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Production and Risk Leveling in the Intertemporal Capital Asset Pricing Model

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ABSTRACT

This paper extends Merton's intertemporal capital asset pricing model with multiple consumers to include a description of the supply of traded securities. The production decisions of firms are described in a model with stochastic investment opportunities and incomplete markets. Firms maximize the welfare of their stockholders based on the sum of dollar values placed on the projects by shareholders of the firm. The monetary value to stockholders of a marginal change in the contract structure due to changing firm production is analyzed. In this setting, the competitive market achieves an appropriately defined Nash-Constrained Pareto Optimum. Sufficient conditions for investor unanimity, market-value maximization by firms, and the equilibrium to achieve a Constrained Pareto Optimum and full Pareto Optimum are derived.

WHEN A FIRM CHANGES its production plan, it alters the stochastic nature of its future profits and changes the set of available securities. Therein lies the firm's problem for valuing its action if markets are incomplete. Its proposed choice, viewed as a welfare issue, must involve the welfare effects of the new contract structure relative to the original set of contracts as well as profitability at prevailing prices. Such a consideration does not arise in perfect markets since the set of contracts (being always complete) is unchanged before and after any production decision.

To illustrate, take the following example from research firms which specialize in R&D in genetic engineering. Investors currently can invest in such a firm, let us say Genentech, and expect returns whose size will be determined by the fortunes of new techniques in gene splicing. Let such firms now remove all of their capital from genetic engineering and invest it in equally valuable stocks such as IBM, and General Motors, and so on. What will be the effect of this on Genentech stockholders? Should they wish to redistribute their portfolio to suit their investing needs they will, of course, be able to do so, but they will no longer be able to purchase assets whose returns are tied to R&D in genetic engineering. Their choices are reduced. In practice, the decisions made by firms do not have such an all or nothing character about them. Yet if the firm were simply deciding what type of genetic research to pursue, how could one determine whether the effect of this seemingly smaller change is beneficial or harmful on net? This paper addresses this issue in a general equilibrium setting which includes uncertain investment opportunities and incomplete markets.

The motivation for this investigation comes from a number of sources. One of

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the central issues associated with general equilibrium under uncertainty has been that of finding a coherent theory of the firm when markets are incomplete and relating it to the workings of the financial markets. Merton [16] in a seminal paper introduced an intertemporal capital asset pricing model which analyzed portfolio choice and firm prices when asset returns were taken as exogenous to the model. Lucas [14] in a discrete-time model later gave conditions under which rationally consistent equilibrium prices and asset returns would exist in a model with no production and a single representative consumer. Brock [4] extended this work to cover production, but again restricted himself to an economy with one consumer. Much of the interesting portfolio behavior which was the focus of Merton's paper was, therefore, lost in order to deal with the difficult issue of existence. Moreover, questions relating to the effect of the firm's actions on risk-spreading between investors cannot be considered in such a model since no risk-spreading is present with identical consumers.

This paper considers risk-spreading and production decisions in a model with different types of consumers. In the absence of complete securities markets, investors may disagree about the production choices of firms. The demand side of the model resembles the intertemporal capital asset pricing model introduced by Merton [16] and subsequently studied in various forms by Long [13], Cox, Ingersoll, and Ross [5], Breeden and Litzenberger [3], Breeden [1, 2], and others. Breeden [1] and Cox, Ingersoll, and Ross [5] were also interested in describing the supply side of the capital asset pricing model. Cox, Ingersoll, and Ross [5] study a restricted technology to allow them to explicitly solve their model for reduced forms. This paper examines similar issues, but I focus on describing market optimality and the supply decisions which guide firms in incomplete markets in the absence of investor unanimity.

The approach taken in this paper assumes that the set of available contracts is limited to purchases of claims to firms' returns. New contracts become available in accordance with changes in the investment opportunities undertaken by the productive units in the economy, but assets which are not linear combinations of traded securities are not permitted. Firm actions are then chosen to maximize a shareholder-weighted measure of consumers' welfare, subject to the restriction that trading must be conducted through the medium of existing contracts. The resulting rule for firms, leading to a Nash-Constrained optimum similar to that introduced by Diamond [6], breaks down into two parts. The first is a "profitability" term related to the market value of the project at existing prices and the second is a "risk-spreading" term which places a value on the project for its associated improvement in the contract structure. If the sum of both values is positive, then the project is undertaken.

I also analyze the welfare properties of the model relative to the outcomes achieved by a constrained social planner. Section I describes the elements of the economy and its allocations relative to the allocations which would be reached by a central planner with appropriately chosen constraints on his or her feasible choices. Two optimality theorems are proved. Section II then relates social insurance to the workings of the competitive firm, and Section III makes concluding remarks.

I. The Elements of the Economy

A. Productive Capital

Consider an economy with m consumers indexed by $i = 1, \cdots, m$ which inherits from the past a fixed stock K of a single consumption good. Since the good can be eaten at any time or costlessly stored for use in producing more of the consumption good, the initial endowment can also be thought of as the capital stock inherited from the past, or initial wealth.

The production choices available to the economy at a point in time are limited by its available capital stock. For each t, the available capital stock $K(t)$ can be allocated between n distinct productive technologies or "industries," subject to the requirement that the allocation $K_j(t)$ satisfy

$$\textstyle\sum_j s_j K_j(t) = K(t), \qquad K_j(t) \geq 0 \tag{1}$$

where $s_j \geq 0$ is the number of identical units or firms operating in each industry. Firms have free access to the set of technologies and each industry exhibits free entry or exit. Capital services are homogeneous and costlessly interchangeable between activities at each point in time. For example, if production of computer hardware in Silicon Valley represents an industry technology then the activities of a firm in that industry can be duplicated by a new entrant locating next door. Firms may exit an industry at any time, liquidating part or all of their capital stock for an equivalent quantity of numeraire. The assumption of free entry and exit is a perfect markets assumption which simplifies the workings of the model.[1]

Output, $dK_j(t)$, for firms in industry j is described by the Itô diffusion process,

$$\begin{aligned} dK_j(t) &= f_j dt + \sigma_j\, dw_j, \qquad j = 1, \cdots, n \\ dS(t) &= h(S(t), t)\, dt + \sigma_s(S(t), t)\, dw_s \end{aligned} \tag{2}$$

where total industry output is given by $s_j dK_j(t)$. w_j and w_s are standard Brownian motion with unit instantaneous variances and $\sigma_{jl}, \sigma_{js}; j, l = 1, \cdots, n$, instantaneous correlations. $S(t)$ is an exogenous stochastic process representing the state of nature at time t and past history, such as weather or the effects of uncertain technological progress on production. $S(t)$ can be vector valued or a scalar. In the case where $S(t)$ is vector valued, each component of $S(t)$ represents a different state variable affecting output. For example, weather conditions at two geographically distinct locations or technological progress in each of two different production processes are different state variables. For the moment, I treat $S(t)$ as if it were a scalar process and generalize to vector valued $S(t)$ later. The levels of f_j,

[1] With costless revision of capital assignments, firms prefer to be able to adjust their use of capital at each instant, whether they actually avail themselves of this option or not. In reality, transactions costs and adjustment costs do exist to varying degrees. One reason for the use of discrete time in standard models is to give implicit recognition to those costs. In a stochastic environment, the trading intervals would typically be endogenous to the model and almost surely of variable length. During periods in which capital in place is not perfectly exchangeable with new capital or capital in other uses, it would bear a different market value in different assignments. If PK_j denotes the market value per unit of capital in activity j ($P_{Kj}K_j$ equal to the market value of firm j), free entry and exit arbitrage imply $P_{Kj} \equiv 1$.

σ_j, σ_{sj}, and σ_{1j} through σ_{nj} are choice variables of firm j, determined by its production decisions and use of capital stock. For the economy as a whole, the choices $y \equiv (s_1, \cdots, s_n; \sum_j s_j f_j, \sum_j s_j \sigma_j \sigma_{j1} \sigma_1, \cdots, \sum_j s_j \sigma_j \sigma_{jn} \sigma_n, \sum_j \sigma_{sj} \sigma_j, -\sum_j s_j K_j)$ must come from a closed convex production set Y containing the origin and depending on the state of nature $S(t)$. The firms' choices, $y_j \equiv (f_j, \sigma_{sj}\sigma_j, \sigma_1\sigma_{1j}\sigma_j, \cdots, \sigma_n \sigma_{nj} \sigma_j, -K_j)$, therefore also come from a closed convex production set Y_j containing the origin and depending on the state of nature $S(t)$ as well as the choices of other firms.[2] Capital is necessary for production; if K_j is equal to zero this implies $y_j \equiv 0$. The dependence of Y_j on the choices of other firms is due to the presence of the covariance terms $\sigma_j \sigma_{jk} \sigma_k$ in firm j's production choices. The assumption of convexity applied to f_j and $-K_j$ expresses the usual requirement that expected output is the result of a production process with declining marginal product of capital.

