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The Trading Decision and Market Clearing under Transaction Price Uncertainty

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ABSTRACT

This paper models an individual's trading decision, given: (1) his/her demand function to hold shares of an asset, (2) his/her expectation on what the market clearing price will be, and (3) the design of the market which determines how orders will be translated into trades. The particular market design we consider is the batched trading (periodic call) regime. Assuming investors are distributed according to their propensities to hold shares, we model the aggregation of orders to obtain market clearing values of price and volume and to show the way in which, with trading friction, these solutions differ from Pareto efficient values. The importance of this analysis for various issues concerning market design is noted.

WHEN MARKETS ARE NOT frictionless and transaction prices are uncertain, the act of trading requires analysis. This paper models an individual's trading decision in secondary financial asset markets. In so doing, we study the order flow as an endogenous part of the system. We show how, given an underlying demand curve, the trader will choose an optimal (price, quantity) order to transmit to the market, and we model the aggregation of orders to obtain market clearing values for price and volume. These models are used to show the way in which the prices and trading volume established on the market will in general differ from their Pareto efficient values.

Much of the microstructure literature that studies the behavior of transaction prices in a nonfrictionless market has assumed that orders are generated by some exogenously given stochastic process. This has been true, for instance, in the analyses of dealer markets (see, e.g., Garman [8], Amihud and Mendelson [1], Stoll [16], and Ho and Stoll [10, 11]), continuous auction matching markets (see Cohen, Maier, Schwartz, and Whitcomb [3]), call market trading regimes (see Garbade and Silber [7] and Mendelson [13]), and the effect of market fragmentation on the volatility of clearing prices (see Mendelson [14]).[1] Also, the

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[1] Cohen, Maier, Schwartz, and Whitcomb [5], in their analysis of how an investor's decision to place a market or limit order affects the market spread, have also recognized that the order flow depends on individual trading strategies. However, their analysis does not solve for the optimal orders of the individual traders, nor do they model the market clearing solutions.

microstructure literature has not, thus far, provided an economic model that relates observed transaction prices and volume to the unobserved "equilibrium" values of price and volume. Goldman and Beja [9] have used a stochastic diffusion process to model the deviation of actual prices from an "underlying value;" but their analysis, although generating important implications concerning the variance and serial correlation of returns, does not present an explicit model of the elements of the market's microstructure which cause market clearing prices and equilibrium prices to deviate from each other in the first place. Our paper is largely motivated by this gap in the literature.

We structure the analysis for a batched trading regime where orders are accumulated for simultaneous execution at a single price at periodic market "calls." The call market system is used by various European exchanges (for example in Belgium, France, and Germany) and by the Tel Aviv Exchange in Israel. Further, the opening procedure used in various continuous market trading systems (most notably the New York Stock Exchange) is also a batched trading arrangement. For this system, we first derive the optimal size and price of an order for each trader, given his/her demand function to hold shares and his/her expectations concerning what the market clearing price will be. We then aggregate the orders to determine the market clearing transaction price and volume.

We also derive the Walrasian price and volume that would be observed if the market were frictionless. We show that two conditions must be met in order that transaction prices equal Walrasian prices. These conditions are:

(1) symmetry in the distribution of individual buy/sell orders, and
(2) accurate investor expectations concerning the market clearing price.

We further show that even when conditions (1) and (2) are met, transaction volume will in general differ from its Walrasian value. For this reason, under transaction price uncertainty, a Pareto efficient distribution of shares will not be realized after any round of trading (i.e., there will be a demand to recontract).

In certain respects, our approach is similar to that of the auction bidding literature, and so too is our interest in showing that market results depend upon the specification of the trading procedure.[2] The primary differences between our analysis and the bidding literature are that we consider a market in which both buyers and sellers are competitive, and allow quantity as well as price to be a decision variable for the individual trader. In addition, our analysis provides a framework for analyzing a broader array of issues concerning market design than has been considered by the auction literature. These issues include: the importance of disseminating "floor" information, the role of a market maker, and the relative advantages/disadvantages of the batched and the continuous trading regimes. However, a detailed examination of these issues is beyond the scope of the current paper, and will be left for future research.

[2] Starting with Vickrey [17], much of the auction literature has focused on the one-sided sealed bid auction (which is representative of, for example, Treasury Bill and mineral lease auctions). The literature has also developed implications for the open auction (both the English ascending-price and the Dutch descending-price varieties). See Wilson [18] and Milgrom [15] for examples of some of the major contributions, and Engelbrecht-Wiggans [6] for a survey of the extensive literature.

The paper is organized as follows. Section I presents the model of the individual investor's trading decision. This includes a specification of the investor's objective function, and a formulation of his/her optimal price, quantity decision in the batched regime. Section II then shows how orders are aggregated to give the market clearing price and volume, and models the market clearing price and volume for both a frictionless Walrasian market and a nonfrictionless regime with batched trading. In Section III, we use the models developed in Sections I and II to examine the effect of trading friction and market design. We show that, under asymmetric buy/sell propensities across traders and/or biased expectations concerning the market clearing price, prices and volume will deviate from their Pareto efficient values. Section IV contains our concluding remarks.

I. The Individual Trading Decision

Assume many atomistic investors, who trade a constant aggregate number of shares among themselves within an interval of time, T_0 to T_1. For each investor, let the share holding attained at T_1 be maintained for some future investment period which is unique to that investor. Assume no short selling restriction, and that transaction size Q and price per share P are continuous variables.

A variety of transaction costs cause the market to be nonfrictionless. These range from information and decision costs to order handling and settlement costs. Essentially, the impact of all of them is the same—they restrict the ease with which the investor can interact with the market. This restriction is both in terms of the frequency with which the investor seeks to trade, and the number of orders placed when he/she does seek to trade. We reflect the incidence of these costs by letting the investor participate at each market call, but by constraining the number of orders that he/she can submit. For the purpose of model simplicity, we have the investor submit one buy and one sell order.

Let the trading system be a batched regime where the market is "called" once in the interval T_0 to T_1. One clearing price P^c is established at the end of the period such that the number of shares offered for sale at P^c or below equals the number sought for purchase at P^c or above.[3] If an investor submits an order to buy Q shares at P, and if $P \geq P^c$, he/she receives Q shares at P^c. If, on the other hand, $P < P^c$, then the individual receives no shares. The reverse is true for sell orders. Assume that no recontracting is allowed within the trading period, and that all accounts are settled at T_1. Within this context, we first specify an individual investor's demand function, and then model his/her trading decision.

[3] There will always be an exact and unambiguous determination of P^c and trading volume if and only if the aggregate buy and sell functions are continuous and nowhere vertical or horizontal, a condition that is satisfied by our analysis which takes price and quantity to be continuous variables, all traders to be atomistic, and the distribution of traders to be continuous. In reality, of course, steps exist in the market buy and sell functions, and exact market clearing is typically not possible. For this reason, most call markets allocate excess supply or demand on the "heavy" side by random selection, pro rata allocation, or by prioritizing orders (by type, size, or time of arrival). A few exchanges (notably Tokyo's *gekitaku* trading) encourage floor traders and market makers to enter balancing orders by disallowing trading unless supply exactly equals demand.

