, 1)$ (since $\Sigma A(t, i) = 1$) would leave $p(t, 1)$ unaffected. The same conclusions apply to cross-effects. The effect of an innovation in $A(t, s)$ on the expected excess return on asset n is given, not by $R\sigma_{n-1,s-1}$, but by $R \sum \sigma_{n-1,j-1} c_{js}(0)$. Assets n and s are q-substitutes if this expression is positive (Walsh [34]). An innovation in $A(t, s)$, if asset s serves as a substitute for asset n, reduces the demand for n-period bonds. The expected excess return on

[4] See also Roley [24] and Walsh [35].

n-period bonds must rise to restore equilibrium. Such substitution effects cannot be estimated solely from asset covariances or from the parameters of asset demand systems such as (13). Knowledge of the moving average component of the $z(t)$ process is also required.

When analyzing the impact of innovations in the $A(t, j)$'s, or changes in the deterministic elements in z', it is legitimate to treat the elements of Σ as fixed. Any change in $B(L)$, $C(L)$, or Ω, the characteristics of the time series process governing the evolution of $z(t)$, will generally not leave Σ unaffected. Since bond prices must adjust to maintain market equilibrium, a new path for the composition of the aggregate portfolio implies a new stochastic structure to bond prices. If this affects the covariances among returns, the demand functions will shift.

Bodie, Kane, and McDonald [1] have used an equation similar to (3) in order to estimate risk premia from U.S. data. Consistent with the argument made here, they find a sizable shift in these premia when the 1980–81 period is compared to the 1973–79 period (see their Table 2). If the Federal Reserve's change in its operating procedures in October 1979, for example, affected the time series properties of debt proportions or the short-term rate of interest, a shift in risk premia would occur. Asset demand functions would not be policy invariant (Walsh [36]).

IV. Numerical Solutions for the Term Structure

Given the analytical intractability of the rational expectations solution when R is nonzero, it is instructive to numerically solve the model under alternative assumed values for the parameters describing the evolution of $z(t)$ in order to determine the manner in which the term structure depends on the parameters of the model. For given values of $B(L)$, $C(L)$, and Ω, the model can be solved numerically for the polynomials $H(L)$ and $Q(L)$ and the matrix Σ which characterize equilibrium asset prices.

Numerical solutions are easy to obtain by exploiting the structure of the model as given by Equations (8b) and (9b). Noting that $h(1, i) = b(1, i)$ and $q(1, i) = c(1, i)$, these two equations can be solved recursively for $H(L)$ and $Q(L)$, given an initial value for Σ. Once a solution is obtained, Equation (10) can be used to form a new estimate of Σ. Iterating between (8b), (9b), and (10) in this manner led to rapid convergence in all cases.[5] This iteration process corresponds to the way in which markets might approach the rational expectations equilibrium. Given investor views about the riskiness of assets, asset demands would be characterized by the solution to (8) and (9). If the resulting behavior of asset returns is not consistent with investors' initial assessments of risk, investors will revise their views on the covariances among returns. This leads to a new solution to (8) and (9). The process would continue until the observed behavior of prices leads to no further revisions in investors' expectations about the relative riskiness of returns.

In order to make use of parameter values which might have some relevance to

[5] The starting value chosen for Σ was $\sigma_{11} = \omega_{11}$, $\sigma_{ii} = 1$ for $i = 2, \cdots, k$, and $\sigma_{ij} = 0$ for $i \neq j$. Convergence was usually achieved after four iterations.

the observed term structure, a time series model for the 1-year return on U.S. Treasury bills was estimated to obtain values for $b_1(i)$, $c_1(i)$, and ω_{11}. Thus, the time period implicit in the numerical solutions is 1 year. The value of k is taken to be 10, so that the term structure is derived for bonds with 1 to 10 years to maturity.