Even though the only source of uncertainty in the model is randomness in the returns to the productive units, this need not imply perfect hedging or duplication of movements in the state variables $S(t)$. A class of specific examples is worked out in the Appendix.

B. The Consumer

The utility function for individual i, $U^i(C^i(t))$, is defined on nonnegative consumption levels $C^i(t)$ where $U_c^i > 0$, $U_{cc}^i < 0$, and $\lim_{c\to 0} U_c^i = +\infty$. (Subscripts denote differentiation with respect to $C^i(t)$.) The individual's expected lifetime utility is given by

$$E_t\left[\int_t^T U^i(C^i(s))e^{-\rho s}\, ds\right], \tag{3}$$

where ρ is a positive discount factor and E_t is the expectation operator applied to the integral of discounted utility over the individual's lifetime from t to T.

Each consumer is endowed with claims to the output of the various productive firms. Consumers are able to trade their claims where each firm has one outstanding perfectly divisible equity share.

Shares to the productive output of firms in industries 1 through n are traded at competitively determined prices $P(t) = (P_1(t), \cdots, P_n(t))$ which are taken as given by consumers. The market value of wealth owned by the consumer at time t consists entirely of the value of his or her claims to output,

$$W^i(t) = \textstyle\sum_j \theta_j^i P_j(t) \tag{4}$$

where θ_j^i is the sum of fractions of output of firms in industry j owned by the consumer. In general, the wealth of individual i grows according to the value of capital gains plus firm production less consumption. Letting dW^i denote the

[2] An alternative formulation is to write f_j, σ_j, and σ_{sj} as functions of $(K_j(t), S(t), t)$ directly, and σ_{jk} as a function of $(K_j(t), K_k(t), S(t), t)$. Such cross-dependence does not imply the presence of production externalities in the usual sense since the "dependence" only refers to the correlation between the output of different firms and not to direct effects on other firms' output. Such correlations also arise in discrete-time models, for example.

change in the wealth of individual i,

$$\begin{aligned} dW^i &= \textstyle\sum_j \theta_j^i[dK_j(t) + K_j(t)\ dP_{Kj}(t)] - C^i(t)\ dt \\ &= \textstyle\sum_j \theta_j^i dK_j(t) - C^i(t)\ dt \\ W^i(0) &= W_0^i, \end{aligned} \tag{5}$$

where $dK_j(t)$ is given in Equation (2) and capital gains on capital in j, $dP_{Kj}(t)$, are zero by the assumption of homogeneous capital and free entry.[3] Merton assumes in his intertemporal asset pricing model that the sum of components for each return in (5) is described by a continuous Itô process without describing the individual processes such as capital gains, production choices, and so on, which make it up. Since $dK_j(t)$ is an Itô process under the specifications of this model, (5) is consistent with Merton's assumptions.

Given (4) and (5), the consumer seeks to maximize (3) subject to the choice of controls $(\theta_1^i, \cdots, \theta_n^i, C^i)$. The maximized value of (3) given the description of the economy in $S(t)$ and the choices of the consumer is the optimal value function for consumer i denoted by $J^i(W^i(t), S(t), t)$.

C. The Definition of Market Equilibrium

Equilibrium in the market for equity is defined by the requirement that demand equals supply in market claims θ_j^i for each industry,

$$\textstyle\sum_{i=1}^m \theta_j^i = s_j, \qquad j = 1, \cdots, n. \tag{6}$$

In the market for physical capital, Equation (1) defines equilibrium where the demand for capital by firms is assumed to be determined from the choice of y_j to maximize,

$$\max_{y_j \in Y_j} (P_j - K_j) + \textstyle\sum_{i=1}^m \theta_j^i R_j^i, \tag{7}$$

where

$$\begin{aligned} P_j &= (f_j + \rho_1 M_j + \rho_2 N_j)/r,\ M_j = \textstyle\sum_k \sigma_k \sigma_{kj} \sigma_j,\ N_j = \sigma_j \sigma_{js} \sigma_s, \\ \rho_1, \rho_2 &= \text{market risk coefficients for } M_j, N_j, \\ \textstyle\sum_{i=1}^m \theta_j^i R_j^i &= \text{share-weighted average of the risk-spreading value of firm } j\text{'s} \\ &\quad\ \text{equity as part of the contract structure.} \end{aligned}$$

R_j^i will be examined in some detail in Section II. Entry into industry j causes expansion or contraction to keep the price of capital equal to unity. For the moment, the choice rule (7) for firms is taken as exogenous to the model. Subsequently, we will derive it endogenously from the assumption of maximization of stockholders' welfare and describe its properties relative to the choices which would be made by a centralized social planner. P_j is the market price for

[3] See Merton [16, pp. 870–71] for a discussion of asset returns under uncertainty. (5) is consistent with Merton by letting the holding period for the return go to zero and incorporating depreciation into f_j and σ_j.

equity in firm j which is familiar from the intertemporal capital asset pricing model.

The definition of a competitive market equilibrium is as follows:

Definition: A competitive market equilibrium $\mathscr{S}$ is a set of prices $\{P_1(t), \cdots, P_n(t), r(t)\} \in R^n_+ \times R^1_{++}$ and optimal value functions $J^i(W^i(t), S(t), t)$, $i = 1, \cdots, m$, with $J^i(W^i(T), S(T), T) \equiv 0$, J^i continuously differentiable in t and twice continuously differentiable in $(W^i(t), S(t))$, such that the evolution in $S(t)$ is described in (2) and the evolution in $W^i(t)$ is described in (5), where at each moment in time firm choices $y_j(t)$ $j = 1, \cdots, n$ and consumer choices $\theta^i_j(t)$, $C^i(t)$ $i = 1, \cdots, m; j = 1, \cdots, n$ satisfy the following conditions:

(i) Producer Optimization
$y_j(t)$ maximizes (7) for $j = 1, \cdots, n; t \leq T$.

(ii) Consumer Optimization
$(\theta^i_1(t), \cdots, \theta^i_n(t), C^i(t))$ maximizes (3) subject to (4) and (5) for $i = 1, 1, \cdots, m; t \leq T$.

(iii) Market Clearing
$\sum_i \theta^i_j(t) = s_j$ for $j = 1, \cdots, n; t \leq T$,
$\sum_j s_j K_j(t) = K(t); s_j, K_j(t) \geq 0$ for $j = 1, \cdots, n; t \leq T$.

This definition corresponds to the usual notion of competitive equilibrium in an Arrow-Debreu economy. The functions J^i describe the consumer's welfare over the horizon from t to T while the instantaneous utility function defines welfare only over current consumption. Since there are two markets in this economy apart from consumption goods, equilibrium is described in (iii) by the requirements that the market for equity clears at each point in time and that the sum of firm demands for capital equals the available supply given by $K(t)$. P_j and r are prices which clear the market for equity and physical capital, respectively. Conditions (i) and (ii) insure that firms maximize the perceived value of output at each point in time (including risk-spreading value), and that consumers make portfolio decisions to maximize their lifetime welfare. Rational expectations (Equation (5)) imply that the growth in output which is perceived by the consumer is equal to the actual growth generated by the model.

D. The Centrally Planned Economy

The objective of the central planner at each moment in time is the maximization of a weighted average of consumers' expected lifetime utilities,

$$\textstyle\sum_i w^i E_t \int_t^T U^i(C^i(s))e^{-\rho s}\, ds, \tag{8}$$

where w^i are positive scalars determined by the distribution of wealth and state of nature $(W^1(t), \cdots, W^m(t), S(t), t)$. Planning decisions are made at each t based on the state variables of the system at that time.

The controls available to the planner at time t are given by $u(t) = (s_1, \cdots, s_n, y_1(t), \cdots, y_n(t), \theta^1_1(t), \cdots, \theta^1_n(t), \cdots, \theta^m_n(t), C^1(t), \cdots, C^m(t))$ subject to the restrictions in (1), (2), (5), and (6). Here $u(t) = (y(t), \theta(t), C(t))$ refers to the feedback control rule $u(W^1(t), \cdots, W^m(t), S(t), t)$ which depends on the state variables $(W^1(t), \cdots, W^m(t), S(t), t)$ and similarly for each of its components $s_j(t), y_j(t), \theta_j^i(t)$, and $C^i(t)$.

The feasible set from which $u(t)$ is chosen is not always convex for economies with incomplete markets.[4] This limits the decentralization properties of the economy since the first-order conditions for the central planner's problem are not sufficient for a Constrained Pareto Optimum (cf. Diamond [6]). By examining the restrictions on a central planner which result in the same allocation as that achieved by a competitive stock market with incomplete markets, we gain insight into the limitations present in the private market. Consider, therefore, a central planner with somewhat limited powers, chosen in a manner so that a competitive stock market with incomplete markets can achieve the same allocation. In addition to distribution determined by stock ownership, assume the Nash-type planning constraint that share assignments and consumption plans (θ_j^i, C^i) must be taken as given when making centralized production choices (y_j), and that production choices must be taken as given when making centralized choices about θ_j^i and C^i. Call the resulting allocation a Nash-Constrained Pareto Optimum.[5]

Although the central planner, subject to the Nash-type planning constraint, has somewhat less freedom than an unconstrained central planner, he or she still has considerably more latitude than an individual firm or household. In particular, he or she can centrally coordinate the production decisions of the entire economy to maximize aggregate welfare given share ownerships, as well as centrally assign ownership shares and consumption levels to maximize aggregate welfare given production. Neither of these could be accomplished by an individual firm or household. We shall see that the private market achieves a Nash-Constrained Pareto Optimum, although not necessarily a Constrained Pareto Optimum or Pareto Optimum.