To do so, we present a set of sufficient conditions for an individual to have a negatively sloping demand function to hold shares of a risky asset. We then show how he/she picks orders in relation to this demand function when trading is not a frictionless process.

A. The Trader's Demand Function

Consider a representative trader with an investment period T_1 to T_2[4] with wealth at T_2 of W_2, and a utility function, $U(W_2)$, twice continuously differentiable with $U' > 0$ and $U'' < 0$. At T_1 (immediately after the accounts are settled), let the investor have cash, C_1, and N_1 shares of the risk asset. His/her expected (derived) utility is therefore given by

$$EU(C_1 r_1 + N_1 \tilde{X}) \tag{1}$$

where E is the expectation operator, $(r_1 - 1)$ is the risk-free rate of interest over the investment period (T_1 to T_2), and $\tilde{X}$ is the stochastic share value of the risk asset at the end of the investment period (T_2).

At T_0 (the start of the trading period), let the investor have cash, C_0, and N_0 shares of the risk asset. If during the trading period, he/she were to trade Q shares of the risk asset at price P (and since all accounts are settled at T_1), the derived utility at T_1 would be expressed as[5]

$$\Phi(P, Q \mid N_0, C_0) = EU[(C_0 r_0 - QP)r_1 + (N_0 + Q)\tilde{X}] \tag{2}$$

where $(r_0 - 1)$ is the risk-free rate of interest over the trading period, and where, by convention, $Q > 0$ denotes a purchase and $Q < 0$ denotes a sale.

We make two simplifying assumptions that facilitate the economic interpretation of the model: let the investor's utility function be quadratic so that $U'''(W) = 0$, and let the squared deviation of the expected rate of return on the risk asset from the risk-free rate be small.[6] Then Equation (2) becomes (see Appendix A for the derivation)

[4] The purpose of this analysis is to show how an investor's (price, quantity) order is specified in relation to a demand curve. To this end, we use a single-period investment model. For the simple buy and hold scenario, the investor makes a trading decision at the beginning of the holding period. At the end of the period, he/she simply liquidates the position at the market price (an act which does not involve a trading decision).

[5] For simplicity and without loss of generality concerning the individual decision model, we assume no explicit cost of placing one order. Further, again for simplicity, we assume that the trader can borrow at the rate $(r_1 - 1)$ at T_1. This assumption may be relaxed to allow the trader to borrow at a rate $(r^B - 1) > (r_1 - 1)$. In this case, the trader's demand function need not be differentiable, although it remains continuous. Since the function is downward sloping (as is indicated later in the section), the qualitative conclusions of the paper would not be altered.

[6] The assumption of quadratic utility is sufficient for our purpose of focusing on the demand curve as it expresses the tradeoff between risk and return for an asset. To interpret the second simplifying assumption, let $h = \bar{X} - Pr_1$ be the "risk premium." As P changes, ceteris paribus, so too does terminal wealth. As a result, the risk-aversion parameter will, in general, change, and so too will the slope of the demand curve. By assuming h^2 sufficiently small in Equation (A1), we ignore this second-order effect.

$$\Phi(P, Q \mid N_0, C_0) = \kappa + \gamma Q(a - bQ - P) \tag{3}$$

where

$\kappa = U(W) + \tfrac{1}{2}U''(W)N_0^2\text{Var}(\tilde{X})$ is the expected utility if the investor does not trade,

$\gamma = U'(W)r_1$,

$a = (U'(W)\bar{X} + U''(W)N_0\text{Var}(\tilde{X}))/U'(W)r_1$, and

$b = -U''(W)\text{Var}(\tilde{X})/2U'(W)r_1$.

We use Equation (3) to obtain two alternative demand functions to hold shares of the risk asset. The first is a *reservation (price) demand function* given by

$$\Phi(P^R, Q \mid N_0, C_0) = \kappa \tag{4}$$

Since, for any transaction size Q, the reservation price P^R is the price at which the individual would be indifferent between trading Q shares and not trading at all, it follows from Equation (3) that the reservation demand function is

$$P^R = a - bQ \tag{5}$$

The second is an *ordinary (price) demand function* given by

$$\frac{\partial \Phi}{\partial Q}(P^0, Q \mid N_0, C_0) = 0 \tag{6}$$

Since, for the ordinary demand function, Q is chosen at each P such that the trader's expected utility Φ is maximized, differentiating the expression for Φ in Equation (3) and equating it to zero, we get

$$P^0 = a - 2bQ \tag{7}$$

When we constrain the investor to submitting just one order to buy and one to sell, we can analyze the specification of each order separately (note that in the batched regime only one of these orders will execute at any market call).

B. An Investor's Optimal Trading Decision

To specify the trader's optimal trading decision, we refer, for expositional simplicity, to buy orders only unless otherwise noted. From Equation (5), the reservation buy function is

$$P^{RB} = a - bQ \qquad \text{for} \quad Q > 0 \tag{8}$$

where P^{RB} is the reservation price for the buy order.

Substituting Equation (8) into (3), we may rewrite derived utility as

$$\Phi = \kappa + \gamma Q(P^{RB}(Q) - P) \tag{9}$$

where P and Q are realized transaction price and size. Prior to the trade, however, P and Q are uncertain.[7] In relation to this uncertainty, the investor specifies an order to maximize the expected derived utility, $E\Phi$.

[7] Transaction price uncertainty exists in all exchanges. It is reduced in continuous-trading exchanges by on-line dissemination of transaction prices and/or bid-ask quotations. It may be reduced in batch trading markets by the transmission of "indicated" prices, at which the market would clear if the call were to be held at that instant. Among batch trading markets, only the Amsterdam Exchange disseminates "indicated" prices beyond the exchange floor.

In a batched regime, the investor transacts order size Q at the uncertain clearing price, $\tilde{P}^c$, when the buy (sell) order price, P, is greater (less) than or equal to $\tilde{P}^c$. Now let $G(x)$ be the subjective probability of the clearing price $\tilde{P}^c$ being less than x. We assume that P^c is distributed normally with mean μ and standard deviation σ.[8] Thus,

$$G(x) = \text{prob}(P^c < x) = N((x - \mu)/\sigma) \tag{10}$$

where $N(\cdot)$ is a standardized cumulative normal distribution. Since, in a batched regime there is no transaction for $x > P$, it follows that the objective function (from Equation (9)) reduces to

$$\max_{P,Q} \int_{-\infty}^{P} Q(P^{RB}(Q) - x)G'(x)\,dx \tag{11}$$

Recognizing that $Q(P^{RB}(Q) - x)$ is a measure of consumer surplus, Equation (11) can be interpreted as the maximization of expected consumer surplus. From Equation (11), it is clear that, for a buy order, the optimal price given Q is $P^b(Q) = P^{RB}(Q)$. This is because each integrand is positive for $P < P^{RB}(Q)$ and negative for $P > P^{RB}(Q)$. Hence (11) becomes

$$\max_{Q} \int_{-\infty}^{P^{RB}(Q)} Q(P^{RB}(Q) - x)G'(x)\,dx \tag{12}$$

Differentiate Equation (12) with respect to Q, set it equal to zero, and simplify (see Appendix B); then, the buy order comprises the optimal bid price P^b and the optimal order size Q^b, which are solutions to the following simultaneous equations:

$$2\pi^b - \alpha = -\nu(-\pi^b) \tag{13a}$$

and

$$\frac{bQ^b}{\sigma} = \alpha - \pi^b \tag{13b}$$

where $\nu(x)^{-1} = N(-x)/f(x)$ is Mill's ratio,[9] a function of one parameter, x, with $f(x) = N'(x)$, $\alpha = (a - \mu)/\sigma$, and $\pi^b = (P - \mu)/\sigma$.