Data on the rate of return on U.S. Treasury bills for 1-year holding periods were taken from Ibbotson and Sinquefield [12].[6] Flavin [6], in her analysis of interest rate volatility, assumes that the short rate (taken as a 3-month rate) follows a first-order autoregressive process with parameter 0.95. Estimating the same model for the 1-year rate (sample period 1950–81) produced quite similar results:

$$\underset{}{R(t,\ 1)} = \underset{(0.014)}{0.047} + \underset{(0.08)}{0.953}R(t-1,\ 1) \qquad \text{S.E.} = 0.0146 \tag{16}$$

where standard errors are in parentheses.

Campbell and Shiller [3] report that the 3-month rate is better characterized as a difference stationary process, and they estimate an ARIMA(1, 1, 1) model for the short rate. For the 1-year rate, this model produces the following estimated equation:

$$(1 + \underset{(0.479)}{0.568}L)(1 - L)R(t,\ 1) = \underset{(0.002)}{0.005} + (1 + \underset{(0.0375)}{0.799}L)e(t) \qquad \text{S.E.} = 0.0135 \tag{17}$$

A check of the residuals suggested that they were white noise. As discussed by Nelson and Plosser [21], and as will be demonstrated below, these two alternative representations of the time series behavior of the short rate have implications for the variance of longer term rates which differ substantially.

To serve as a baseline reference for the later numerical solutions, the term structure implied by the pure expectations model with $R = 0$ can be derived when the short rate follows either an AR(1) or an ARIMA(1, 1, 1) process. Using the parameter estimates reported above for these two processes, the implied equilibrium solution for the prices of discount bonds with 2 to 10 years to maturity is given in Table I. The same results expressed in terms of yields to maturity at annual rates are presented in Table II. For example, from Table II, the yield to maturity of a 10-year bond is given by

$$R(t,\ 10) = 0.047 + 0.775R(t-1,\ 1) - 0.813e(t,\ 1)$$

when $R(t, 1)$ follows the estimated AR(1) process given in (16).

The top and bottom halves of Table II differ in the manner one should expect, given the different specifications of the time series process for $p(t, 1)$. When $p(t, 1)$ follows a stationary AR(1) process, the coefficient on $e(t, 1)$, the innovation in the short rate, declines with maturity; when $p(t, 1)$ is ARIMA(1, 1, 1) it increases. In this latter case, the conditional variance of one-period returns on

[6] That these are not the ideal data is clear since the model assumes that the one-period return on the shortest bill is known with certainty. This is not the case for the annual yield on a portfolio of Treasury bills with less than 1 year to maturity. However, the objective here is not to estimate a model of the risk-free rate, but simply to obtain illustrative numbers for the subsequent simulations.

Table I

Term Structure under Risk Neutrality: Prices

	Maturity									
	1	2	3	4	5	6	7	8	9	10
(1) AR(1) $p(t, 1) = -0.047 + 0.953p(t-1, 1) + e(t, 1)$										
p'	−0.045	−0.139	−0.273	−0.448	−0.662	−0.913	−1.20	−1.52	−1.87	−2.25
$p(t-1, 1)$	0.953	1.861	2.727	3.552	4.338	5.087	5.801	6.481	7.129	7.747
$e(t, 1)$	1.0	1.953	2.861	3.727	4.552	5.338	6.087	6.081	7.481	8.129
σ_{ii}	0.0002	0.0008	0.0016	0.0028	0.0041	0.0057	0.0074	0.0092	0.0112	0.0132
(2) ARIMA(1, 1, 1) $(1 + 0.568L)(1 - L)p(t, 1) = -0.005 + (1 + 0.799L)e(t, 1)$										
p'	−0.005	−0.012	−0.023	−0.037	−0.054	−0.074	−0.098	−0.124	−0.154	−0.187
$p(t-1, 1)$	0.432	1.187	1.758	2.433	3.050	3.700	4.331	4.972	5.608	6.247
$p(t-2, 1)$	0.568	0.813	1.242	1.567	1.950	2.300	2.669	3.028	3.392	3.753
$e(t, 1)$	1.0	2.211	3.302	4.461	5.582	6.724	7.85	8.992	10.125	11.266
$e(t-1, 1)$	0.799	1.116	1.703	2.148	2.675	3.155	3.661	4.153	4.652	5.147
σ_{ii}	0.0002	0.0010	0.0022	0.0040	0.0062	0.0090	0.0123	0.0162	0.0205	0.0254

