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Hedging Interest Rate Risk with Futures Portfolios under Term Structure Effects

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ABSTRACT

This study develops and tests a methodology for reducing interest rate risk in a fixed spot portfolio of assets and liabilities with default-free cash flows. A minimum variance hedge is constructed by adding a portfolio of financial futures to the spot portfolio. Theorems are given which establish necessary and sufficient conditions for the existence of unique and zero-variance hedges. The risk reduction characteristics of the methodology are demonstrated by an empirical analysis.

UNEXPECTED INTEREST RATE fluctuations may have significant impact on the market value of both bond portfolios and interest-sensitive institutions. The early literature on this subject includes the seminal works of Samuelson [27], Redington [26], and Wallas [29], all of whom consider the impact of unexpected interest rate changes on financial institutions. Briefly, their results indicate that interest rate risk is minimized when the "mean time" of the assets and liabilities are equal (assuming equal initial asset and liability values). Redington coined the term "immunization" to refer to the absence of interest rate risk.

The immunization solutions, in their simplest form, depend on the *duration* (mean time) of asset and liability cash flows. The concept of duration was apparently developed independently by Macaulay [22] and Hicks [18]. Macaulay, interested in the "effective" maturity of a bond, defined duration as

$$D = [\textstyle\sum_{t=1}^{n} t \cdot A_t/(1 + r)^t]/[\textstyle\sum_{t=1}^{n} A_t/(1 + r)^t] \tag{1}$$

where r is the yield, n is the number of payments, and A_t is the payment in period t. Hicks was more interested in the elasticity of capital and developed a formula similar to (1) to describe the proportional response of capital flows to a change in the discount factor $X = 1/(1 + r)$.

Fisher and Weil [13] develop and test a bond portfolio strategy based on duration. Under certain assumptions, strategies can be developed so that bond portfolio yields are greater than or equal to promised yields when the duration of the portfolio is equal to the investor's planning horizon. An extensive literature has evolved in this area, depending on specific assumptions about the shape of the yield curve, shifts in the yield curve, distributional assumptions, and instantaneous versus interval immunization. The work includes the investigations of Bierwag [2], Bierwag and Kaufman [3], Bierwag, Kaufman, and Toevs [4],

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Bierwag and Khang [5], Cox, Ingersoll, and Ross [9], Ingersoll, Skelton, and Weil [19], and Boyle [6]. See Hawawini [17] for an extensive bibliography.

Recent applications literature has focused on the use of interest rate futures to immunize firms' balance sheets and investors' portfolios. Ederington [11] develops a portfolio approach to hedging by minimizing the variance of a portfolio composed of a single debt instrument and a futures contract in T-bills or GNMA certificates. Frankle [14] has commented on the work of Ederington; and Frankle and Senchack [15] have developed models for anticipatory hedges (to minimize the risk of anticipated positions in the spot market).

A set of hedging models using duration concepts and futures instruments has been developed by Gay and Kolb [16], Kolb and Chiang [20], and Kolb, Gay, and Corgel [21]. These papers develop promising methodology which, in general, is largely untested. They assume that the spot position is fixed and develop solutions for the number of futures contracts required to reduce (eliminate, ideally) interest rate risk. The operational form of these models is deterministic and assumes that yield curve shifts are uniform and/or depend importantly on a single state variable such as a Treasury Bill rate or a futures rate.

The hedging approach to reduction of interest rate risk, using price sensitivity models such as postulated by Kolb and others, results in a solution such as "buy X T-bill contracts." This raises an obvious question. Which one? Indeed, shouldn't the hedge be constructed from a portfolio of futures contracts? Consistent with the notion of gap management in the financial institutions literature, the a priori expectation is that the value of cash inflows and outflows are most effectively hedged by futures contracts with underlying financial instruments promising cash flows with equivalent (overlapping) terms-to-maturity.

The purpose of this study is to develop and test a methodology for reducing interest rate risk in a fixed (spot) portfolio of assets and liabilities.[1] A minimum variance hedge is constructed by adding a portfolio of financial futures to the spot portfolio. Theorems are subsequently developed which establish necessary and sufficient conditions for the existence of unique and zero variance hedges.