π^b is the buy order price normalized with respect to the expected market clearing price μ. By the definition of $G(x)$, and because π^b is a normalized variable, $N(\pi^b)$ is the subjective probability that the order will execute. Since at price a the investor would not seek to transact, α can be viewed as a measure of his/her desire to buy.

In similar fashion, we can derive the optimal sell price P^s and sell order size Q^s from:

$$\alpha - 2\pi^s = -\nu(\pi^s) \tag{14a}$$

and

$$-\frac{bQ^s}{\sigma} = \pi^s - \alpha \tag{14b}$$

where

$$\pi^s = \frac{P^s - \mu}{\sigma}$$

[8] Use of the normality assumption enables us to obtain an explicit solution for an individual's optimal buy/sell orders.

[9] See Kendall and Stuart [12, p. 156].

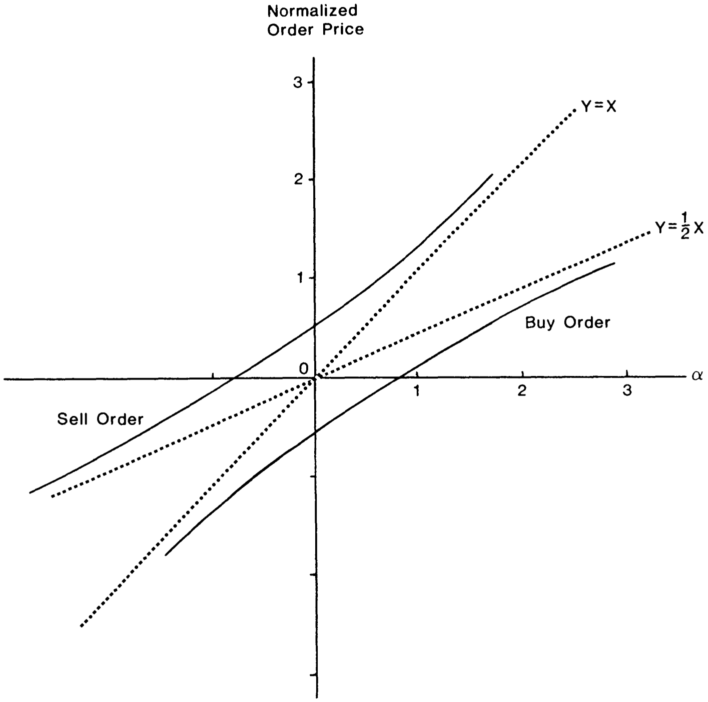

Figure 1. The Trading Orders in the Batched Regime

Because of the symmetry of the arguments for the optimal buy and sell orders, Equations (14a) and (14b) may be obtained from (13a) and (13b), respectively, by letting $\pi^s = -\pi^b$ and changing the sign of α.

The results are summarized in Figure 1. π^b is asymptotic to the line $\pi^b = \alpha$ for small values of α and is asymptotic to the line $\pi^b = \alpha/2$ for large values of α. π^s is asymptotic to the line $\pi^s = \alpha/2$ for small values of α and is asymptotic to the line $\pi^s = \alpha$ for large values of α.

In the absence of transaction price uncertainty, the investor's order size at any price would be given by a point on his/her normal demand function.[10] That is, if the clearing price is μ and the order size is $\bar{Q}$, we would have

$$\mu = a - 2b\bar{Q} \tag{15}$$

However, given transaction price uncertainty with μ being the expected clearing price, the optimal order size in the batched regime would in general be different from that given by Equation (15). We now consider how the optimal transaction size compares with $\bar{Q}$.

[10] It should be noted that our demonstration, that the investor in a batched regime submits a price from a linear reservation buy function, depends on the assumption that the investor may submit only a single P, Q order. If multi-part orders are allowed (e.g., Q_1 shares at P_1 and $Q_2 - Q_1$ additional shares at $P_2 < P_1$, etc.), it can be shown that the reservation buy function becomes piecewise linear (under our assumptions) and approaches the ordinary buy function in the limit as the number of order parts allowed increases without bound, and hence as the distance between order points becomes infinitesimal.

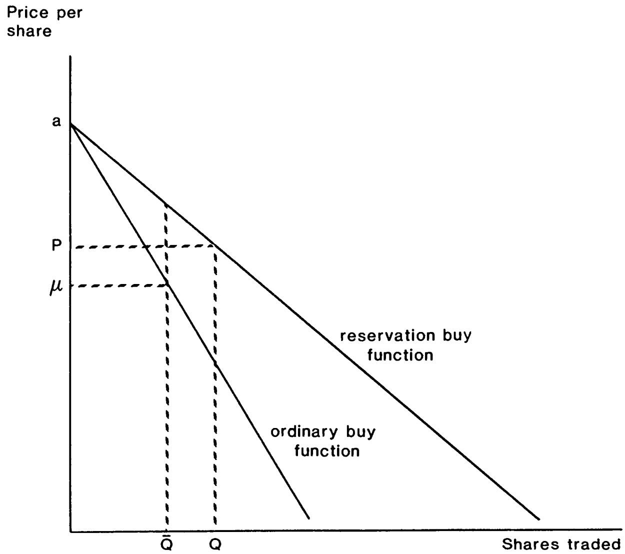

Figure 2. The Order Price (P) and Order Size (Q) under Transaction Price Uncertainty

By the definition of α, Equation (15) may be written as

$$\alpha = \frac{2b\bar{Q}}{\sigma} \tag{16}$$

In the batched regime, from Equations (13b) and (15), we get

$$\frac{b}{\sigma}(\bar{Q} - Q) = \pi^b - \frac{\alpha}{2} \tag{17}$$

Therefore, it follows from Equation (13a) that

$$(\bar{Q} - Q) = -\frac{\sigma}{2b}\nu(-\pi^b) < 0 \tag{18}$$

Equation (18) shows that in the batched regime, the investor would always submit a larger order size in the presence of demand uncertainty.

This result can be understood intuitively with reference to Figure 2. In the batched regime, the investor obtains some "insurance" against transaction price uncertainty by submitting an order from his/her reservation buy function (i.e., $P > \mu$ in Figure 2). But with this protection, the investor will "gamble" on getting the larger consumer surplus associated with an execution at some point further down his/her ordinary buy function. Accordingly, the investor lowers the price below the reservation price at $\bar{Q}$, and increases the size of the order from $\bar{Q}$ to Q.