Table II

Term Structure under Risk Neutrality: Rates (in Percentage Points)

	Maturity									
	1	2	3	4	5	6	7	8	9	10
(1) AR(1) $R(t, 1) = 4.7 + 0.953R(t-1, 1) - e(t, 1)$										
R'	4.7	4.7	4.7	4.7	4.7	4.7	4.7	4.7	4.7	4.7
$R(t-1, 1)$	0.953	0.931	0.909	0.888	0.868	0.848	0.829	0.810	0.792	0.775
$-e(t, 1)$	1.0	0.977	0.954	0.932	0.910	0.890	0.870	0.850	0.831	0.813
$\sqrt{\sigma_{ii}}$	1.41	0.69	0.45	0.33	0.26	0.21	0.18	0.15	0.13	0.11
(2) ARIMA(1, 1, 1) $(1 + 0.568L)(1 - L)R(t, 1) = 0.5 - (1 + 0.799L)e(t, 1)$										
R'	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$R(t-1, 1)$	0.432	0.593	0.586	0.608	0.610	0.617	0.619	0.621	0.623	0.625
$R(t-2, 1)$	0.568	0.407	0.414	0.392	0.390	0.383	0.381	0.379	0.377	0.375
$-e(t, 1)$	1.0	1.106	1.101	1.115	1.116	1.121	1.122	1.124	1.125	1.126
$-e(t-1, 1)$	0.799	0.558	0.568	0.537	0.535	0.526	0.523	0.519	0.517	0.515
$\sqrt{\sigma_{ii}}$	1.41	0.78	0.52	0.39	0.32	0.26	0.23	0.20	0.18	0.16

Table III

Expected Excess Returns in Percentage Points ($R = 2$)

$\mathrm{EXH}(n) = E_t H(t, n) + p(t, 1)$

	Maturity									
	1	2	3	4	5	6	7	8	9	10
(1) AR(1) $p(t, 1) = -0.047 + 0.953p(t-1, 1) + e(t, 1)$										
EXH	0	0.09	0.17	0.24	0.32	0.39	0.46	0.52	0.58	0.64
$\sqrt{\sigma_{ii}}$	0	1.41	2.76	4.05	5.27	6.44	7.55	8.61	9.62	10.58
(2) ARIMA(1, 1, 1) $(1 + 0.568L)(1 - L)p(t, 1) = -0.005 + (1 + 0.799L)e(t, 1)$										
EXH	0	0.10	0.23	0.34	0.46	0.58	0.70	0.81	0.93	1.05
$\sqrt{\sigma_{ii}}$	0	1.41	3.13	4.67	6.31	7.89	9.51	11.11	12.72	14.32

bonds with more than one period to maturity is greater, for all maturities, than when $p(t, 1)$ is AR(1). Note that the nonstationarity of $p(t, 1)$ imparts nonstationarity to the yields to maturity for all maturities.

To assess the role of risk premia generated solely by the stochastic nature of the short-term yield, the model can be solved for a nonzero R but under the assumption that asset stocks for bonds of maturity 2 to 10 years are nonstochastic. Thus, the innovation in $p(t, 1)$ is the only source of uncertainty. Rather than report the complete solution values for $H(L)$ and $Q(L)$, it will be convenient to express the results in terms of expected excess holding period yields, defined as $E_t p(t + 1, n - 1) - p(t, n) + p(t, 1) = E_t H(t, n) + p(t, 1)$. Under the pure expectations model of risk-neutral investors, these expected excess returns should be zero.

As Table III shows, these expected excess returns are relatively small when $R = 2$.[7] They would double if $R = 4$.[8] As expected, the excess returns implied by the assumption that $p(t, 1)$ is nonstationary exceed those implied by the AR(1) assumption by progressively larger amounts as maturity increases. On a 10-year bond, uncertainty about the short rate adds 64 basis points to the expected one-period yield if $p(t, 1)$ is AR(1) and 105 basis points if $p(t, 1)$ is ARIMA(1, 1, 1).