This study is organized as follows. Section I develops the basic portfolio model, the general minimum variance solution, and existence theorems. Section II describes the data and develops the experimental design. Section III is a discussion of the results of the analysis, followed by a Summary in Section IV.

I. The Portfolio Formulation

Consider a spot position with market price $P_0 \neq 0$ and unknown price $\tilde{P}_1$ at the end of period one. Let F_0 be the price of a financial futures contract and $\tilde{F}_1$ the unknown price at the end of period one. Assume the settlement date is at least one period away. The holding period yield on a portfolio with an initial investment

[1] In an efficient market with risk-adverse investors, relevant risk will be priced so that less risk, per se, is not always desirable. However, if the spot portfolio is fixed (perhaps because of tax considerations, regulatory or trust restrictions, or management options) investors prefer hedged portfolios when expected future wealth is not diminished. In the context of firm value, it is not clear, ceteris paribus, that reducing risk is desirable. It may be justified pragmatically as a means of reducing bankruptcy, agency, and regulatory costs (see Posner [24] and Buser, Chen, and Kane [8]).

P_0 and a long position in a futures contract is

$$\tilde{r}_p = \frac{\tilde{P}_1 - P_0 + \tilde{F}_1 - F_0}{P_0} = \tilde{r}_s + \frac{F_0}{P_0}\tilde{r}_f \equiv \tilde{r}_s + \gamma\tilde{r}_f, \tag{2}$$

where transactions costs are ignored and it is assumed that sufficient collateral is available to establish the futures position with no additional capital requirements. The total yield is thus the holding period yield on the spot position (r_s) plus the ratio of the futures price to spot position ($\gamma \equiv F_0/P_0$) times the *price* yield on the futures contract ($\tilde{r}_f \equiv (\tilde{F}_1 - F_0)/F_0$). In general, for a portfolio of n cash and m futures positions, the portfolio yield equation can be written as

$$\tilde{r}_p = \textstyle\sum_{i=1}^{n} \alpha_i\tilde{r}_{si} + \sum_{j=1}^{m} \gamma_j\tilde{r}_{fj} = \underline{\alpha}'\underline{\tilde{r}} + \underline{\gamma}'\underline{\tilde{f}}, \tag{3}$$

where α_i are spot portfolio weights, $'$ denotes transpose, and underline denotes an array. For net present value positive, $\sum_{i=1}^{n} \alpha_i = 1$. For net present value negative, $\sum_{i=1}^{n} \alpha_i = -1$.[2] For a long (short) position, the futures weight γ_i corresponds to the ratio of futures price (−price), contract i, to the net present value of the spot position. Assuming that liquidity premiums are negligible and that futures contracts are priced like forward contracts, it follows that $E[\tilde{F}_1] = F_0$, so that $E[\underline{\tilde{f}}] = \underline{0}$.[3] This assumption is consistent with results shown in Table III. By partitioning holding period yields on cash positions into an expected component and unexpected (zero mean) component, $E[\underline{\tilde{r}}]$ and $\underline{\tilde{r}}_u$, respectively, Equation (3) can be rewritten as

$$\tilde{r}_p = \underline{\alpha}'E[\underline{\tilde{r}}] + \underline{\alpha}'\underline{\tilde{r}}_u + \underline{\gamma}'\underline{\tilde{f}} = E[\underline{\tilde{r}}_p] + \underline{\alpha}'\underline{\tilde{r}}_u + \underline{\gamma}'\underline{\tilde{f}}, \tag{4}$$

(since $E[\underline{\alpha}'\underline{\tilde{r}}_u] = 0 = E[\underline{\gamma}'\underline{\tilde{f}}]$). The unexpected holding period yield component, $\underline{\tilde{r}}_u$, is hereafter referred to as the yield, or unexpected yield, when its meaning is clear from the context.