II. Market Clearing

Let a set of investors make their trading decisions at time T_0 in accordance with the model developed in Section I. We assume that each of the investors has a linear demand function with the same slope b, and that the demand function varies across individuals according to the intercept parameter a.[11] We also assume that the investors' expectations of the clearing price are homogeneous, i.e., that μ and σ are the same for all investors. This assumption enables us to focus on the individual propensities to trade in relation to the distribution of the parameter a across investors. Finally, we assume the investors are continuously distributed according to the intercept parameter, a. At any price x, the subset of investors with

$$a < x \tag{19}$$

will wish to sell in a frictionless environment. Let $F(x)$ denote the proportion of investors in this subset.

We utilize $F(x)$ to represent the distribution of the investors' propensities to transact. Such a distribution will in general differ at each market call. This is because the investors may revise their expectations of the asset's returns and/or because their liquidity needs may change. We assume that $F(x)$ is continuous and piecewise differentiable, such that

$$F'(x) = \phi(x) \tag{20}$$

As noted above, each investor's transaction uncertainty results from not knowing the orders which the others will submit. We can represent this by assuming that at any market call, investors do not know the specification of $F(x)$. In this context, we first show the market clearing solution for the frictionless market, and then for the batch trading environment with friction.

A. The Aggregate Buy and Sell Functions in a Frictionless Setting

Consider an investor as being of type t if his/her normalized intercept α has the value t. t may be either positive or negative; t-positive investors are more apt to buy, while t-negative investors are more apt to sell. We relate the buy-sell orders of an individual to the sign and magnitude of his/her t-type classification.

In the frictionless market, each investor would submit his/her full demand function to the market. At the normalized price

$$\pi = \frac{P - \mu}{\sigma} \tag{21}$$

any t-type investor would buy, from Equation (15),

$$Q = \frac{\sigma}{2b}(t - \pi) \tag{22}$$

Since there are $\phi(t)\,dt$ individuals of each type, the aggregate buy function is

$$D(\pi) = \frac{\sigma}{b}\int_{\pi}^{\infty} \frac{1}{2}(t - \pi)\phi(t)\,dt \tag{23a}$$

[11] Note that, with linear functions and in a frictionless environment, trades will occur if and only if the value of a differs for some investors. Independent variation of the parameters a and b across investors is possible because W and $\bar{X}$ can vary independently (since N_0 and C_0 can vary across investors). Now, taking b to be the same for all investors can be viewed as holding $U(\cdot)$ and W constant (see footnote 6). Under this constraint, any differing combination of C_0, N_0, and X will, by Equation (3), result in a change of the parameter a.

for any value π. Similarly, the aggregate sell function is

$$S(\pi) = \frac{\sigma}{b} \int_{-\infty}^{\pi} \frac{1}{2} (t - \pi)\phi(t)\, dt \tag{23b}$$

Let π^c be the (normalized) clearing price such that

$$D(\pi^c) + S(\pi^c) = 0 \tag{24}$$

Substituting Equations (23a) and (23b) into (24), we have

$$\int_{-\infty}^{\infty} (t - \pi^c)\phi(t)\, dt = 0$$

or

$$\int_{-\infty}^{\infty} t\phi(t)\, dt = \pi^c \tag{25}$$

Therefore, the mean of the investors' distribution is the unique clearing price. We call this the Pareto efficient price, π^*, because after all the desired trades have been made at this price, there would be no immediate demand to recontract. Recontracting is inherently more costly than order revision during a tatonnement because actual transactions give rise to clearance and settlement costs.[12] For this reason, the frictionless Walrasian auction is taken as the standard of comparison for actual systems.

B. The Aggregate Buy and Sell Functions with Friction

Using Equations (13a) and (13b) from Section I and the definition of t, the buy order for each type-t investor is given by

$$2\pi^b - t = -\nu(-\pi^b) \tag{26}$$

where

$$\pi^b = (P_t^b - \mu)/\sigma \tag{27}$$

$$Q_t^b = \frac{(t - \pi^b)}{b}\, \sigma > 0 \tag{28}$$

$$t = (a_t - \mu)/\sigma \tag{29}$$

Write the inverse of Equation (26) explicitly as

$$\tau^b(\pi^b) = 2\pi^b + \nu(-\pi^b) \tag{30}$$

Equation (30) enables us to identify the t-classification of an investor from the observable buy price that he/she specifies.

From the definition of $F(x)$, we have that the number of investors of type-t is $\phi(t)\, dt$, and since from Equation (28) the transaction buy size for each investor is

[12] Institutionally, this is reflected in brokerage fees being based on transactions rather than on orders, although excessive order revisions would also have to be charged for. Recontracting can also result in transferring larger amounts to the taxing authority.

$$Q^b = \frac{(t - \pi^b(t))\sigma}{b}$$

the aggregate buy order for the type-t investors is $Q^b\phi(t)\,dt$. The buy function for investors is obtained by summing across all investors (types) who will submit an order at a normalized price of π (i.e., those investors whose limit order prices are π or greater). Recognizing that at any normalized price π the marginal investor type is $\tau^b(\pi)$, the aggregate demand function is

$$D(\pi) = \int_{\tau^b(\pi)}^{\infty} Q^b(t)\phi(t)\,dt \tag{31}$$

This expression for the aggregate buy function may be simplified. Substituting (28) into (31), we have

$$D(\pi) = \int_{\tau^b(\pi)}^{\infty} \frac{\sigma}{b}(t - \pi^b(t))\phi(t)\,dt \tag{32}$$

Analogously, the aggregate sell function can be written as

$$S(\pi) = \int_{-\infty}^{\tau^s(\pi)} \frac{\sigma}{b}(t - \pi^s(t))\phi(t)\,dt \tag{33}$$

where π^s and τ^s are given by

$$t - 2\pi^s = -\nu(\pi^s) \tag{34}$$

$$\pi^s = (P_t^s - \mu)/\sigma \tag{35}$$

$$Q_t^s = \frac{-(\pi^s - t)\sigma}{b} < 0 \tag{36}$$

$$\tau^s(\pi^s) = 2\pi^s - \nu(\pi^s) \tag{37}$$

Finally, define the demand imbalance as

$$I(\pi) = D(\pi) + S(\pi) \tag{38}$$

Substituting Equations (32) and (33) into (38) and simplifying, we get[13]

[13] Note that for any t-type investor, the buy curve, which is the mirror image of the sell curve, would have (from Equations (26) and (34))

$$\tau^b(\pi) = -\tau^s(-\pi) \tag{f1}$$

Further, from Equations (28) and (32), we have

$$\pi^b(t) = -\pi^s(-t) \tag{f2}$$

For simplicity write

$$\tau = \tau^b \tag{f3}$$

Then, first substituting (f1) into (33) and next substituting (32) and (33) into (38) gives

$$I(\pi) = \int_{\tau(\pi)}^{\infty} \frac{\sigma}{b}(t - \pi^b(t))\phi(t)\,dt + \int_{-\infty}^{-\tau(-\pi)} \frac{\sigma}{b}(t - \pi^s(t))\phi(t)\,dt \tag{f4}$$

Then substituting (f2) into (f4), reversing the order of integration in the second integral on the RHS of (f4), and appropriately reversing the sign of t and dt gives the result.

$$I(\pi) = \int_{\tau(\pi)}^{\infty} \frac{\sigma}{b}(t - \pi^b(t))\phi(t)\,dt - \int_{\tau(-\pi)}^{\infty} \frac{\sigma}{b}(t - \pi^b(t))\phi(-t)\,dt \tag{39}$$

Note that π is the (normalized) call market clearing price when $I(\pi) = 0$.