When the vector $A(t)$ is stochastic, the equilibrium structure of bond prices will depend on the time series process describing the joint evolution of $p(t, 1)$ and $A(t)$. The simplest case to consider is one in which each $A(t, n)$ is equal to a constant plus a mean zero white noise process: $A(t, n) = A'(n) + e(t, n)$. In the simulations, it is assumed that $E_t e(t, n)^2$ equals 0.0005 for $n = 2, \cdots, 10$. Also, to introduce contemporaneous correlation between the innovations in the portfolio composition variables and the price of one-period bonds, suppose innovations in $p(t, 1)$ are accompanied by open market operations in an attempt

[7] Bodie, Kane, and McDonald [1] provide figures for the composition of government debt by maturity (see their Table A-1). Their figures suggested letting $A' = (0.34\ 0.13\ 0.09\ 0.03\ 0.05\ 0.03\ 0.02\ 0.02\ 0.07)$. These numbers for the deterministic component of the $A(t, n)$'s were used in constructing Tables III–VI.

[8] Grossman and Shiller [10] find that a value of R as large as 4 is necessary in order to explain the observed volatility of stock returns.

by the monetary authority to partially stabilize the short-term rate of interest. If such open market operations are carried out in one- and two-period bonds, a negative correlation might arise between $e(t, 1)$ and $A(t, 2)$ as a positive innovation in $p(t, 1)$ induces a sale by the monetary authority of one- for two-period bonds. Similarly, any debt operations between one- and two-period bonds, which would introduce a negative correlation between innovations in $A(t, 1)$ and $A(t, 2)$, would lead to a positive correlation between $p(t, 1)$ and $A(t, 2)$. Such contemporaneous correlations can be analyzed by letting off-diagonal elements of $C(0)$ be nonzero. For example, suppose

$$(1 - 0.953L)p(t, 1) = -0.047 + e(t, 1) + 0.5e(t, 2) \tag{18a}$$

or

$$(1 + 0.568L)(1 - L)p(t, 1) = -0.005 + e(t, 1) + 0.5e(t, 2) \tag{18b}$$

and

$$A(t, 2) = A_2' - 0.5e(t, 1) + e(t, 2) \tag{19}$$

$$A(t, n) = A_n' + e(t, n), \qquad n = 3, \cdots, 10 \tag{20}$$

Equilibrium expressions for expected excess returns for the stochastic structure given by Equations (18)–(20) are presented in Table IV.

As a comparison of Tables III and IV reveals, the additional uncertainty introduced when asset supplies are stochastic causes a rise in the deterministic component of expected excess returns (i.e., the unconditional equilibrium risk premia). For example, the unconditional expected excess return on a 10-period bond rises from 64 to 104 basis points in the AR(1) case, and it rises by 25 basis points in the ARIMA(1, 1, 1) case. More importantly, risk premia are no longer constant but depend on the realizations of all the stochastic shocks.

With the parameter values underlying Table IV, all assets are q-substitutes. However, substitution effects are small. A one standard deviation innovation in $A(t, 2)$, for example, raises the expected excess return on the 10-period bond by only one basis point. As Table IV shows, the variance of excess returns increases dramatically with maturity for both the AR(1) and the ARIMA(1, 1, 1) specifications of the one-period bond price.

Because the spread between long and short rates, $-p(t, 10)/10 + p(t, 1)$, is correlated with the $e(t, i)$'s, this spread will also be correlated with excess returns. Such a result for U.S. data is reported by Campbell and Shiller [3].