Collecting the requirements for the central planning problem, the planner chooses $u(t) = (\theta(t), C(t), y(t))$ to maximize (8) subject to the Nash-type planning constraint (1), (2), (5), (6) and the technical requirement that the control come from a set of admissible controls, V, where, in addition to satisfying the constraints above, V is restricted so that the Itô process is defined for $u \in V$. That is, $u \in V$ is defined such that there exists a Brownian motion w_j, w_s for which Equations (2) and (5) describe the movement in $S(t)$ and $W^i(t)$. For example, a sufficient condition on u which is appropriate for certain problems is the requirement that u satisfy a Lipschitz condition.

Applying the principle of dynamic programming to the planner's problem, the solution for the optimal control u is given by the dynamic programming equation,

$$0 = J_t + \max_{u \in V} \left[\textstyle\sum_i w^i U^i(C^i(t))e^{-\rho t} + \mathscr{A}(t)^u J\right],$$

[4] See Drèze [6a]. Let $d\bar{K}_j > dK_j > 0$ and $d\bar{K}_k = dK_k = 0$, $k \neq j$, $\bar{\theta}_j^i > \theta_j^i > 0$ and $(d\bar{W}^i + d\bar{C}^i) - \sum_j \bar{\theta}_j^i dK_j = 0$, $(dW^i + dC^i) - \sum_j \theta_j^i dK_j = 0$. Then for $0 < \lambda < 1$, $\lambda(d\bar{W}^i + d\bar{C}^i) + (1-\lambda)(dW^i + dC^i) - (\lambda\bar{\theta}_j^i + (1-\lambda)\theta_j^i)(\lambda d\bar{K}_j + (1-\lambda)dK_j) = \lambda(1-\lambda)[\bar{\theta}_j^i d\bar{K}_j + \theta_j^i dK_j - \bar{\theta}_j^i dK_j - \theta_j^i d\bar{K}_j] = \lambda(1-\lambda)(\bar{\theta}_j^i - \theta_j^i)(d\bar{K}_j - dK_j) > 0$.

[5] From the nesting of constraints, Pareto Optimal allocations are a subset of the set of Constrained Pareto Optimal allocations which are a subset of the set of Nash-Constrained Pareto Optimal allocations.

$$0 \equiv J(W^1(T), \cdots, W^m(T), S(T), T), \tag{9}$$

where $\mathscr{A}(t)^u$ is the differential generator for control u discussed by Fleming and Rishel [8, pp. 120–23] (cf. Merton [15] and Cox, Ingersoll, and Ross [5]), $J = \sum_i w^i J^i(W^i(t), S(t), t)$, J^i $j = 1, \cdots, m$ are continuously differentiable with respect to t, twice continuously differentiable with respect to $(W^i(t), S(t))$, and J^i has negative second derivative with respect to $W^i(t)$.[6] In the remainder of the paper, I assume that the constraint qualification for (9) holds and that the solution to the central planner's problem for u and J exists, satisfying the requirements of (9). The assumption of differentiability, in particular, is necessary for the application of the dynamic programming technique and the analysis which follows.[7] Performing the maximization in (9) generates the necessary first-order conditions for the central planner's problem:

$$w^i U_c^i e^{-\rho t} - J_{W^i} = 0, \tag{10a}$$

$$J_{W^i} f + J_{W^i W^1} \Sigma \theta^1 + \cdots + J_{W^i W^m} \Sigma \theta^m + J_{W^i s} N - \mu = 0,$$
$$i = 1, \cdots, m, \tag{10b}$$

$$\sum_i J_{W^i} \theta_j^i \frac{\partial f_j}{\partial \lambda_j} + \frac{1}{2} \sum_i \sum_h J_{W^i W^h} (\theta^i)' \frac{\partial \Sigma}{\partial \lambda_j} \theta^h$$
$$+ \sum_i J_{W^i_s} \theta_j^i \sigma_s \frac{\partial \sigma_{sj} \sigma_j}{\partial \lambda_j} - \delta \frac{\partial K_j}{\partial \lambda_j} \le 0, \tag{10c}$$

$$\mu_j - \delta K_j \le 0, \qquad j = 1, \cdots, n \tag{10d}$$

where μ is a nonnegative $n \times 1$ vector of Lagrangian multipliers $[\mu_j]$ for the constraint on portfolio claims, $\delta > 0$ is the multiplier for the capital constraint, Σ is the $n \times n$ covariance matrix $[\sigma_j \sigma_{jk} \sigma_k]$, θ^i is the $n \times 1$ vector $[f_j]$, N is the $n \times 1$ vector $[N_j]$, and J stands for the optimal value function for the social planner. By convention the derivative with respect to λ_j represents an infinitesimal change in firm j's current production $y_j(t)$ in the direction of a feasible choice $\hat{y}_j(t) \in Y_j(t)$. Defining $z(\lambda) \equiv \lambda \hat{y}_j(t) + (1 - \lambda) y_j(t)$ as the weighted average of $\hat{y}_j(t)$ and $y_j(t)$ allows changes in production with respect to λ to be defined in terms of movements in $z(\lambda)$. $z(\lambda) \in Y_j$ by the convexity of Y_j. The effect of a small change in y_j can then be considered in terms of a differentiable change in λ.

[6] See Fleming and Rishel [8], Theorem 4.1. The condition that J at T be identically zero also implies that $J_{W^i}(W^1(T), \cdots, W^m(T), S(T), T) = 0$ as well as for higher derivatives. Given $u \in V$ and an associated controlled diffusion process dx, $x \in R^n$, where the instantaneous drift term for the j^{th} component of dx at time t is given by $b_j(t)\,dt$ and its covariance matrix at time t is given by $[a_{ij}(t)]$, then $\mathscr{A}(t)^u F(x, t)$ is the differential generator for control u given by,

$$\mathscr{A}(t)^u F(x, t) \equiv \sum_{j=1}^n b_j(t) \frac{\partial F}{\partial x_j} + \frac{1}{2} \sum_{i,j} a_{ij}(t) \frac{\partial^2 F}{\partial x_j \partial x_i}.$$

[7] The assumption of differentiability is standard in this type of analysis. Virtually all techniques for solving dynamic programming problems in continuous time depend on the dynamic programming equation or "Bellman equation" as it is sometimes called.

E. *Optimality*

The following example shows that satisfaction of the conditions for a competitive market equilibrium need not lead to a Constrained Pareto Optimum. Theorems 1 and 2 then show that the set of competitive market equilibria coincides with the set of Nash-Constrained Pareto Optima defined by the planner's problem in (9).

In the example, the economy consists of two firms $j = 1, 2$, two households, and one state variable s. Production is characterized by $\sigma_1^2 = \sigma_2^2 = \sigma$ for constant σ when $K_1 = K_2 = 1$, $\sigma_{12} = 0$. Feasible choices for expected return f_j and σ_{js} for firm 1 are given by *Oabc* in Figure 1 when $K_1 = 1$, and by *Odef* for firm 2 when $K_2 = 1$. $\theta_1^1 = \theta_2^2 = 1$ implies that household 1 fully owns firm 1 and household 2 fully owns firm 2. Preferences in f_j, σ_{js} space for the two households are given by curves *au* and *ev*, respectively. Directions of welfare improvement are shown by arrows.

Given the production choices *Oabc*, and ownership by household 1, firm 1 maximizes the value of output to its stockholder when production is at point a, given $K_1 = 1$ and $\sigma_1 = \sigma$. Similarly, firm 2 maximizes the value of its output at point e, given $K_2 = 1$ and $\sigma_2 = \sigma$. With production at a and e, neither stockholder has an incentive to exchange shares of stock for ownership shares in the other firm. Hence, each firm produces optimally, *given* the preferences of its owner, and each household holds an optimal portfolio, *given* the production decisions of

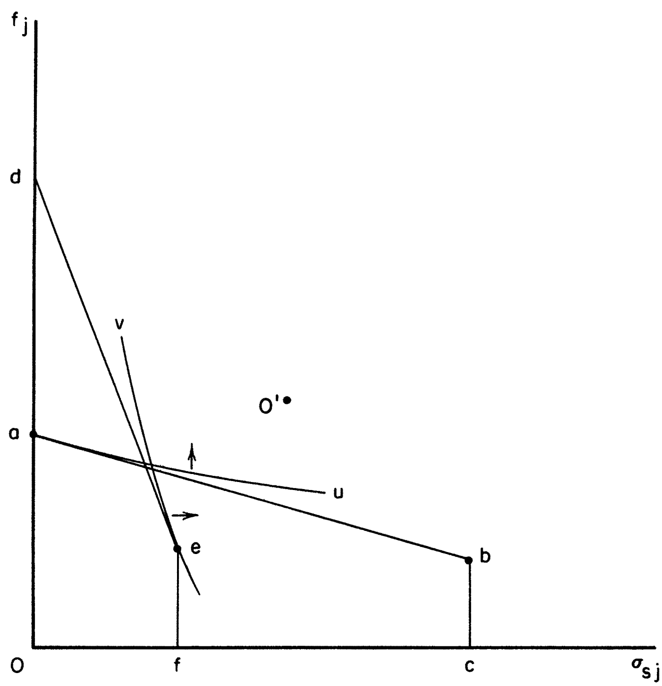

Figure 1. Example of Inefficient Equilibrium

firms. The inefficiency of the equilibrium is obvious, however. By producing b and d and dividing ownership shares of firms equally between households, a central planner could improve the welfare of both at point O'. The equilibrium in Figure 1 fails to achieve a Constrained Pareto Optimum since, without the convexity of the overall choice set, the first-order conditions in (10) are necessary for a Constrained Pareto Optimum but not sufficient.