A. The Minimum Variance Solution

Assuming risk aversion and *fixed* choice of $\underline{\alpha}$, the investor's problem is to choose $\underline{\gamma}$ so as to minimize risk, assumed to be measured by $\mathrm{Var}(\tilde{r}_p)$.[4,5] Mathematically, the structure of the problem is

$$\begin{aligned}\min_{\underline{\gamma}} \mathrm{Var}(\tilde{r}_p) &= \min_{\underline{\gamma}} \mathrm{Var}(\underline{\alpha}'\underline{\tilde{r}}_u + \underline{\gamma}'\underline{\tilde{f}}) \\ &= \min_{\underline{\gamma}}\{\underline{\alpha}'\Omega_U\underline{\alpha} + \underline{\gamma}'\Omega_F\underline{\gamma} + 2\underline{\alpha}'\Omega_{UF}\underline{\gamma}\}\end{aligned}$$

[2] The array $\underline{\alpha}$ includes short positions (liabilities) when some $\alpha_i < 0$.

[3] In the context of their market efficiency studies, Rendleman and Carabini [25] and Elton, Gruber, and Rentzler [12] find that the mark-to-market effect of futures contracts does not significantly differentiate their results from those obtained via a forward pricing formulation.

[4] A sufficient condition for using $\mathrm{Var}(\tilde{r}_p)$ as a risk-proxy is normality of $\tilde{r}_p$.

[5] $\mathrm{Var}(\tilde{r}_p) = \mathrm{Var}(\tilde{r}_p - E(\tilde{r}_p))$ since $E[\tilde{r}_p]$ is a parameter. Thus, the population variance of yield equals that of the unexpected component of yield. However, $E[\tilde{r}_p]$ changes over observations whereas $\tilde{r}_p - E[\tilde{r}_p]$ has zero mean. Therefore estimates obtained from sample data are unbiased estimates of the variance of the unexpected component of yield but are inappropriate as estimates of $\mathrm{Var}(\tilde{r}_p)$ due to the changing mean of $\tilde{r}_p$.

where

Ω_U is the $n \times n$ covariance matrix of unexpected spot holding period yields,
Ω_F is the $m \times m$ covariance matrix of futures price yields, and
Ω_{UF} is the $n \times m$ matrix of covariances between spot and futures yields.

The first-order conditions for an optimal solution are

$$2\underline{\gamma}'\Omega_F + 2\underline{\alpha}'\Omega_{UF} = \underline{0}. \tag{5}$$

For nonsingular Ω_F, Equation (5) is necessary and sufficient for a minimum variance solution since the covariance matrix Ω_F is positive definite. The unique solution in this case is

$$\underline{\gamma}' = -\underline{\alpha}'\Omega_{UF}\Omega_F^{-1}. \tag{6}$$

Using (6) in the expression for $\mathrm{Var}(\tilde{r}_p)$ shows the variance of portfolio yield to be

$$\mathrm{Var}(\tilde{r}_p) = \underline{\alpha}'\Omega_U\underline{\alpha} - \underline{\gamma}'\Omega_F\underline{\gamma} = \underline{\alpha}'(\Omega_U - \Omega_{UF}\Omega_F^{-1}\Omega_{UF}')\underline{\alpha}. \tag{7}$$

The first expression in (7) reveals that the variance of the *optimally chosen* portfolio is the variance of the spot portfolio yield less the variance of the futures portfolio yield.

Solutions of this nature have been developed by a number of investigators, including, for example, Anderson and Danthine [1] in a production planning context.[6] However, Equations (6) and (7) are of little utility unless they are tied to a more fundamental valuation model. For example, Equation (6) cannot be used to minimize the variance of a firm's unexpected economic yield because the covariance matrix of spot yields and futures price yields is not generally available.

B. Using the Term Structure

To operationalize Equations (6) and (7), spot interest rates are posited as state variables so that both cash positions and futures prices are tied to a common set of variables. All promised cash flows are assumed to occur on a subset of periods one through $T + 1$ ($k = 1, 2, \cdots, T + 1$). For futures contract j, the settlement date is s_j, $s_j \geq 1$ (hereafter denoted by s to simplify notation). Cash flows from underlying futures instruments are necessarily zero for $k = 1, 2, \cdots, s$.