Recently, Mascaro and Meltzer [19] have attempted to assess the effect of risk on the level of long- and short-term interest rates within an IS-LM framework. Using a CAPM model of the term structure, the effect of increased uncertainty can also be analyzed. Tables V and VI report the equilibrium solutions for expected excess returns after increases in the variance of $e(t, 1)$ and $e(t, 2)$, respectively. Thus, Table V uses the same specification as Table IV except that ω_{11} is increased from 0.0002 to 0.002. This increase in the volatility of the riskless one-period interest rate has a very large impact on expected excess returns across the entire term structure. The unconditional expected excess return on 10-year bonds rises from 1.04% to 6.80% in the AR(1) case, and bond prices of all

Table IV

Expected Excess Returns with Stochastic Asset Stocks

$c_1(0) = (1\ 0.5\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$

$c_2(0) = (-0.5\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$

	Maturity									
	1	2	3	4	5	6	7	8	9	10
AR(1) $p(t, 1) = -0.047 + 0.953p(t-1, 1) + e(t, 1) + 0.5e(t, 2)$										
Constant	0	0.14	0.27	0.40	0.52	0.63	0.74	0.84	0.94	1.04
$e(t, 1)$	0	−0.03	−0.06	−0.09	−0.12	−0.15	−0.17	−0.20	−0.22	−0.24
$e(t, 2)$	0	0.06								
$e(t, 3)$	0	0.13	0.25							
$e(t, 4)$	0	0.19	0.36	0.53						
$e(t, 5)$	0	0.24	0.47	0.69	0.90					
$e(t, 6)$	0	0.30	0.58	0.85	1.10	1.34				
$e(t, 7)$	0	0.35	0.68	0.99	1.29	1.58	1.85			
$e(t, 8)$	0	0.40	0.77	1.13	1.47	1.80	2.11	2.41		
$e(t, 9)$	0	0.44	0.86	1.26	1.65	2.01	2.36	2.69	3.00	
$e(t, 10)$	0	0.49	0.95	1.39	1.81	2.21	2.59	2.96	3.31	3.64
$\sqrt{\sigma_{ii}}$	0	1.80	3.52	5.16	6.72	8.20	9.62	10.97	12.26	13.48
ARIMA(1, 1, 1) $(1 + 0.568L)(1 - L)p(t, 1) = -0.005 + (1 + 0.799L)e(t, 1) + 0.5e(t, 2)$										
Constant	0	0.15	0.29	0.44	0.58	0.72	0.87	1.01	1.16	1.30
$e(t, 1)$	0	−0.03	−0.06	−0.09	−0.12	−0.16	−0.19	−0.22	−0.25	−0.28
$e(t, 2)$	0	0.06								
$e(t, 3)$	0	0.12	0.25							
$e(t, 4)$	0	0.19	0.37	0.56						
$e(t, 5)$	0	0.25	0.49	0.74	0.99					
$e(t, 6)$	0	0.31	0.62	0.92	1.23	1.54				
$e(t, 7)$	0	0.37	0.74	1.11	1.48	1.85	2.22			
$e(t, 8)$	0	0.43	0.86	1.29	1.73	2.16	2.59	3.02		
$e(t, 9)$	0	0.49	0.99	1.48	1.97	2.47	2.96	3.45	3.94	
$e(t, 10)$	0	0.55	1.11	1.66	2.22	2.77	3.33	3.88	4.44	4.99
$\sqrt{\sigma_{ii}}$	0	1.80	3.51	5.27	7.02	8.78	10.53	12.29	14.04	15.80

maturities become much more responsive to shocks to either the one-period bond price or to any of the asset stock variables.

Similar results are evident in Table VI which differs from the specification in Table IV only in that the variance of $e(t, 2)$ is increased from 0.0005 to 0.005. While this has a smaller impact on equilibrium returns than a tenfold rise in the variance of $e(t, 1)$, the results are qualitatively similar. Again, changes in the stochastic behavior of prices or asset stocks at the short-term end of the term structure have a large impact on long-term bond prices and expected excess returns when investors are risk averse.