We now turn to the type of optimality which is achieved by the market.

Definition: A control rule $u(t) = (\theta(t), C(t), Y(t))$ with associated optimal value functions $J^i(W^i(t), S(t), t)$, $i = 1, \cdots, m$, defined on the set of state variables $(W^1(t), \cdots, W^m(t), S(t), t)$ is said to be supported as a competitive market equilibrium if there exist prices $\mathscr{P} = \{P_1(t), \cdots, P_n(t), r(t)\} \in R^n_+ \times R^1_{++}$ such that choices $\theta^i_j(t)$, $C^i(t)$, $i = 1, \cdots, m; j = 1, \cdots, n$ by consumers and choices $s_j, y_j(t), j = 1, \cdots, n$, by firms constitute a competitive market equilibrium relative to $\mathscr{P}$.

Theorems 1 and 2 now show the equivalence between a competitive market equilibrium and the centrally planned economy with the Nash-type planning constraint. They play the role of the First and Second Fundamental Theorems of Welfare Economics in an Arrow-Debreu economy. The proofs are in the Appendix.

THEOREM 1. *Let $u(t) = (y(t), \theta(t), C(t))$ be an optimal solution to the central planning problem satisfying the conditions in* (9), *then u is supported as a competitive market equilibrium.*

The reverse implication also holds:

THEOREM 2. *Let $\mathscr{S}$ be a competitive market equilibrium and let $\mathscr{S}$ be a solution to the central planning problem satisfying the conditions in* (9), *then there exists a choice of weights $\mathscr{W} = \{w^i\}$ for $\mathscr{S}$ such that the market economy achieves the optimum to the central planning problem with weights $\mathscr{W}$.*

Having described the optimality of equilibrium, the next section examines in greater detail the firms' supply and the effect of changes in the contract structure.

II. Production and the Capital Market Structure

In an economy with different types of consumers there is no guarantee that the equilibrium allocation is Pareto Optimal since opportunities for risk-sharing implicit in trading claims to the n productive activities may not be sufficient to support a full Pareto Optimum. In addition, investors will generally have differing ideas about the marginal value of a change in the capital market structure. Since consumers are different, they hold portfolios which consist partially of stocks held for hedging purposes; some investors will be long in the contracts ("buyers") and some investors will be short in the contracts ("sellers"). Because the terms of the contracts are tied to the returns of traded securities, a production decision which changes the return structure of a given security affects the ability of private consumers to offer contracts to one another. Even though a project is profitable in terms of existing market prices, its effect on altering the types of contracts which the market can offer may be great enough to make it socially undesirable to all consumers.

The discussion of multiple consumers as it relates to firms therefore requires an explicit policy governing the types of "inside" contracts which consumers may offer to one another. An inside contract is one for which the sum of payments over all consumers in the economy adds up to zero in every state of nature. It is a pure risk-spreading arrangement. One policy might be to allow the costless introduction of inside contracts, whose payoff is tied to various state variables of the system, in addition to the traded securities of productive units. It is known, however, that such a policy would result in perfect hedging where, at most, C such contracts are needed and C is the dimensionality of the vector $S(t)$.[8] This result is further considered in the case of imperfect hedging in Theorem 4 where the social benefits of an additional inside contract are evaluated.

A more realistic alternative would be to assume that a perfect set of markets is not available, either because the costs of issuing new contracts are prohibitive relative to their benefits, or that costs of observing the relevant state variables are too high. Instead investors can trade only in contracts whose payoffs are tied to the returns of traded securities but not to other variables. Since stockholders may disagree about the value of the firm's alternatives, some assumption must be made about how differences are resolved. In this paper, the firm chooses a production plan which maximizes the sum of values placed on the project by the stockholders of the firm themselves. This objective agrees with market-value maximization in the one-consumer case. Its rationale when there is more than one consumer is that it is the rule which would be chosen if shareholders who favor one production plan over another were able to compensate those who do not. Even when the compensations are not actually carried out, as in the Kaldor-Hicks welfare criterion, the firm chooses the project with the highest net value after compensations are netted out. If the resulting value is positive the plan is undertaken.

To the investor, changes in the capital market structure have welfare consequences which may not be captured solely in market value. Since the firm values its activities as its shareholders do, terms valuing changes in the contract structure appear in the formulae used by firms to choose their production plans. These can be isolated for study.

A. Production, Portfolio, and Pricing Rules

Solving (10a) and (10b) for θ^i and substituting into (10c)[9] gives the production rule for firms introduced in (7),

$$\frac{\partial(P_j - K_j)}{\partial\lambda_j} + \frac{\partial(\sum_i \theta_j^i R_j^i)}{\partial\lambda_j} = 0, \tag{11}$$

[8] See Breeden [1].

[9] Without loss of generality, let Σ refer only to the variance-covariance matrix of the firms operating at nonzero levels where (10c) and (10d) are strict equalities.

and the equilibrium pricing and portfolio relations,

$$P_j = \frac{1}{r}\,[f_j + \rho_1 M_j + \rho_2 N_j], \tag{12}$$

$$\theta^i = \alpha^i \Phi - \eta^i \Psi, \tag{13}$$

which are familiar from Merton's intertemporal capital asset pricing model where the notation is condensed using,

$$R^i_j = \beta^i \eta^i (N_j - \Psi' \textstyle\sum_j),$$

$$\beta^i = \rho_1 / r\alpha^i,$$

$$\eta^i = \phi_i \frac{J_{W^i s}}{J_{W^i}} - \alpha^i \textstyle\sum_i \phi_i \dfrac{J_{W^i s}}{J_{W^i}},$$

$$\Psi = \Sigma^{-1} N,$$

$$\textstyle\sum_j = \begin{bmatrix} \sigma_1 \sigma_{1j} \sigma_j \\ \vdots \\ \sigma_n \sigma_{nj} \sigma_j \end{bmatrix},$$

$$\rho_1 = (\textstyle\sum_i \phi_i)^{-1},$$

$$\rho_2 = \rho_1 \textstyle\sum_i \phi_i \dfrac{J_{W^i s}}{J_{W^i}}, \quad \text{and}$$

$$\alpha^i = \rho_1 \phi_i.$$

$$\phi_i = J^i_{W^i} / J^i_{W^i W^i} < 0$$

is the inverse of individual i's coefficient of risk aversion, and the firm treats Ψ as a constant in computing (11).[10] Since (11) is derived from the maximization of investors' utilities, the firm's objective function in (7) can also be derived endogenously from the maximization of shareholder welfare.[11]

Both (13), describing portfolio selection, and (12), the pricing relation for firm equity, are well known from the intertemporal capital asset pricing model and therefore do not need much elaboration. As in Merton [16], the typical consumer, in addition to the riskless asset, holds a portion of the market portfolio indicated

[10] Equation (11) implicitly assumes the presence of riskless borrowing and lending, say activity j_0, at the market clearing rate $r(t)$. For $j = j_0$,

$$dK_{j_0} = r(t) K_{j_0}\, dt$$

$$\textstyle\sum_i B^i_{j_0} = K_{j_0}$$

where $B^i_{j_0} \equiv \theta^i_{j_0} K_{j_0}$. The portfolio equation for the case of no riskless asset can be analyzed on the same principles as the riskless asset case leading to an additional term in (11). The shadow price related to μ_j in (12) has been replaced by its equivalent market price P_j.

[11] That is, firm j solves the problem $\max_{y_j} \sum_i \theta^i_j (1/U^i_c e^{-\rho t}) \int_t^T U^i(C^i(s)) e^{-\rho s}\, ds$ given the portfolios of shareholders.

by the weights $\alpha^i\phi$, and a second portfolio whose weights are proportional to Ψ. This represents a portfolio whose returns are maximally correlated with the state variable $S(t)$. Portfolios with weights proportional to Ψ are hedges against changes in $S(t)$. (If $S(t)$ is a vector of dimension C there will be C portfolios purchased instead of one, each one representing a hedge against innovations in one of the components of $S(t)$. See Breeden [1].) Since it sums to zero over i, η^i in Equation (11) reveals that the net quantity of such portfolios outstanding is zero; some consumers will be purchasers and others will be sellers. Ψ corresponds to the hedging component of the consumer's portfolio.