We can now assess Equation (39) at π^*, the (normalized) Pareto efficient clearing price in the frictionless environment, to determine whether the (normalized) call market clearing price, π, is the same as π^*. We do this in Section III.

III. Model Results

In this section we consider the way in which the price and quantity established in the market depend on the distribution of trading propensities across investors. We describe the set of investor demand propensities by an exogenously specified distribution, ϕ. We have structured the analysis by holding the slope, b, of investor demand curves constant while allowing the intercept, a, to vary as described by ϕ; that we are free to vary a with b constant has been noted in footnote 11.

We have already shown that the frictionless market clearing price π^* is the mean of the distribution of the t-classifications, which is independent of transaction price uncertainty. The reason is that, with transaction price uncertainty, the investor simply submits his/her entire buy/sell order function to the frictionless market. Therefore, transaction price uncertainty has no effect on the frictionless market. In this section we show that this result, in general, does not hold for the market with friction.

The comparative results for the market with friction are obtained by studying the implications of two conditions on the distribution of investors:

C.1 ϕ is a symmetrical function; that is, for some value, π^*,

$$\phi(\pi^* + t) = \phi(\pi^* - t)$$

for all values of t. Note that, since $\int_{-\infty}^{\infty} t\phi(t)\,dt = \pi^*$, by Equation (27) π^* is the Pareto efficient price.

C.2 The normalized Pareto efficient price is zero; that is

$$\pi^* = 0$$

Equivalently, the expected clearing price, μ, is the Pareto efficient price, P^*.

C.2, which is implied by "rational expectations," will be satisfied if the market environment is sufficiently repetitive for investors to draw accurate inferences on the market clearing price.

It is easy to demonstrate from Equation (39) that, when C.1 and C.2 both hold, the clearing price with friction is the Pareto efficient price. We show the effects of relaxing each of the conditions in turn in the following subsections. To interpret our results intuitively, it is useful to define two classes of orders: class A comprises those orders which are expected to transact (the probability of the order executing is greater than 0.5), and class B comprises those orders which are not expected to transact (the probability of the order executing is less than 0.5). Then, according to Equations (26), (27), (34), and (35), the association between investors classified by t-type and their orders is:

t-type	buy order class	sell order class
$t > \nu(0)$	A	B
$-\nu(0) \leq t \leq \nu(0)$	B	B
$t < -\nu(0)$	B	A

A. Transaction Price: The Market Asymmetry Effect

We first relax the symmetry condition and show that, in general, the clearing price is different in the two regimes. We obtain this result by assessing Equation (39), the demand imbalance equation, at $\pi = 0$:

$$I(0) = \frac{\sigma}{b} \int_{\tau(0)}^{\infty} (\phi(t) - \phi(-t))(t - \pi^b(t))\, dt \tag{40}$$

Because $\int_{-\tau(0)}^{\tau(0)} t\phi(t)\, dt = \int_{\tau(0)}^{\infty} t(\phi(-t) - \phi(t))\, dt$, we can impose a mean-preserving shift on $\phi(t)$ and $\phi(-t)$ in the region $-\tau(0)$ to $\tau(0)$. This shift causes skewness in the distribution of ϕ, but preserves the value of the Pareto efficient price, thus giving us a clear ceteris paribus result.

Now recognizing that $(t - \pi^b(t))$ must be positive for all t (see Equation (26)), then $\phi(t) - \phi(-t)$ being positive (negative) for all $t > \tau(0)$ would imply $I(0)$ positive (negative), which implies that the transaction price must be higher (lower) than the frictionless market clearing price.[14]

This result can be explained intuitively. When the investor frequency function is symmetrical ($\phi(t) = \phi(-t)$), the number of investors of type t on the buy side of the market equals their counterpart number on the sell side. On the other hand, ($\phi(t) \neq \phi(-t)$) reflects asymmetry of the buy/sell orders. Using Equation (40) we can analyze the impact of asymmetry on the transaction price in relation to the asymmetry in the group associated with $|t| > \tau(0)$. Referring to Section II *B*, and noting that $\nu(0) = \tau(0)$, Equations (26) and (34) show that this group comprises the class A orders of those investors whose desire to transact is great enough so that they place orders which they expect will transact. Now, for instance, let $\phi(t) > \phi(-t)$ for each $t > \tau(0)$; then there are more class A buy orders than class A sell orders, and thus some of the class A buys transact against class B sells. Because order sizes are larger for class A than for class B orders, the imbalance in the order flow exerts an upward pressure on price, thereby causing the transaction price to be above its frictionless market value. Similarly, the transaction price is lower when the class A orders are preponderantly on the sell side.

Our formal analysis is a one-period model. Nonetheless, the results concerning the asymmetry effect can be interpreted in a way that generates an interesting implication for autocorrelation in security returns. Suppose a sequence of calls

[14] Our results do not depend on the inequality holding for all $t > \tau(0)$; we consider this case simply to facilitate the exposition.

where, at each call, participants behave in accordance with our single-period decision model. It is reasonable to assume that the skewness of the distribution of ϕ is random at each call. Therefore, the clearing price will fluctuate randomly around the Pareto efficient price. Then, if the Pareto efficient price follows a random walk, the clearing price will generate a returns series that is negatively autocorrelated.[15]

B. Transaction Price: The Guess Wrong Effect

There is a second reason why the transaction price will differ from its Pareto efficient value. We show this by restoring the assumption of market symmetry (C.1) and by relaxing the assumption that the expected clearing price is the Pareto efficient price (C.2). In other words, we relax the rational expectations assumption that the market environment is sufficiently repetitive for investors to draw accurate inferences on the market clearing price.

By the symmetry assumption, ϕ can be expressed as

$$\phi(t) = \Omega(t - \pi^*)$$

for some function Ω where

$$\Omega(\nu) = \Omega(-\nu) \quad \text{for any value } \nu \tag{41}$$

Then, using the transformations $s = t - \pi^*$ and $y = t + \pi^*$, Equation (39) can be expressed as

$$\begin{aligned} I(\pi) = & \int_{\tau(\pi)-\pi^*}^{\infty} \frac{\sigma}{b}(s + \pi^* - \pi^b(s + \pi^*))\Omega(s)\, ds \\ & - \int_{\tau(-\pi)+\pi^*}^{\infty} \frac{\sigma}{b}(y - \pi^* - \pi^b(y - \pi^*))\Omega(-y)\, dy \end{aligned} \tag{42}$$

Thus with symmetry, by (41), rearranging (42) and assessing at π^* gives

$$\begin{aligned} I(\pi^*) = & \frac{\sigma}{b}\int_{\tau(\pi^*)-\pi^*}^{\infty} (2\pi^* + \pi^b(s - \pi^*) - \pi^b(s + \pi^*))\Omega(s)\, ds \\ & - \frac{\sigma}{b}\int_{\tau(-\pi^*)+\pi^*}^{\tau(\pi^*)-\pi^*} (s - \pi^* - \pi^b(s - \pi^*))\Omega(s)\, ds \end{aligned} \tag{43}$$

To determine the sign of the two terms on the right-hand side of Equation (43), first take $\pi^* > 0$. Since $0 < \partial\pi^b(x)/\partial x < 1$ (see Section I), the Mean Value Theorem ensures that the integrand of the first term is positive. Since $\pi^b(x) < x$, the integrand of the second term is positive. By equation (37), $(\tau(\pi^*) - \pi^*) - (\tau(-\pi^*) + \pi^*) = 2\pi^* + \nu(-\pi^*) - \nu(\pi^*) > 0$; therefore, the upper limit is greater than the lower limit for the second integration. Hence, both terms of Equation (43) are positive. Identically, for $\pi^* < 0$, both terms are negative.