One important implication of the model of the term structure developed here is that comparative statics cannot generally be done under the assumption that investors' beliefs about the riskiness of assets will remain unchanged. In calculating the solution in Table VI, for example, the increase in the variance of $e(t, 2)$ results in a new set of asset demand equations as investors adjust to the new

Table V

Expected Excess Returns with Stochastic Asset Stocks and Increased Variance of $p(t, 1)$

$$c_1(0) = (1\ 0.5\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$c_2(0) = (-0.5\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$\omega_{11} = 0.002$$

	Maturity									
	1	2	3	4	5	6	7	8	9	10
AR(1) $p(t, 1) = -0.047 + 0.953p(t-1, 1) + e(t, 1) + 0.5e(t, 2)$										
Constant	0	0.91	1.77	2.60	3.39	4.14	4.85	5.53	6.18	6.80
$e(t, 1)$	0	−0.21	−0.42	−0.61	−0.79	−0.97	−1.14	−1.30	−1.45	−1.59
$e(t, 2)$	0	0.42								
$e(t, 3)$	0	0.83	1.62							
$e(t, 4)$	0	1.22	2.38	3.49						
$e(t, 5)$	0	1.59	3.10	4.54	5.92					
$e(t, 6)$	0	1.94	3.79	5.55	7.23	8.83				
$e(t, 7)$	0	2.27	4.44	6.51	8.48	10.36	12.15			
$e(t, 8)$	0	2.59	5.06	7.42	9.67	11.81	13.85	15.80		
$e(t, 9)$	0	2.89	5.66	8.29	10.81	13.20	15.48	17.65	19.72	
$e(t, 10)$	0	3.18	6.23	9.12	11.89	14.52	17.03	19.42	21.70	23.87
$\sqrt{\sigma_{ii}}$	0	4.61	9.01	13.21	17.20	21.01	24.65	28.11	31.40	34.55
ARIMA(1, 1, 1) $(1 + 0.568L)(1 - L)p(t, 1) = -0.005 + (1 + 0.799L)e(t, 1) + 0.5e(t, 2)$										
Constant	0	1.08	2.35	3.52	4.75	5.94	7.15	8.35	9.56	10.76
$e(t, 1)$	0	−0.21	−0.46	−0.69	−0.93	−1.16	−1.40	−1.63	−1.87	−2.10
$e(t, 2)$	0	0.42								
$e(t, 3)$	0	0.92	2.01							
$e(t, 4)$	0	1.38	3.01	4.49						
$e(t, 5)$	0	1.86	4.05	6.06	8.18					
$e(t, 6)$	0	2.32	5.07	7.59	10.23	12.81				
$e(t, 7)$	0	2.79	6.11	9.13	12.32	15.42	18.57			
$e(t, 8)$	0	3.26	7.13	10.67	14.39	18.01	21.69	25.33		
$e(t, 9)$	0	3.73	8.17	12.21	16.48	20.61	24.82	28.99	33.19	
$e(t, 10)$	0	4.20	9.19	13.75	18.55	23.21	27.95	32.65	37.37	42.07
$\sqrt{\sigma_{ii}}$	0	4.61	10.26	14.99	20.22	25.30	30.47	35.59	40.74	45.90

equilibrium values for the covariance matrix of returns. One way to assess the importance of these shifts in behavioral relationships is to repeat the analysis of the impact of a change in the time series properties of $p(t, 1)$ and/or $A(t)$ under the assumption that the parameters of the asset demand equations given in Equation (13) remain unchanged. The top half of Table VII reports the results of introducing correlation between $e(t, 1)$ and $e(t, 3)$ while increasing A_3' and decreasing A_3', but with investors not revising their estimate of the covariance matrix of asset returns.[9] That is, investor demands are based on the covariances obtained from the equilibrium solution found in Table IV. The bottom half of

[9] A_2' is raised from 0.13 to 0.18, while A_3' is reduced from 0.09 to 0.04. Only the case in which $p(t, 1)$ is ARIMA(1, 1, 1) is reported.