In the pricing relation, (12), ρ_1 and ρ_2 constitute the risk coefficients for market risk M_j and state variable risk N_j. Were there to be other state variables such as would be the case if $S(t)$ were a vector, the arguments of Equation (12) would expand to include each component of $S(t)$. Instead of the term $\rho_2 N_j$ would be the sum $\sum_v \rho_v N_{vj}$ where v ranges over the elements of the vector $S(t)$.

Since the stochastic process for security returns and the capital market structure in Section II satisfy all of the assumptions used by Merton [16] in his original exposition of the intertemporal capital asset pricing model, the analysis in that paper of the demand functions for assets and the equilibrium yield relationships among assets applies here. Equations (12) and (13) imply Merton's Equations (16) and (25) from which the rest of his analysis follows.[12] The additional element in the present model, however, is the presence of Equation (11) which describes the investment choices of firms. Since Merton treats asset returns as exogenous, he does not analyze the supply side of the model and there is no corresponding equation in his model. Equation (11) separates the condition

[12] Let $\mathrm{diag}[P_j]$ denote the diagonal matrix whose diagonal elements are $P_1, \cdots, P_n$. Then from Equations (12) and (13),

$$\begin{aligned}\theta^i &= \alpha^i\Phi - \eta^i\Psi = -\phi_i\rho_1\Phi - \left(\phi_i\frac{J^i_{W^is}}{J^i_{W^i}} - \phi_i\rho_2\right)\Sigma^{-1}N \\ &= -\phi_i\Sigma^{-1}(-\rho_1\Sigma\Phi - \rho_2 N) - \frac{J^i_{W^is}}{J^i_{W^iW^i}}\Sigma^{-1}N \\ &= -\phi_i\Sigma^{-1}\mathrm{diag}[P_j]\Sigma^{-1}\mathrm{diag}[P_j]\begin{bmatrix} f_1/P_1 - r \\ \vdots \\ f_n/P_n - r\end{bmatrix} - \frac{J^i_{W^is}}{J^i_{W^iW^i}}\Sigma^{-1}N.\end{aligned}$$

Individual i's vector of demands for assets is, therefore,

$$\mathrm{diag}[P_j]\theta^i = A\Omega^{-1}\begin{bmatrix}\alpha_1 - r \\ \vdots \\ \alpha_n - r\end{bmatrix} + H_k\Sigma^{-1}N \qquad (11')$$

which is Merton's Equation (16) where

$$A = -\phi_i, \qquad \Omega^{-1} = \mathrm{diag}[P_j]\Sigma\ \mathrm{diag}[P_j], \qquad \alpha_j = f_j/P_j$$

and $H_k = -\dfrac{J^i_{W^is}}{J^i_{W^iW^i}}$. Assuming as Merton does that the return of the n^{th} asset is perfectly negatively correlated with changes in the state variable, (11′) leads directly to Merton's Equation (25) for aggregate demands.

under which a firm will be satisfied with its investment choice at time t into two components, market value and the value of the induced change in contract structure to investors. In general, the subjective value of a marginal change in the contract structure to shareholder i is given by $\frac{\partial}{\partial \lambda_j}(\theta_j^i R_j^i)$. The share θ_j^i determines individual i's participation level in the project and R_j^i is made up of the term,

$$\eta^i N_j - \eta^i \Psi' \textstyle\sum_j, \tag{14}$$

multiplied by the scaling factor β^i converting units of risk into present value terms. From (11) $\eta^i N_j$ is the difference of individual i's $S(t)$-induced risk and the fraction α^i of the market sum over all individuals of $S(t)$-induced risk. Since individual i diversifies by holding the fraction α^i of each type of risk, $\eta^i N_j$ is the difference between desired level and market holdings (undiversified risk). The difference between this and

$$\eta^i \Psi' \textstyle\sum_j$$

is the residual risk after using all available market opportunities for diversification. A change in the value of this residual with respect to λ_j is, therefore, the amount the investor would be willing to pay in addition to market value for the beneficial effect of changing the set of contracts available for spreading risk.[13]

Theorem 3 summarizes the demand and supply behavior for the economy contained in conditions (10)–(13) above. Since the description of the economy from the point of view of consumers and asset trading is identical to Breeden [1], the demand-side description of consumers and equilibrium pricing is the same as derived in his paper. Theorem 3 adds the supply-side production decisions of firms based on shareholder preferences. Theorems 1 and 2, of course, have already described the optimality of market equilibrium.

THEOREM 3. *Let $u(t) = (y(t), \theta(t), C(t))$ be the optimal control for the economy in Theorems* 1 *and* 2. *Then,*

(i) *Consumers. All individuals in the economy regardless of preferences may obtain their optimal portfolio position from Equation* (13) *by investing in at most three mutual funds. These funds may be chosen to be the assets consisting of* (1) *the portfolio made up of traded securities whose return is most highly correlated with $S(t)$,* (2) *the market portfolio, and* (3) *the instantaneously riskless asset.*

If $S(t)$ is a vector of dimension C, at most $C + 2$ funds are necessary where the funds may be chosen to be the market portfolio, the instantaneously riskless asset, and C portfolios constructed from traded assets each of whose returns is chosen to be maximally correlated with a different element of the vector $S(t)$.

[13] Since $N_j - \Psi' \sum_j$ is proportional to the covariance between $dK_j(t)$ and the residual from a regression of $S(t)$ on the returns of marketed securities, the covariance corresponding to $\frac{\partial}{\partial \lambda_j}[N_j - \Psi' \sum_j]$ is a measure of the partial correlation between the marginal project and $S(t)$, when the effect of marketed securities has already been removed from both series by projection. In a recently published paper, Breeden [2] discusses hedging behavior in a continuous-time model.

(ii) *Prices. The equilibrium market price of firm j is given by* $P_j = \frac{1}{r}[f_j + \rho_1 M_j + \rho_2 N_j]$ *where* r, ρ_1, M_j, *and* N_j *are as defined in Equation* (12).

(iii) *Firms. Firms make production choices so as to maximize net market value and the value of the contract structure to their stockholders,*

$$[P_j - K_j] + \sum_i \theta_j^i R_j^i$$

where the term $\frac{\partial}{\partial \lambda_j} \theta_j^i R_j^i$ *defined in Equation* (11) *is the marginal value to individual* i *of a change in the contract structure due to a change in* λ_j.

The significance of Theorem 3 is that it provides a rule for firms to follow in incomplete markets which takes into account the welfare effect of the investment on the contract structure.

The following corollary is immediate.

COROLLARY 3. *Market-value maximization maximizes shareholders' welfare if and only if*

$$\sum_i \theta_j^i \frac{\partial R_j^i}{\partial \lambda_j} = 0. \tag{15}$$

The next section investigates sufficient conditions for (15) to occur.

B. Spanning, Market Value Maximization, and Optimality

Condition 1: *Spanning.* When the output of the firm dK_j, for any choice of production, can be written as a linear combination of a fixed set of basis vectors, we say that the output of firm j is spanned by that set. For example, if the firm's output can be replicated by a linear combination of the output of other firms $\sum_j \alpha_j dK_j$ for some choice of scalars α_j, then $\{dK_j\}$ form a set of basis vectors.

Replacing dK_j by a linear combination of the output of existing firms shows that a sufficient condition for (15) is that the output of firm j be spanned by traded securities.[14] In this case, the contract structure is unchanged by the firm's production choices (i.e., the basis vectors are unaffected), implying no change in contract structure. Market value at existing prices, therefore, becomes the sole criterion for accepting or rejecting the project. Unanimity among shareholders about the firm's choices also follows when the output of firms is spanned.[15]

[14] $\sigma_s \frac{\partial \sigma_{sj} \sigma_j}{\partial \lambda_j} - \Psi' \begin{bmatrix} \sigma_1 \frac{\partial \sigma_1 1_j \sigma_j}{\partial \lambda_j} \\ \vdots \\ \sigma_n \frac{\partial \sigma_{nj} \sigma_j}{\partial \lambda_j} \end{bmatrix}$ is identically zero when $\frac{\partial \sigma_{sj} \sigma_j}{\partial \lambda_j}$ and $\frac{\partial \sigma_{kj} \sigma_j}{\partial \lambda_j}$, for $k = 1, \cdots, n$, are replaced by $\sigma_{sj}\sigma_j$, $\sigma_{kj}\sigma_j$, $k = 1, \cdots, n$, for any $j = 1, \cdots, n$.

[15] Writing (11) in the case where the firm is 100 percent owned by a single investor, we see that disagreements between investors arise solely over the relative merits of changes in the contract structure. When the term related to changes in the contract structure is zero, as in the case of spanning, market value determines investor ranking of projects.