For the spot portfolio, the discounted cash flow valuation model for long position i with cash flows in the next $T + 1$ periods is

$$P_{0i} = \textstyle\sum_{k=1}^{T+1} [A_{ki}/(1 + R_{k0})^k], \tag{8}$$

where A_{ki} is the default-free cash flow from position i in period k and $\tilde{R}_{kj}$ is the k-period spot rate existing at period j. The period one value of the position is

$$\tilde{P}_{1i} = \textstyle\sum_{k=0}^{T} A_{k+1,i}/(1 + \tilde{R}_{k1})^k. \tag{9}$$

[6] The model $\tilde{r}_p = E[\tilde{r}_p] + \underline{\alpha}'\tilde{r}_u + \underline{\gamma}'\tilde{\underline{f}}$ can be rewritten as the regression model

$$\underline{\alpha}'\tilde{\underline{r}}_u = -\underline{\gamma}'\tilde{\underline{f}} + \tilde{\eta}$$

where $\tilde{\eta} = \tilde{r}_p - E[\tilde{r}_p]$, and $E[\tilde{\eta}] = 0$. The Ordinary Least Squares solution is identical to (6) when $\tilde{\underline{f}}$ is assumed to be nonstochastic and sample moments replace parameters.

Thus, let $\tilde{P}_{1i} = \tilde{P}_{1i}(\tilde{R}_{11}, \tilde{R}_{21}, \cdots, \tilde{R}_{T1})$ and let $\bar{R}_{k1}$ be the "expected" k-period rate in period one, $k = 1, 2, \cdots, T$. The unexpected holding period yield component due to a change in spot rates is

$$\tilde{r}_{ui} = (\tilde{P}_{1i} - \bar{P}_{1i})/P_{0i} \cong P_{0i}^{-1} \textstyle\sum_{k=1}^{T} \dfrac{\partial \tilde{P}_{1i}}{\partial \tilde{R}_{k1}} d\tilde{R}_{k1}, \tag{10}$$

where $\bar{P}_{1i}$ is the expected price and the right-hand side of (10) is a linearized approximation with partials evaluated at $\bar{R}_{k1}$ where $d\tilde{R}_{k1} = \tilde{R}_{k1} - \bar{R}_{k1}$. The Taylor series approximation in (10) is valid so long as the $\tilde{R}_{k1}$ are functions of other common state variables, e.g., a long and short rate (see Brennan and Schwartz [7]).[7] Taking partials of Equation (9) gives

$$\tilde{r}_{ui} \cong P_{01}^{-1} \textstyle\sum_{k=1}^{T} \dfrac{(-k \cdot A_{k+1,i})}{(1 + \bar{R}_{k1})^{k+1}} d\tilde{R}_{k1}, \tag{11}$$

which reduces to a Macaulay duration result at period one if it is assumed that $\bar{R}_{k1}$ and $d\tilde{R}_{k1}$ are constant for all k. To facilitate later derivations define

$$a_{ki} = -P_{0i}^{-1} \cdot k \cdot A_{k+1,i}/(1 + \bar{R}_{i1})^{k+1}, \qquad k = 1, 2, \cdots, T \tag{12}$$

so

$$\tilde{r}_{ui} = \textstyle\sum_{k=1}^{T} a_{ki}\, d\tilde{R}_{k1} = \underline{a}_i'\ \underline{d\tilde{R}}, \tag{13}$$

where $\underline{d\tilde{R}}$ is an array of the unexpected component of spot rates in period one.

The sensitivity of futures holding period yields are analyzed using the forward pricing formulation of futures prices. Specifically, let the futures price of contract j on an underlying instrument promising default-free cash flows C_{kj}, $j = 1, 2, \cdots, T + 1$, be given by

$$F_{0j} = \textstyle\sum_{k=1}^{T+1} C_{kj}(1 + R_{s0})^{s}/(1 + R_{k0})^{k}, \tag{14}$$

where $C_{kj} = 0$ for $k = 1, 2, \cdots, s$, since contract j has settlement date at $k = s$. This formulation can be explained via an arbitrage argument in a risk-neutral context [25]; or equivalently, using the forward pricing model it can be seen that the *present value* of a cash flow C_{s+k} at $s + k$ is $C_{s+k}/(1 + R_{s+k,0})^{s+k}$ and the time s (future) value is $[C_{s+k}/(1 + R_{s+k,0})^{s+k}] \cdot [1 + R_{s0}]^{s}$.