Table VI

Expected Excess Returns with Stochastic Asset Stocks and Increased Variance of $e(t, 2)$

$$c_1(0) = (1\ 0.5\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$c_2(0) = (-0.5\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$\omega_{22} = 0.005$$

	Maturity									
	1	2	3	4	5	6	7	8	9	10
AR(1) $p(t, 1) = -0.047 + 0.953p(t-1, 1) + e(t, 1) + 0.5e(t, 2)$										
Constant	0	0.62	1.20	1.76	2.29	2.80	3.28	3.74	4.18	4.59
$e(t, 1)$	0	−0.15	−0.28	−0.41	−0.54	−0.66	−0.77	−0.88	−0.98	−1.08
$e(t, 2)$	0	0.29								
$e(t, 3)$	0	0.56	1.10							
$e(t, 4)$	0	0.83	1.61	2.36						
$e(t, 5)$	0	1.08	2.10	3.07	4.00					
$e(t, 6)$	0	1.31	2.56	3.75	4.88	5.96				
$e(t, 7)$	0	1.54	3.00	4.40	5.73	6.99	8.20			
$e(t, 8)$	0	1.76	3.42	5.01	6.53	7.97	9.35	10.66		
$e(t, 9)$	0	1.96	3.83	5.60	7.29	8.90	10.44	11.91	13.30	
$e(t, 10)$	0	2.16	4.21	6.16	8.02	9.80	11.49	13.10	14.63	16.10
$\sqrt{\sigma_{ii}}$	0	4.81	7.42	10.86	14.14	17.27	20.25	23.09	25.79	28.37
ARIMA(1, 1, 1) $(1 + 0.568L)(1 - L)p(t, 1) = -0.005 + (1 + 0.799L)e(t, 1) + 0.5e(t, 2)$										
Constant	0	0.53	0.83	1.27	1.64	2.04	2.42	2.81	3.20	3.59
$e(t, 1)$	0	−0.15	−0.22	−0.34	−0.43	−0.54	−0.64	−0.74	−0.84	−0.95
$e(t, 2)$	0	0.29								
$e(t, 3)$	0	0.45	0.70							
$e(t, 4)$	0	0.68	1.07	1.62						
$e(t, 5)$	0	0.86	1.37	2.08	2.68					
$e(t, 6)$	0	1.08	1.71	2.60	3.34	4.17				
$e(t, 7)$	0	1.28	2.03	3.08	3.97	4.95	5.87			
$e(t, 8)$	0	1.48	2.36	3.58	4.61	5.75	6.83	7.94		
$e(t, 9)$	0	1.69	2.69	4.08	5.25	6.54	7.76	9.03	10.27	
$e(t, 10)$	0	1.89	3.01	4.57	5.89	7.34	8.71	10.13	11.52	12.93
$\sqrt{\sigma_{ii}}$	0	3.81	5.93	9.01	11.59	14.43	17.13	19.92	22.66	25.42

Table VII shows the new rational expectations equilibrium expressions for expected excess returns when investors do revise their estimate of Σ.

Comparing the top half of Table VII with the bottom half of Table IV, the change in the stochastic behavior of the aggregate portfolio is predicted to result in a small decline in unconditional expected excess returns under the assumption that the parameters of the asset demand equations are constant. In addition, the change is predicted to produce no change in the responsiveness of expected excess returns to any of the stochastic shocks except $e(t, 1)$. Innovations in the price of one-period bonds reduce expected excess returns across all maturities by an amount that is approximately three times that found in Table IV.

Because the change in the time series properties of $p(t, 1)$ and $A(t)$ does not