Condition 2: *Perfect Hedging of State Variables.* It is also possible that $\partial R_j^i/\partial\lambda_j$ is identically zero for all i and j because $N_j - \Psi' \sum_j$ is identically zero. Following the discussion of (14), this happens only when state variable risk is perfectly hedged by a properly chosen portfolio of traded securities.[16] Thus, $N_j - \Psi' \sum_j$ identically zero indicates that perfect risk-sharing is provided by the stock market using existing securities. With perfect hedging, no premium or discount relative to market value is necessary in evaluating production choices, as is indicated by (11) when (15) holds.

C. Inside Assets and Hedging

In addition to changing the contract structure through changing production choices, the contract structure can be changed directly through other devices such as the introduction of inside assets.[17] The analysis of the firm's production decision can be applied to shed light on this problem. For example, the introduction of an inside asset using no physical resources but offering stochastic returns different from the returns of already traded assets will have welfare implications due to its effect on risk-spreading. Rather than analyzing a discrete change in market structure, one can imagine introducing the new asset in a continuous fashion by choosing a bound $B \in [0, \infty]$ such that investors who are long or short in the asset are constrained to have holdings less than the bound B. When $B = 0$ the initial equilibrium is described, when $B = \infty$ the new equilibrium is described, and with other cases $(0 < B < \infty)$ in between. The marginal effect of introducing the new asset then corresponds to the effect of increasing B at the initial point $B = 0$. The following theorem shows that if the new asset improves hedging, measured in terms of the correlation between $S(t)$ and the market hedge

$$\Psi' \begin{bmatrix} dK_1 \\ \vdots \\ dK_n \end{bmatrix},$$

it is, in principal, possible to improve everyone's welfare.[18] The proof in the Appendix exploits the analogy between R^2 (correlation coefficient) of a linear regression and the measure of change in contract structure $\partial R_j^i/\partial\lambda_j$ introduced earlier.

[16] That is, $dS(t) - \Psi' \begin{bmatrix} dK_1 \\ \vdots \\ dK_n \end{bmatrix} \equiv 0.$

[17] Mergers, financing decisions, and recapitalizations are other ways that the contract structure could be altered without physical changes in production initially, although physical production might subsequently be adjusted to the changed market.

[18] This is not to say that the actual introduction of a new asset will improve the welfare of any given household. Hart [10] shows, for example, that the introduction of a new asset, which does not result in a complete set of markets, may lead to welfare losses for investors. The present analysis differs from his in that the introduction of a new asset is accompanied by the use of appropriate lump sum transfers between investors and the assignment of ownership rights to the new asset. If transfers and assignments are not carried out, welfare losses could occur for any given investor or set of investors.

THEOREM 4. *Let $\mathscr{S}$ be a competitive market equilibrium satisfying* (9) *and* (11)–(13), *and let I represent the terms of a contract which is being considered for introduction in zero quantity in aggregate. From the individual's point of view, purchase of one unit of I entitles him to the random output $\sigma_I dw_I$ where w_I is standard Brownian motion. Let $\rho_s(t)$ represent the highest correlation which is possible between $S(t)$ and a portfolio of traded securities. Then there exists an allocation of claims to I such that every individual's welfare rises relative to the initial equilibrium upon the introduction of I if and only if $\rho_s(t)$ increases.*

As a corollary it follows that if $\rho_s(t)$ equals 1 initially, no allocation of claims to I can improve the welfare of individuals relative to the initial equilibrium. If $\rho_s(t)$ rises to 1 after the introduction of I, no further introduction of assets can improve the welfare of every individual relative to the equilibrium which includes I and the original set of contracts.[19]

When correlation is less than perfect, it can be shown that as $\rho_s(t)$ rises so does the ability of the investor to diversify and smooth his fluctuations in utility. Since changes in utility are given by $dU^i e^{\rho t} = U^i_c dC^i$, it is sufficient to consider changes in individual i's optimal consumption $C^i(W^i, S, t)$. Using stochastic calculus and substituting from the consumer's first-order conditions:

$$\begin{aligned} dC^i &= C^i_W dW^i + C^i_s\, dS \\ &= \alpha^i C^i_W[dM + adS] + (C^i_s - \alpha^i a C^i_W)\left[dS - \Psi'\begin{bmatrix} dK_1 \\ \vdots \\ dK_n \end{bmatrix}\right] \\ &= \alpha^i C^i_W \tilde{Z} + (C^i_s - \alpha^i a C^i_W)\tilde{Y} \end{aligned} \tag{16}$$

where a is given by $\sum_i \phi_i \dfrac{J^i_{W^i s}}{J^i_{W^i}}$, dM is a variation in the market portfolio, $\sum_j dK_j$, and $\tilde{Z}$ and $\tilde{Y}$ are random variables, identical for all consumers, defined by

$$dM + adS \quad \text{and} \quad dS - \Psi'\begin{bmatrix} dK_1 \\ \vdots \\ dK_n \end{bmatrix},$$

respectively. $\tilde{Z}$ represents risk systematically related to market and economy-wide variables. $\tilde{Y}$ represents nonsystematic state-variable risk after removing market-related variations from $dS(t)$. With perfect hedging for $S(t)$, $\tilde{Y}$ is identically zero. In this case, every consumer's utility varies solely in proportion to changes in scalar $\tilde{Z}$, the economy-wide aggregate.[20]

[19] Breeden [2] reaches the same conclusions about the case of perfect correlation.

[20] The aggregate-consumption beta of asset j is the same whether $\tilde{Y} \equiv 0$ or not;

$$\begin{aligned} \mathrm{cov}(\textstyle\sum_i dC^i, dK_j) &= \textstyle\sum_i \alpha^i C^i_W \mathrm{cov}(\tilde{Z}, dK_j) + \sum_i (C^i_s - \alpha^i a C^i_W)\mathrm{cov}(\tilde{Y}, dK_j) \\ &= (\textstyle\sum_i \alpha^i C^i_W)\mathrm{cov}(\tilde{Z}, dK_j). \end{aligned}$$

As the residual from a "regression" of $dS(t)$ on $\{dK_1, \cdots, dK_n\}$, $\tilde{Y}$ is orthogonal to dK_j ($\mathrm{cov}(\tilde{Y}, dK_j) = 0$), for all j.

When the contract structure does not permit perfect hedging, an increase in hedging measured by $\rho_s(t)$ can be shown to reduce the variance of $\tilde{Y}$. Using the fact that $\tilde{Y}$ is uncorrelated with marketed securities,

$$\sigma_Y^2 = \sigma_s^2 - \Psi'N = \sigma_s^2\left[1 - \left(\frac{\Psi'N}{\sqrt{\Psi'N}\sqrt{\sigma_s^2}}\right)^2\right] = \sigma_s^2(1 - \rho_s^2(t)). \tag{17}$$

Thus, an increase in ρ_s implies a reduction in σ_Y^2.

Another implication of a change in the contract structure is the effect that larger ρ_s has on the correlations between marginal utilities of different investors. Using (16),

$$\rho_{C^i,C^k} = \frac{\alpha^i\alpha^k C_W^i C_W^k \sigma_X^2 + C_s^i C_s^k \sigma_Y^2}{\sqrt{(\alpha^i\alpha^k C_W^i C_W^k \sigma_X^2 + C_s^i C_s^k \sigma_Y^2)^2 + (\alpha^i C_W^i C_s^k - \alpha^k C_W^k C_s^k)^2 \sigma_X^2 \sigma_Y^2}} \tag{18}$$

where ρ_{C^i,C^k} is the correlation between dC^i and dC^k, and

$$\tilde{X} = dM + a\Psi'\begin{bmatrix} dK_1 \\ \vdots \\ dK_n \end{bmatrix} = \tilde{Z} - a\tilde{Y}$$

is constructed to have zero correlation with $\tilde{Y}$. It follows that,

$$\sigma_Z^2 = \sigma_X^2 + a^2\sigma_Y^2. \tag{19}$$

Rewriting (16), variation in dC^i can be written in terms of the orthogonal components $\tilde{X}$ and $\tilde{Y}$,

$$dC^i = \alpha^i C_W^i \tilde{X} + C_s^i \tilde{Y}.$$

Increasing ρ_s through the introduction of an inside asset reduces the variance of $\tilde{Y}$ and increases the variance of $\tilde{X}$ by (19). Two investors for whom $\tilde{X}$ represents the major component of variation in their marginal utilities, therefore, will have marginal utilities more highly correlated with one another as the contribution of nonsystematic risk is reduced in their holdings.[21] This is confirmed by taking the derivative of (18) with respect to ρ_s using (17) and (19),

$$\frac{d\rho_{C^iC^k}}{d\rho_s} = K\left(\sigma_X^2 - \frac{C_s^i C_s^k}{\alpha^i C_W^i \alpha^k C_k^i}\,\sigma_Y^2\right) \tag{20}$$

where K is a positive constant of proportionality given by

$$\left[\frac{\alpha^i\alpha^k C_W^i C_W^k}{D}\right][(\alpha^i C_W^i C_s^k - \alpha^k C_W^k C_s^i)^2 \sigma_s^2 \rho_s \sigma_Z^2]$$

and D is the denominator of (18). When σ_Y^2 is small relative to σ_X^2, the right-hand side of (20) is positive, showing that an improvement in hedging opportunities does, indeed, lead to higher correlation between investors' marginal utilities.[22]

[21] If $\tilde{Y}$ were the major component generating covariation in the two portfolios, its deemphasis could cause the correlation to drop. Exact conditions for this reversal can be read from (20).