The value of the futures contract j in period one is

$$\tilde{F}_{1j} = \textstyle\sum_{k=s}^{T} C_{k+1,j}(1 + \tilde{R}_{s-1,1})^{s-1}/(1 + \tilde{R}_{k1})^{k}. \tag{15}$$

Proceeding as in (10), the futures one-period price yield is

$$\tilde{r}_{fj} = (\tilde{F}_{1j} - F_{0j})/F_{0j} \cong F_{0j}^{-1} \textstyle\sum_{k=1}^{T} (\partial \tilde{F}_{1j}/\partial \tilde{R}_{k1})\, d\tilde{R}_{k1}, \tag{16}$$

where the partials are evaluated at expected spot rates $E[\tilde{R}_{k1}] \equiv \bar{R}_{k1}$ and the unexpected component of spot rate changes is $d\tilde{R}_{k1} = \tilde{R}_{k1} - \bar{R}_{k1}$. Taking partials in (16) gives

[7] The Itô calculus could be used here to quantify local changes in P_{0i}, assuming spot rates are Itô random variables. Under this formulation the unexpected or random part of dP_{0i} would be captured by terms of the form $\sum_k^T (\partial P_{0i}/\partial R_{k0})\, d\tilde{R}_{k0}$. Thus, the Taylor series expansion in (10) in discrete time is consistent with the continuous time Itô expansion in the random component (see [9], Equations (6), (7), and (8b)).

$$\tilde{r}_{fj} \cong \sum_{k=1}^{T} c_{kj}\, d\tilde{R}_{k1} = \underset{\sim}{c}_j'\, \underline{d\tilde{R}}, \tag{17}$$

where the elements of the $\underset{\sim}{c}_j'$ array are

$$c_{ij} \begin{cases} = 0, & i = 1, 2, \cdots, s-2 \\ = F_{0j}^{-1}(s-1) \sum_{k=1}^{T} [C_{k+1,j}(1+\bar{R}_{s-1,1})^{s-2}/(1+\bar{R}_{k1})^k], & i = s-1 \\ = -F_{0j}^{-1} \cdot i \cdot C_{i+1,j}(1+\bar{R}_{s-1,1})^{s-1}/(1+\bar{R}_{i1})^{i+1}, & 1 = s, \cdots, T. \end{cases} \tag{18}$$

The unexpected yield of spot position i and the one-period yield of futures price j are approximately given by Equations (13) and (17), respectively. A matrix formulation for the *portfolio* of spot and future positions can be defined as follows:

$$A = [\underset{\sim}{a}_1 \underset{\sim}{a}_2 \cdots \underset{\sim}{a}_n]', \qquad C = [\underset{\sim}{c}_1 \underset{\sim}{c}_2 \cdots \underset{\sim}{c}_n]',$$

$$\text{and} \quad \underline{d\tilde{R}} = [d\tilde{R}_{11} d\tilde{R}_{21} \cdots d\tilde{R}_{T1}]', \tag{19}$$

where T is the period one term-to-realization of the most distant cash flow. From (4), (13), and (19), the unexpected one-period yield component of the spot portfolio can be written as

$$\underset{\sim}{\alpha}' \underset{\sim}{\tilde{r}}_u \cong \underset{\sim}{\alpha}' A\, \underline{d\tilde{R}}. \tag{20}$$

Similarly, the one-period price yield of the futures portfolio can be written as

$$\underset{-}{\gamma}' \underset{-}{\tilde{f}} \cong \underset{-}{\gamma}' C\, \underline{d\tilde{R}}, \tag{21}$$

so that now both yields are related via partials to common state variables $\underline{d\tilde{R}}$—unexpected spot interest rate changes. The variance of the unexpected component of spot portfolio yield is