[15] The standard argument is as follows. Suppose that the Pareto efficient price is constant. Then, as the transaction price fluctuates around the Pareto efficient price, we would on expectation have: (1) high (low) prices associated with high (low) returns, and (2) high (low) prices followed by lower (higher) prices. The combined effect of this generates negative autocorrelation in the transaction-to-transaction returns. The argument extends to the case where the Pareto efficient price moves randomly over time.

Because both terms on the right-hand side of (43) have the same sign as π^*, a specific relationship between π^* and $\Omega(s)$ may exist that would satisfy $I(\pi^*) = 0$. However, because the distribution of investors about π^* need not have a fixed relationship to π^*, it is apparent that, in general, we will not have $I(\pi^*) = 0$. To assess the demand imbalance at π^* (and thus to determine the effect on the market clearing price of the expected clearing price differing from its Pareto efficient value), consider two alternative distributions of Ω:

Case 1. The investor classifications are distributed closely around π^*. In the extreme, assume

$$\Omega(s) = 0 \quad \text{for all} \quad s \geq \tau(\pi^*) - \pi^*$$

Then, the first term on the right-hand side of (43) is zero and hence

$$I(\pi^*)\begin{cases} < 0 & \pi^* > 0 \\ > 0 & \pi^* < 0 \end{cases}$$

It follows that

$$\begin{pmatrix}\text{expected clearing} \\ \text{price}\end{pmatrix} \gtreqless \begin{pmatrix}\text{market clearing} \\ \text{price}\end{pmatrix} \gtreqless \begin{pmatrix}\text{Pareto efficient} \\ \text{price}\end{pmatrix}$$

Case 2. The investor classifications are sufficiently dispersed around π^*. As an extreme case assume

$$\Omega(s) = 0 \quad \text{for all} \quad \tau(-\pi^*) + \pi^* \leq s \leq \tau(\pi^*) - \pi^*$$

Then the second term on the right-hand side of (43) is zero, and hence

$$I(\pi^*)\begin{cases} > 0 & \pi^* > 0 \\ < 0 & \pi^* < 0 \end{cases}$$

It follows that

$$\begin{pmatrix}\text{expected clearing} \\ \text{price}\end{pmatrix} \gtreqless \begin{pmatrix}\text{Pareto efficient} \\ \text{price}\end{pmatrix} \gtreqless \begin{pmatrix}\text{market clearing} \\ \text{price}\end{pmatrix}$$

The results for these two cases can be explained intuitively by considering what happens when the expected clearing price is less than its Pareto efficient value ($\pi^* > 0$). Decreasing the expected clearing price, by itself, induces investors to lower their order prices, which exerts downward pressure on the market clearing price. But the deviation of price from its Pareto efficient value also affects the proportion of investors who expect to transact, and its does so asymmetrically on the buy and the sell sides of the market. Now $\pi^* > 0$ increases the proportion of investors who expect to transact on the buy side of the market, which increases the proportion of class A buys relative to class A sells. As a result, some class A buys transact against class B sells; this exerts an upward pressure on the market clearing price (as was true for the asymmetry effect considered in Section III*B*).

Because the analysis treats normalized variables, changing the distribution of $\Omega(s)$ is equivalent to changing π^* by varying either μ or σ. When μ is large relative to σ, Case 1 applies, the first effect predominates (all prices are lowered), and the clearing price is less than the Pareto efficient price. When μ is small relative to σ, Case 2 applies, the second effect predominates (class A buys transact against class B sells), and the clearing price is greater than the Pareto efficient price.

The intuitive explanations for both the market asymmetry effect and the guess wrong effect are similar. The predominant result in both cases follows from the way in which orders interact to give the clearing price. Furthermore, our analysis of each effect suggests that, over time, the market clearing price will tend to fluctuate randomly around the Pareto efficient price (since, over time, there should be no systematic tendency for the distribution of investors to be skewed toward the high or toward the low values of t, or for investors to over- or underestimate the Pareto efficient price). Our analyses of both the asymmetry and guess wrong effects suggest, therefore, that if change in the Pareto efficient price is serially uncorrelated, we would, in accordance with Goldman and Beja's [9] model, expect to observe negative serial correlation in the market clearing (transaction) prices.

We noted in relation to the asymmetry effect (Section III *A* above) that the random behavior of skewness (of distribution ϕ) from call to call would cause call-to-call returns to be negatively autocorrelated if the Pareto efficient price follows a random walk. In similar fashion, call-to-call returns would be negatively autocorrelated if investors' expectations of the clearing price fluctuate randomly about the Pareto efficient price from call to call. With regard to the guess wrong effect, overshooting (Case 1) would reinforce the negative autocorrelation, and undershooting (Case 2) could mitigate (or even reverse) it.

C. Transaction Price Uncertainty

We now show that transaction price uncertainty (σ) is not entirely exogenous to the market regime. That is, we show, for a call market regime, that transaction price uncertainty also results from each investor submitting one (P, Q) order rather than his/her complete buy/sell order function. We show this by demonstrating that the market aggregate buy/sell functions become highly inelastic as σ tends to zero around the Pareto efficient price. Hence, any slight change in the location of the curves relative to each other results in substantial price variation. We structure the analysis by restoring condition C.2 and by relaxing condition C.1.

Because the analysis involves varying σ, we cannot treat normalized price. Therefore, we start by classifying investors according to $T = a - \mu$ (i.e., $T = \sigma t$). Let $\Phi(T)\,dT$ denote the number of investors belonging to type T. It then follows from Equation (31) that the demand function is

$$D(\bar{P}) = \int_{\bar{\tau}(\bar{P})}^{\infty} \frac{\Phi(T)}{b} (T - \bar{P}^b(T))\, dT \tag{44}$$

where

$\bar{P}^b(T) = \sigma\pi^b(T/\sigma)$,
$\bar{P} = \sigma\pi$ for normalized price π, and
$\bar{\tau}$ is the inverse function of $\bar{P}^b$

Note that $\bar{P}^b$, $\bar{P}$, and $\bar{\tau}$ all depend on the value of σ.