[22] When $\rho_s = 1$, ρ_{C^i,C^k} is identically 1. This agrees with Breeden [1], Theorem 2.

III. Summary and Evaluative Discussion

This paper describes the production decisions of firms in a model with stochastic investment opportunities and incomplete markets. The model extends Merton's intertemporal capital asset pricing model with multiple consumers to include a description of the supply of traded securities.

In the model, firms maximize the welfare of their stockholders based on the sum of dollar values placed on projects by shareholders of the firm. Since undertaking a new project may change the available contract structure, there will, in general, be disagreements between shareholders about project values. This paper shows that investments are evaluated by investors in terms of profitability at market prices and the effect that the project has on the contract structure. The value of a change in the contract structure was found to be proportional to the covariance of the project's return with changes in the state variable $S(t)$ after subtracting from it the returns of the highest correlated hedging portfolio made up of existing securities. Conditions for improvements in the contract structure based on this measure, to lead to higher correlations between the marginal utilities of investors with access to the new contracts, were given.

Necessary and sufficient conditions for market value maximization to coincide with the firm rule are given in Equation (15). When markets are perfect or the project does not alter the set of available contracts for risk-spreading, the component of the firm's rule which values changes in the contract structure goes to zero, and the firm acts as a pure market-value maximizer. Conditions for this to occur are equivalent to the conditions for investor unanimity about the value of firm alternatives.

The welfare properties of the market equilibrium were discussed in detail by comparing the set of market equilibria to the outcomes achieved by a suitably constrained central planner. Because markets are incomplete and spanning does not hold, the market equilibria achieved by firms and households in this model need not be Pareto Optimal or Constrained Pareto Optimal, in general, except in special cases. It was shown that the set of market equilibria are equivalent to the set of Nash-Constrained Pareto Optima. Nash-Constrained Pareto Optima are the set of allocations achieved by a central planner who, in addition to the distribution constraints imposed by share ownership, treats production plans as given when deciding on consumption and ownership assignments and treats consumption and ownership as given when choosing production.

To place these findings in the context of what is already known about market structure with incomplete markets, refer to the papers by Nielsen [17] (see also Leland [11]) and Litzenberger and Sosin [12]. Nielsen examines the conditions for investor unanimity and constrained pareto optimality of value maximizing investment decisions by firms. He shows that the condition for each are the same, and that both are equivalent to the conservation of security values with respect to mergers and financing decisions.[23] Further, he shows that if the marginal investment decisions of each firm do not alter the state space spanned by existing securities then value maximizing investment decisions by firms are

[23] Nielsen [17, p. 596].

Constrained Pareto Optimal.[24] Investigating recapitalization decisions specifically, Litzenberger and Sosin show that recapitalizations leave firms' total market values unchanged if the space spanned by marketed securities is unaltered by the recapitalization.

In the present model, we examined "recapitalizations" in the guise of inside assets which had no effect on total market returns, but offered new contracts. In the case where the inside assets do not alter the contract structure (are a linear combination of already-traded securities), Theorem 4 shows that no improvement in risk-spreading takes place and the equilibrium is unchanged, as is predicted by Litzenberger and Sosin's analysis.

The model also confirms the conclusions of Nielsen when the conditions of spanning hold. In that case the choice set available to the central planner is convex and the market and planner achieve a Constrained Pareto Optimum.[25] Since investment projects are linear combinations of the output of traded securities, market-value maximization is the firm rule (from (15)) and unanimity obtains, as predicted by Nielsen. By the same token, a merger or financing decision which does not change the asset structure would be evaluated by firms and households at market value since the various components would be linear combinations of existing securities. This paper adds to their analysis, however, by considering cases where spanning and investor unanimity do not hold.

In the absence of spanning, an investment or recapitalization which changes the contract structure will have a welfare impact due to the change in risk-spreading opportunities. As the analysis in this paper shows, market value is no longer a sufficient measure of the total welfare impact of the change, unanimity does not hold between shareholders, and the market achieves a Nash-Constrained Pareto Optimum rather than a Constrained Pareto Optimum. As Nielsen notes: "If the doctrine of conservation of value is not satisfied local unanimity will in general not obtain, and the significant aspect of a change in the j^{th} firm's decision variable will not be the marginal change in supply but the creation of a new allocation instrument."[26] The present paper has been primarily devoted to analyzing the welfare properties of the model when the creation of "new assets" is allowed, and in describing the measure of the marginal welfare effect of the new asset to the firm and investor.

[24] Nielsen [17, p. 589], Theorem 2.

[25] The unconstrained central planner chooses y, from the closed convex set Y, and $C^i(t)$ such that

$$\textstyle\sum_i s_j dK_j(t) - \sum_i C^i(t) = dK(t), \qquad \sum_j s_j K_j(t) \le K(t).$$

Aggregate production $\sum_j s_j dK_j$ (hence dK_j) comes from a convex choice set and dC^i is unrestricted with respect to individual dK_j. In addition to the constraints above, any choice of dK_j, $j = 1, \cdots, n$, for the constrained central planner (and hence also for the Nash-Constrained central planner) must come from a linear combination of the output of all firms $\{dK_1, \cdots, dK_n\}$ and individual ownership constraints imply dC^i must be a linear combination as well. Since the space spanned by $\{dK_1, \cdots, dK_n\}$ is a linear (convex) space, its intersection with the choice set of the unconstrained central planner results in a convex choice set for the constrained central planner. If the production decisions of firms for any choice of production lie in the subspace spanned by $\{dK_1, \cdots, dK_n\}$, then the private market achieves a Constrained Pareto Optimum. A sufficient condition on firms for production to be spanned by $\{dK_1, \cdots, dK_n\}$, for example, is the requirement used by Diamond [6] that firms exhibit constant stochastic returns to scale. Further discussion can be found in Radner [18].

[26] Nielsen [17, p. 596].

Appendix

Example of less-than-perfect correlation between a portfolio of marketed securities and changes in the state variables: Let Σ be the $n \times n$ variance-covariance matrix of returns on marketed securities, and let N be the $n \times 1$ vector of covariances between each of the returns on marketed securities and changes in the state variable $S(t)$. Let $\rho_{p,s}$ be the correlation between the returns to a portfolio of marketed securities with membership weights $w(n \times 1)$ and changes in $S(t)$. Then the maximization problem,

$$\begin{aligned} &\text{Choose } w \text{ to maximize } \rho_{p,s} \\ &\text{such that } w'\Sigma w = 1 \end{aligned} \tag{A1}$$

is given by:

$$\begin{aligned} w^* &= 1/\sqrt{N'\Sigma^{-1}N\Sigma^{-1}N} \\ \rho^*_{p,s} &= \sqrt{N'\Sigma^{-1}N}. \end{aligned}$$

The following example shows that, even with more marketed securities than state variables and with each security correlated with changes in $S(t)$, $\rho^*_{p,s}$ can be strictly less than 1. Consider the following process of the type described in Section I:

$$\begin{aligned} dK(t) &= f(K, S, t)\, dt + g(K, S, t)\, dw(t) \\ dS(t) &= h(S, t)\, dt + dw_s(t) \end{aligned}$$

where

$$f(K, s, t) = \begin{bmatrix} f_1(K_1, S, t) \\ f_2(K_2, S, t) \end{bmatrix},$$

$$g(K, S, t) = \begin{bmatrix} g_{11}(K_1, S, t) & g_{12}(K_1, S, t) \\ g_{21}(K_2, S, t) & g_{22}(K_2, S, t) \end{bmatrix}, \qquad \begin{aligned} g_{11} + g_{12} &= 1 \\ g_{21} + g_{22} &= 1 \end{aligned}$$

$$dw(t) = \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix}, \qquad \sigma^2_{w_1} = \sigma^2_{w_2} = \sigma^2_{w_s} = 1$$

$$\sigma_{w_1 w_2} = \sigma_{w_2 s} = 0, \quad \sigma_{w_1 s} = a, \quad 0 < a \leq 1.$$

By direct calculation,

$$\Sigma = \begin{bmatrix} g_{11}^2 + g_{12}^2 & g_{11}g_{21} + g_{12}g_{22} \\ g_{11}g_{21} + g_{12}g_{22} & g_{21}^2 + g_{22}^2 \end{bmatrix}, \qquad N = \begin{bmatrix} g_{11}a \\ g_{21}a \end{bmatrix}, \qquad \rho^*_{p,s} = \sqrt{N'\Sigma^{-1}N} = a.$$

Thus, even with two risky assets, $dK_1(t)$ and $dK_2(t)$, correlated with changes in a single state variable, the maximum correlation is between 0 and 1, depending on a.