$$\operatorname{Var}[\underset{\sim}{\alpha}' \underset{\sim}{\tilde{r}}_u] = \underset{\sim}{\alpha}' \Omega_U \underset{\sim}{\alpha} = E[\underset{\sim}{\alpha}' \underset{\sim}{\tilde{r}}_u (\underset{\sim}{\alpha}' \underset{\sim}{\tilde{r}}_u)'] \cong \underset{\sim}{\alpha}' [A \Omega_R A'] \underset{\sim}{\alpha}, \tag{22}$$

where Ω_U is the covariance matrix of unexpected holding period yields (see Section I*A*) and Ω_R is the covariance matrix of unexpected changes in spot interest rates. Similarly, the variance of the portfolio of futures price yields is

$$\operatorname{Var}[\underset{-}{\gamma}' \underset{-}{f}] \cong \underset{-}{\gamma}' \Omega_F \underset{-}{\gamma} = \underset{-}{\gamma}' [C \Omega_R C'] \underset{-}{\gamma}. \tag{23}$$

while the spot-futures yield covariance matrix is

$$\operatorname{Cov}[\underset{\sim}{\alpha}' \underset{\sim}{\tilde{r}}_u, \underset{-}{\gamma}' \underset{-}{\tilde{f}}] = \underset{\sim}{\alpha}' \Omega_{UF} \underset{-}{\gamma} \cong \underset{\sim}{\alpha}' [A \Omega_R C'] \underset{-}{\gamma}. \tag{24}$$

The solution which minimizes portfolio variance can be cast in terms of state variable parameters and constants by using (23) and (24) in (6). The result is

$$\underset{-}{\gamma}' = -\underset{\sim}{\alpha}' A \Omega_R C' [C \Omega_R C']^{-1}. \tag{25}$$

Thus, an optimal hedge is formed by selecting futures portfolio positions defined by the weights $\underset{-}{\gamma}$ given in Equation (25). The solution requires knowledge of spot interest rates and cash flows (elements of A, C) and the covariance matrix of unexpected spot interest rate change, Ω_R. The variance of yield in the optimal portfolio is (from (7) and (25))

$$\operatorname{Var}(\tilde{r}_P) = \underset{\sim}{\alpha}' A[\Omega_R - \Omega_R C' [C \Omega_R C']^{-1} C \Omega_R] A' \underset{\sim}{\alpha}. \tag{26}$$

C. Algebraic Properties

The algebraic properties of the matrices in Equation (26) determine the existence or nonexistence of a unique optimal hedge and if such a hedge can be a zero-variance hedge. Specific statements in the form of theorems and corollaries are developed in the following paragraphs. Proofs use standard results in matrix algebra (see [10] or [28], for example).

Consider spot portfolios with default-free cash flows defined on one or more of periods one through $T + 1$. Suppose there are m futures contacts defining the rows of C and let Ω_R be a $T \times T$, nonnull matrix of rank K, i.e., $\rho(\Omega_R) = K$, $1 \leq K \leq T$. Without loss of generality, relabel the array of unexpected spot rate changes into K and $T - K$ element subarrays $[\underline{d\tilde{R}}_u' \ \underline{d\tilde{R}}_v']$ so that Ω_R can be partitioned as

$$\Omega_R = \begin{bmatrix} U & Q \\ Q' & V \end{bmatrix},$$

where the $K \times K$ submatrix U is the covariance of elements of $\underline{d\tilde{R}}_u$ and is of rank K. Since $\rho(U) = \rho(\Omega_R)$, the submatrix $[Q' \ V]$ is linearly dependent on $[U \ Q]$ implying the existence of a matrix B such that $[Q' \ V] = B[U \ Q]$. Partition the (relabeled) C matrix as $C = [S \ P]$ where S is an $m \times K$ submatrix and P is $m \times (T - K)$. Equations (20) and (21) are assumed to hold as equalities. Then, for fixed C, the condition for the existence of a unique minimum variance hedge is given by Theorem 1.