Differentiating Equation (44) with respect to $\bar{P}$ we have

$$\frac{\partial D(\bar{P})}{\partial \bar{P}} = \frac{-\Phi(\bar{\tau}(\bar{P}))}{b} [\bar{\tau}(\bar{P}) - \bar{P}^b(\bar{\tau}(\bar{P}))] \frac{\partial \bar{\tau}(\bar{P})}{\partial \bar{P}}$$

or

$$\frac{\partial D(\bar{P})}{\partial \bar{P}} = \frac{-\Phi(\bar{\tau}(\bar{P}))}{b} (\bar{\tau}(\bar{P}) - \bar{P}) \frac{\partial \bar{\tau}(\bar{P})}{\partial \bar{P}} \tag{45}$$

When $\sigma \longrightarrow 0$,

$$\bar{\tau}(\bar{P}) \begin{cases} = 2\bar{P} & \text{for } \bar{P} > 0 \\ = \bar{P} & \text{for } \bar{P} \leq 0 \end{cases} \tag{46}$$

and

$$\frac{\partial \bar{\tau}(\bar{P})}{\partial \bar{P}} \begin{cases} = 2 & \text{for } \bar{P} > 0 \\ = 1 & \text{for } \bar{P} \leq 0 \end{cases} \tag{47}$$

Substituting Equations (46) and (47) into Equation (45) gives

$$\lim_{\sigma \to 0} \frac{\partial D(\bar{P})}{\partial \bar{P}} \begin{cases} = \dfrac{-2\Phi(\bar{\tau}(\bar{P}))\bar{P}}{b} & \text{for } \bar{P} > 0 \\ = \quad 0 & \text{for } \bar{P} \leq 0 \end{cases} \tag{48}$$

Since by assumption Φ is bounded, Equation (48) shows that the demand function is relatively inelastic around the expected clearing price. By similar argument, the aggregate supply function is also inelastic around μ.

This result can be explained intuitively. The analysis shows that, in a call regime, when an investor is forced to submit just one (P, Q) order, that order will be picked from his/her reservation price function. Regardless of how small σ might be, each investor will do so to be protected from transaction price uncertainty.[16] This causes buyers to submit prices higher than μ and sellers to submit prices lower than μ.

When σ is small, the marginal transactors on both sides of the market will have a close to μ, which causes the market to be "thin" on the margin. Because of this, any *intramarginal* asymmetry would have a relatively large impact on the clearing price.[17]

D. Trading Volume

While conditions C.1 and C.2 are sufficient for the clearing price to be the Pareto efficient price, trading volume in a call market does not, in general, equal its frictionless level even when C.1 and C.2 both hold. By Equation (32), the aggregate transaction volume for a market with friction is

[16] By "rational expectations" (which we assume here) and the results of this section, the investor cannot believe σ will literally go to 0 because he knows that, as σ gets smaller, the demand curve becomes more inelastic in the neighborhood of the clearing price, and hence that endogenous transaction price uncertainty must be introduced.

[17] If multi-part orders are allowed so that the reservation demand function is piecewise linear (see footnote 10), the tendency of the aggregate buy/sell functions to be become completely inelastic as σ tends to 0 is diminished. Hence, reducing friction reduces that transaction price uncertainty which is endogenous to the market regime.

$$V = \frac{\sigma}{b} \int_{\tau(0)}^{\infty} (t - \pi^b(t))\phi(t)\, dt \tag{49}$$

Contrasting this with the frictionless result given by

$$V' = \frac{\sigma}{b} \int_{0}^{\infty} \frac{\phi(t)t}{2}\, dt \tag{50}$$

we have a difference

$$\Delta = \frac{\sigma}{b}\left[\int_{\tau(0)}^{\infty} \left(\frac{1}{2}\, t - \pi^b(t)\right)\phi(t)\, dt - \int_{0}^{\tau(0)} \frac{1}{2}\, t\phi(t)\, dt\right] \tag{51}$$

To assess Δ, first consider the extreme case, where $\phi(t)$ is such that $\phi(t) = 0$ for all $|t| > \tau(0)$. Then we have $\Delta < 0$. In this case, the individuals' intercepts are all very close to their expectation of the clearing price. Therefore, orders are submitted which, on expectation, will not transact. The reason is that transaction price uncertainty leads investors to speculate on gaining more consumer surplus than that implied by the frictionless market solution. As a result, ex post, transaction volume is reduced.

At the other extreme, let $\phi(t) = 0$ for all $|t| \leq \tau(0)$. We then have $\Delta > 0$. In this case, each investor will submit a larger order at a higher (lower) buy (sell) price, vis-à-vis the specific (P, Q) point on the normal demand curve at which he/she would transact in the frictionless environment. Hence, for all investors, the ex post transaction volume is higher than its frictionless market value.

IV. Conclusion

We have shown that, in coping with his/her price-quantity decision, the expected utility maximizing trader in a nonfrictionless environment with transaction price uncertainty does not behave with the simplicity of motion generally attributed to the atomistic decision maker. Neither will he/she necessarily achieve the optimal holdings that would be attained if the system were frictionless.

Within our microstructure framework—trading friction, imperfect investor knowledge of the location of the aggregate demand functions, and a batched trading regime—we have shown how, given his/her demand curve to hold shares of the asset, the investor will write a buy and a sell order to submit to the market. This treatment of the investor's trading decision has endogenized the order flow. Further, we have aggregated the orders, so as to analyze the behavior of the market price and transaction volume. This has enabled us to show the conditions under which batched trading does not give Pareto efficient prices.[18] These

[18] The fact that Pareto efficient prices and volume are generally not attained in a call market is manifested by a demand to recontract after each call. This demand is observed in, for instance, the "à la criée" system on the Paris Bourse and the "gekitaku" system on the Tokyo Stock Exchange. These systems are verbal auctions which permit order revision (by traders present on the floor) "as the market forms." The CATS opening for the Toronto and Tokyo stock exchanges also permits order revision. However, call market systems that use written orders (e.g., the "par cassier" in the Paris spot market) do not allow for order revision, and hence result in an unsatisfied demand to recontract.

conditions are: (C.1) an asymmetric distribution of traders according to their buy/sell functions and/or (C.2) mistaken investor expectations of the market clearing price. We also show that with batched trading, the volume of trading will not, in general, be Pareto efficient.

Our results can be understood intuitively. With transaction price uncertainty in a call market, the price and quantity of an order depend both on the height of the trader's demand curve (as represented by the parameter a) and on the expected clearing price μ, which we take to be the same for all investors. Further, each order is picked from the trader's reservation price demand curve, and its price and quantity in general differ from values that would attain if the market clearing price were known in advance to be μ. Also, as $(a - \mu)$ increases for buy orders (or $(\mu - a)$ increases for sell orders), orders are priced so as to increase their probability of transacting, and the size of orders is appropriately adjusted.

Now, on each side of the market, orders can be classified into two groups: those that are expected to transact (class A) and those that are not expected to transact (class B). When conditions C.1 and C.2 are met, we find that the class A buy and sell orders offset each other exactly, and that the call price is μ. On the other hand, when either or both of the conditions is violated, the class A orders do not offset each other and the class B orders on the "short" side of the market partially absorb the difference. Now, by Equation (18), order sizes are misspecified vis-à-vis their Pareto efficient value at μ (the investor will submit larger order sizes), and the misspecification is greater for class A than for class B orders. Hence, the size misspecification will be greater for the net class A orders than for the class B counterpart orders that the net in this case transacts against. This is what perturbs the clearing price, causing it to differ from its Pareto efficient value.