Proof of Theorems 1 *and* 2: (1) By the definition of the Nash-type planning constraint, the solution to the central planner's problem $u^*(t) = (C^*(t), \theta^*(t), y^*(t))$ in (9) can be written as the two subproblems,

$$\max_{(C,\ \theta)} \textstyle\sum_{i=1}^{m} w^i U^i e^{-\rho t} + \mathscr{A}(t)^u J, \tag{A2}$$

subject to $y = y^*(t)$ and (6), and

$$\max_{y \in Y} \textstyle\sum_{i=1}^{m} w^i U^i e^{-\rho t} + \mathscr{A}(t)^u J, \tag{A3}$$

subject to $(C, \theta) = (C^*(t), \theta^*(t))$ and (1),

where it is understood that (2) and (5) are used in writing out $\mathscr{A}(t)^u J$. With the constraint qualification satisfied, the first-order conditions given in (10) are necessary and sufficient for an optimum. In problem (A2) the concavity assumptions on $U^i(C^i(t))$ insure a unique interior maximum for $C^*(t)$, and (6) and (10b) define the unique solution for $\theta^*(t)$ in (11). In problem (A3) the first-order conditions for a local maximum y^* also imply global maximality since the set of feasible points in Y satisfying the constraint in (1) are convex. Thus, if a solution to the central planning problem exists, it is characterized by the first-order conditions for a local maximum, (1), (6), and (10).

(2) Given a competitive equilibrium $\mathscr{E}$, set

$$w^i(t) = J^i_{W^i}(W^i(t), S(t), t)^{-1}, \tag{A4}$$

and replace the objective function of consumer i at time t by the scalar equivalent,

$$w^i(t) E_t \int_t^T U^i(C^i(s)) e^{-\rho s}\, ds. \tag{A5}$$

By the conditions of competitive equilibrium, consumer and producer choices $(\hat{C}(t), \hat{\theta}(t), \hat{y}(t))$ satisfy (1), (5), (6), (10), and the boundary conditions in (9), where the objective function of consumers at time t is replaced by the scalar multiple (A5), and prices $rP_1, \cdots, rP_n, r$ appear in place of the shadow values $\mu_1, \cdots, \mu_n, \delta$. $u^*(t) = (\hat{C}(t), \hat{\theta}(t), \hat{y}(t))$ thus achieves an optimum for the corresponding central planning problem with weights given by (A4). By the assumption of competitive equilibrium, the evolution in $S(t)$ and $W^i(t)$ is given by (2) and (5), and J^i, $i = 1, \cdots, m$, $J = \sum_i w^i J^i$ satisfy the conditions in (9). $[J_{WW}]$ is invertible since (10a) implies $J^i_{WW} = e^{-\rho t} U_{cc} C^i_W < 0$.

(3) Given a central planning optimum $u^*(t) = (C^*(t), \theta^*(t), y^*(t))$ satisfying the associated first-order conditions in (10), choose market prices $\mathscr{P} = \{P_1(t), \cdots, P_n(t), r(t)\}$ by $\{\mu_1(t)/\delta(t), \cdots, \mu_n(t)/\delta(t), \delta(t)\}$ where $\mu_j(t)$ is the j^{th} shadow price for the constraint on the allocation of ownership shares and $\delta(t)$ is the shadow price for the constraint on capital. Then by the assumptions for a central planning solution in (9), J^i, $i = 1, \cdots, m$, satisfy the requirements for the optimal value functions; $S(t)$ and $W^i(t)$ are described by (2) and (5) from the definition of the set of admissible solutions for $u^*(t)$ and

(i) $y^*(t)$ maximizes (7), $j = 1, \cdots, n$; $t \le T$ since Y_j is convex and $y^*(t)$ satisfies (10c).

(ii) $(\theta_1^{i*}(t), \cdots, \theta_n^{i*}(t), C^{i*}(t))$ maximizes (3), subject to (4) and (5) for $i = 1, \cdots, m$; $t \le T$ for prices $\mathscr{P}$ since $(\theta^*(t), C^*(t))$ satisfy (10a) and (10b).

(iii) $\sum_i \theta_j^{i*}(t) = s_j$, $\quad j = 1, \cdots, n; t \le T$
$\sum_j s_j K_j^*(t) = K(t)$, $\quad s_j, K_j(t) \ge 0$, $\quad j = 1, \cdots, n; t \le T$
since $(\theta^*(t), y^*(t))$ satisfy constraints (1) and (6).

Proof of Theorem 4: *Sufficiency*: The difference in ρ_s^2, when I is available and when it is not, is given by $r_I^2(1 - \rho_s^2)$ where ρ_s is the correlation when I is not present and r_I is the correlation between $\sigma_I dw_I$ and the "residual" from the least squares projection of $S(t)$ on the returns of traded assets,

$$e = \sigma_s dw_s - \Psi' \begin{bmatrix} \sigma_1 dw_1 \\ \vdots \\ \sigma_n dw_n \end{bmatrix}.$$

(See, e.g., Theil [19, p. 175].) Since ρ_s^2 is less than one by the assumption that it increases upon the introduction of I, $r_I^2(1 - \rho_s^2)$ is positive if and only if $r_I^2 > 0$. This implies that the correlation between e and $\sigma_I dw_I$ is nonzero. Without loss of generality, assume it is positive. Define P_I as the value of the pricing formula (12) applied to asset I, and let ε denote the number of units of asset I assigned to individual i, where $\varepsilon = 0$ is the original equilibrium position. Totally differentiating (9) with respect to ε and solving as in (11)–(13), the lifetime welfare of individual i rises or falls as,

$$\frac{dP_I}{d\varepsilon} + \beta^i \eta^i \frac{d(N_I - \Psi' \sum_I)}{d\varepsilon} - p \gtreqless 0,$$

where the change in covariance between e and the maximally correlated portfolio formed from securities and I,

$$\frac{d(N_I - \Psi' \sum_I)}{d\varepsilon},$$

is positive by assumption and p is the purchase price per unit of I. Let m_1 be the number of investors for which $\eta^i > 0$, $m_2(\geq 1)$ the number for which $\eta^i = 0$, and m_3 the number for which $\eta^i < 0$. Then there exist $\delta > 0$, $\bar{\varepsilon} > 0$, and nonzero assignments,

$$\varepsilon = \begin{cases} -\bar{\varepsilon}/2m_1, & \eta^i > 0 \\ 0, & \eta^i = 0 \\ \bar{\varepsilon}/2m_2, & \eta^i < 0 \end{cases}$$

with payments $p = -\dfrac{dP_I}{d\varepsilon}\dfrac{\bar{\varepsilon}}{2m_1} + \delta$, $-\dfrac{m_1 + m_3}{m_2}\delta$, $\dfrac{dP_I}{d\varepsilon}\dfrac{\bar{\varepsilon}}{2m_3} + \delta$, respectively, such that

$$\frac{dP_I}{d\varepsilon}\varepsilon + \beta^i \eta^i \frac{d(N_I - \Psi' \sum_I)}{d\varepsilon}\varepsilon - p$$

$$= \begin{cases} -\beta^i \eta^i \dfrac{d(N_I - \Psi' \sum_I)}{d\varepsilon}\dfrac{\varepsilon}{2m_1} - \delta > 0, & \eta^i > 0 \\[2ex] \dfrac{m_1 + m_3}{m_2}\delta > 0, & \eta^i = 0 \\[2ex] \beta^i \eta^i \dfrac{d(N_I - \Psi' \sum_I)}{d\varepsilon}\dfrac{\varepsilon}{2m_3} - \delta > 0 & \eta^i < 0. \end{cases}$$

(Recall that $\beta^i < 0$.) Since the assignments sum to zero by construction, each individual is better off. If there are no individuals for whom $\eta^i = 0$, then payments for that group and δ are replaced by zero and the same conclusions follow.

Necessity: If $\rho_s(t) = 1$, there is a portfolio consisting of traded assets not including I such that

$$\alpha' \begin{bmatrix} \sigma_1 dw_1 \\ \vdots \\ \sigma_n dw_n \\ \vdots \\ \sigma_I dw_I \end{bmatrix} = \sigma_s dw_s$$

for the $(n + 1)$ vector of weights $\alpha(t)$, $0 \le t \le T$. In that case $\Psi = \alpha$. Since the sum of individuals' holdings of asset I must sum to zero and the $(n + 1)$st component of Ψ is 0, asset I does not enter the optimal hedging portfolio.

Alternatively, if $\rho_s(t) < 1$ does not rise with the introduction of I, it implies that $r_I^2 = 0$ and I does not enter the optimal hedging portfolio. Since $d(N_I - \Psi' \sum_I)/d\varepsilon = 0$, the marginal change in welfare from holding a claim to I is the same for all individuals. No assignment of claims adding to zero can simultaneously improve everyone's welfare.

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