Table VII

Effect of Induced Shifts in Asset Demand Equations

$c_1(0) = (1\ 0.5\ 0.5\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$

$c_2(0) = (-0.5\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$

$c_3(0) = (-0.5\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$

$A' = (0.34\ 0.18\ 0.04\ 0.03\ 0.05\ 0.03\ 0.02\ 0.02\ 0.07)$

	Maturity									
	1	2	3	4	5	6	7	8	9	10
Expected Excess Returns When Covariance Matrix Is Assumed Fixed										
Constant	0	0.14	0.28	0.43	0.57	0.71	0.85	0.99	1.13	1.28
$e(t, 1)$	0	−0.10	−0.19	−0.28	−0.37	−0.46	−0.56	−0.65	−0.74	−0.83
$e(t, 2)$	0	0.06								
$e(t, 3)$	0	0.12	0.25							
$e(t, 4)$	0	0.19	0.37	0.56						
$e(t, 5)$	0	0.25	0.49	0.74	0.99					
$e(t, 6)$	0	0.31	0.62	0.92	1.23	1.54				
$e(t, 7)$	0	0.37	0.74	1.11	1.48	1.85	2.22			
$e(t, 8)$	0	0.43	0.86	1.29	1.73	2.16	2.59	3.02		
$e(t, 9)$	0	0.49	0.99	1.48	1.97	2.47	2.96	3.45	3.94	
$e(t, 10)$	0	0.55	1.11	1.66	2.22	2.77	3.33	3.88	4.44	4.99
$\sqrt{\sigma_{ii}}$	0	2.12	3.86	5.81	7.67	9.58	11.46	13.36	15.25	17.15
Equilibrium Expected Excess Returns										
Constant	0	0.19	0.34	0.52	0.68	0.85	1.02	1.19	1.36	1.53
$e(t, 1)$	0	−0.13	−0.23	−0.34	−0.45	−0.57	−0.68	−0.79	−0.90	−1.01
$e(t, 2)$	0	0.09								
$e(t, 3)$	0	0.16	0.30							
$e(t, 4)$	0	0.24	0.45	0.67						
$e(t, 5)$	0	0.32	0.59	0.89	1.18					
$e(t, 6)$	0	0.39	0.74	1.11	1.47	1.83				
$e(t, 7)$	0	0.47	0.88	1.33	1.76	2.20	2.63			
$e(t, 8)$	0	0.55	1.03	1.55	2.05	2.56	3.06	3.57		
$e(t, 9)$	0	0.63	1.18	1.77	2.34	2.92	3.50	4.08	4.65	
$e(t, 10)$	0	0.70	1.32	1.99	2.63	3.28	3.93	4.58	5.23	5.88
$\sqrt{\sigma_{ii}}$	0	2.12	3.86	5.81	7.67	9.58	11.46	13.36	15.25	17.15

leave the covariances among asset prices unchanged, investors will revise their estimates of the risk characteristics of the different bonds, and this will induce shifts in the demand functions for bonds of different maturity. Allowing for such shifts, the results in the bottom half of Table VII show the equilibrium expressions for expected excess returns when investors have rational expectations about both the first and second moments of asset returns. As can be seen from a comparison with Table IV, the change considered actual results in an increase in unconditional expected excess returns rather than the decline predicted under the assumption of constant demand functions. Also, expected returns are much more sensitive to innovations in all the exogenous variables. Thus, assuming that the parameters of the asset demand equations are invariant to this particular change in the stochastic behavior of asset stock variables would lead to a failure to predict the rise in longer term interest rates relative to the one-period rate which would occur.

V. Conclusions

One difficulty with recent attempts to empirically estimate asset demand systems is their failure to indicate the types of comparative static exercises which can be analyzed under the assumption of a constant structure. If assets are imperfect substitutes because their risk characteristics differ, then changes in the stochastic behavior of the aggregate portfolio of outside assets are likely to produce shifts in asset demand equations.

This suggests, for example, that it may not be appropriate to analyze alternative debt management policies or monetary policy operating procedures under the assumption that the parameters of the private sector's asset demand functions remain unaffected.

By dealing with assets which differed only in maturity, the CAPM equilibrium relationship between expected returns was used to show how the equilibrium time series behavior of the term structure of bond prices was related to the stochastic behavior of the riskless one-period rate of interest and the composition of the aggregate portfolio. Under the assumption of rational expectations, both the expected returns and the return covariances used by risk-averse investors in choosing their portfolio must agree with the properties of returns predicted by the assumption of market equilibrium. This requirement was shown to place nonlinear restrictions on the parameters characterizing the behavior of equilibrium asset prices.

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