Our paper, by relating the observable call prices to the unobservable equilibrium prices, shows that the order placement strategy of investors would generate random fluctuations of the clearing price around its Pareto efficient value. This result, as noted in Section III, yields an important empirical implication: namely, that transaction-to-transaction returns will be correlated. Serial correlation has been well established for continuous trading (where the impact of the bid-ask spread has generally been advanced as the explanation). We suggest that the observation is a more general manifestation of investors using order placement strategies when they trade in markets that are nonfrictionless and nonstable. Accordingly, one should also expect to observe serial correlation in a call market (where there is no bid-ask spread).

While a development of the issue is beyond the scope of the current paper, our analysis highlights the importance of various aspects of market design. First, market clearing prices will commonly diverge from their Pareto efficient values; this suggests that, along with a well-established role as a provider of immediacy to the market, a dealer, because of superior knowledge of the order flow, has an important function to perform with regard to the *discovery* of appropriate market prices.[19] Second, we have shown that asymmetry in the order flow will cause transaction prices to differ from their Pareto efficient values; this suggests that

[19] For a review of the dealer literature, see Cohen, Maier, Schwartz, and Whitcomb [4].

capital gains taxes, short selling restrictions,[20] and other asymmetrical impediments of the order flow will distort market clearing prices. Third, the call market and continuous market trading regimes are expected to give different price/volume results because of differences in order handling and information dissemination in the two regimes;[21] an investigation of how the systems compare in this regard should give insight into the relative advantages and disadvantages of these two alternative trading mechanisms.

In much of the economics literature, the price formation process is assumed to be a Walrasian tatonnement in which market participants trade at equilibrium prices. But when friction (information, decision, and order handling costs) impedes the trading process, such prices are generally not attained in a single round of trading, and the very act of trading becomes a process that requires analysis. We have shown that, in a nonfrictionless security market, trading arrangements affect the decisions of market participants and the market clearing values for price and volume. An extension of this analysis to other markets should yield insight into a broader array of economic problems.

Appendix A

In this appendix, we derive Equation (3). Expanding the investor's utility of wealth around $W = C_0 r_0 r_1 + N_0 \bar{X}$ (i.e., the expected wealth if the investor does not trade) and assuming quadratic utility, Equation (2) can be written as

$$\begin{aligned}\Phi(P, Q \mid N_0, C_0) &= EU(C_0 r_1 r_0 + N_0 \bar{X} + N_0 \tilde{X} - N_0 \bar{X} + Q\tilde{X} - QPr_1) \\ &= U(W) + U'(W)(Q\bar{X} - QPr_1) + \tfrac{1}{2}U''(W)[N_0^2 \mathrm{Var}(\tilde{X}) \\ &\quad + (2QN_0 + Q^2)\mathrm{Var}(\tilde{X}) + (Q\bar{X} - QPr_1)^2] \qquad \text{(A1)}\end{aligned}$$

where $\bar{X}$ is the expected share value at T_2 and $\mathrm{Var}(\tilde{X})$ is the variance of share value at T_2. Assume that the squared deviation of the expected rate of return on the risk asset from the riskless rate is sufficiently small such that the term $\tfrac{1}{2} U''(W)Q^2(\bar{X} - Pr_1)^2$ may be ignored. Collecting terms as a polynomial in variables P and Q, we have

$$\begin{aligned}\Phi &= U(W) + \tfrac{1}{2}U''(W)N_0^2\mathrm{Var}(\tilde{X}) \\ &\quad + [U'(W)\bar{X} + U''(W)N_0\mathrm{Var}(\tilde{X})]Q \\ &\quad - U'(W)r_1 PQ + \tfrac{1}{2}U''(W)\mathrm{Var}(\tilde{X})Q^2 \qquad \text{(A2)}\end{aligned}$$

[20] Short selling is not allowed, for instance, in Israel, and this has been thought by some students of that market to contribute to the volatility of stock prices. We thank Yehuda Kahane for calling this to our attention.

[21] Although the evidence has not to our knowledge been formally examined, it appears that trading volume tends to be less, ceteris paribus, for batched systems than for continuous markets. This is primarily because public traders participate less in the batched systems (presumably, this is the case, for instance, for the Paris Bourse). The reason for this might well be the informational advantage that continuous markets have over "closed book" batched trading systems.

Using k, γ, a, and b as defined in the text, Equation (A2) becomes,

$$\Phi = k + a\gamma Q - \gamma PQ - \gamma\beta Q^2 \tag{A3}$$

Equation (3) immediately follows from (A3).

Appendix B

In this appendix, we derive Equations (13a) and (13b). To do so, first rewrite Equation (12) as

$$\max_Q J = \max_Q \frac{1}{\sigma}\int_{-\infty}^{P^{RB}} (P^{RB} - x)Qh\left(\frac{x-\mu}{\sigma}\right) dx \tag{B1}$$

for

$$h(y) = \frac{1}{(\sqrt{2\pi})} e^{-y^2/2}$$

Now, by putting $P^b = P^{RB} = a - bQ$, and then directly evaluating $\partial J/\partial Q = 0$, we have

$$\int_{-\infty}^{P^b} (P^b - x - bQ)h\left(\frac{x-\mu}{\sigma}\right) dx = 0$$

or

$$\int_{-\infty}^{P^b} (P^b - bQ)h\left(\frac{x-\mu}{\sigma}\right) dx = \int_{-\infty}^{P^b} xh\left(\frac{x-\mu}{\sigma}\right) dx \tag{B2}$$

Since $bQ = a - P^b$, (B2) can be rewritten as

$$\int_{-\infty}^{P^b} \left[\frac{2(P^b-\mu)}{\sigma} - \left(\frac{a-\mu}{\sigma}\right)\right] h\left(\frac{x-\mu}{\sigma}\right) dx = \int_{-\infty}^{P^b} \frac{(x-\mu)}{\sigma}\, h\left(\frac{x-\mu}{\sigma}\right) dx \tag{B3}$$

Changing the parameter $z = \dfrac{x-\mu}{\sigma}$, the right-hand side of (B3) can be expressed as

$$\sigma \int_{-\infty}^{(P^b-\mu)/\sigma} zh(z)\, dz$$

but $dh/dz = -zh(z)$, and so we have

$$\int_{-\infty}^{P^b} \left[\frac{2(P^b-\mu)}{\sigma} - \left(\frac{a-\mu}{\sigma}\right)\right] h\left(\frac{x-\mu}{\sigma}\right) dx = -\sigma h\left(\frac{P^b-\mu}{\sigma}\right) \tag{B4}$$

Substituting $\pi^b = \dfrac{P^b-\mu}{\sigma}$ and $\alpha = \dfrac{a-\mu}{\sigma}$ into (B4) and rearranging gives Equation

(13a). As noted in Section I *B*, the optimal order is picked from the reservation buy curve; hence, rearranging Equation (8) immediately gives Equation (13b).

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