THEOREM 1. *A necessary and sufficient condition for the existence of a unique minimum variance hedge for every spot portfolio (defined by $\underset{\sim}{a}'A$) is $\rho(S + PB) = m$.*

Proof: A unique solution exists for every spot portfolio if and only if the inverse of $C\Omega_R C'$ exists (see Equation (25)), or equivalently, if and only if $\rho(C\Omega_R C') = m$ since $C\Omega_R C'$ is $m \times m$. Multiplying out the partitioned matrices gives

$$C\Omega_R C' = [S + PB]U[S + PB]'.$$

Since U is symmetric and positive definite, there exists a nonsingular matrix Γ such that $U = \Gamma\Gamma'$, allowing $C\Omega_R C'$ to be written in the form XX', where $X = [S + PB]\Gamma$. But $\rho(XX') = \rho(X)$ so that $\rho(C\Omega C') = \rho([S + PB]\Gamma)$ and $\rho([S + PB]\Gamma) = \rho(S + PB)$, since Γ is nonsingular and multiplication by a nonsingular matrix leaves the rank of a matrix product unchanged. Therefore, $\rho(C\Omega_R C') = \rho(S + PB)$ and $\rho(C\Omega_R C') = m$ if and only if $\rho(S + PB) = m$.

The matrix B corresponds to a linear transformation generating the $\underline{d\tilde{R}}_v$ array from the $\underline{d\tilde{R}}_u$ array. If the second of two linearly independent spot rate changes equals the third rate change, the transformation matrix is $B = [0 \ 1]$ so that $d\tilde{R}_3 = B[d\tilde{R}_1 \ d\tilde{R}_2]' = B\underline{d\tilde{R}}_u$.[8] This theorem gives conditions both necessary and sufficient for the existence of a unique optimal hedge. The conditions required depend on several factors, including the number of futures instruments in the

[8] Random variables $\tilde{X}_i, \cdots, \tilde{X}_n$ are said to be linearly dependent if

$$\textstyle\sum_{i=1}^{n} A_i\tilde{X}_i = 0 \text{ implies } A_i = 0,\ i = 1, 2, \cdots, n\ [10]$$

hedge, the cash flow patterns of instruments underlying the futures contracts, and the extent to which the term of these cash flows corresponds to linearly independent spot rate changes.

COROLLARY 1.1. $\rho(S + PB) \leq K$.

Proof: $S + PB$ is an $m \times K$ matrix and rank cannot exceed the minimum of the number of rows or columns. If, for example, all rates are driven by a long and short rate ($K = 2$), and there are $m > 2$ futures contracts defining the rows of C, there will be redundant contracts and the hedge will not be unique.

COROLLARY 1.2. *Multiplication of $\underline{d\tilde{R}}$ by the nonzero scalar a leaves the minimum variance solution unchanged.*

Proof: Let $\underline{d\tilde{R}}^* = a\underline{d\tilde{R}}$. Then $\Omega_{R^*} = a^2\Omega_R$. From Equation (25) $\gamma'^* = -\underset{\sim}{\alpha}' Aa^2\Omega_R C'[Ca^2\Omega_R C']^{-1} = -\underset{\sim}{\alpha}' A\Omega_R C'[C\Omega_R C']^{-1} = \gamma'$.

COROLLARY 1.3. *Multiplication of $d\tilde{R}$ by the nonzero scalar a changes the variance of the unique optimal hedge to $a^2\mathrm{Var}(\tilde{r}_p)$.*

Proof:

$$\mathrm{Var}(\tilde{r}_p) = \underset{\sim}{\alpha}' A[a^2\Omega_R - a^2\Omega_R C'[Ca^2\Omega_R C']^{-1} Ca^2\Omega_R]A'\underset{\sim}{\alpha} = a^2\mathrm{Var}(\tilde{r}_p).$$

The magnitude of the elements of the covariance matrix Ω_R may change over time. Corollaries 1.2 and 1.3 show that simple magnitude changes (generated by a scalar) will not change the composition of the optimal hedge but will change the variance of the hedge.

Theorem 1 stated the condition for the existence of unique hedges. An additional condition for such hedges to be of zero variance is given in Theorem 2.

THEOREM 2. *Necessary and sufficient conditions for a unique, zero-variance hedge for every spot portfolio are $\rho(S + PB) = m$ and $K = m$.*

Proof of Necessity: A unique minimum variance hedge exists for every $\underset{\sim}{\alpha}'A$ implies $\rho(C\Omega_R C') = m$ and $\rho(S + PB) = m$ (Theorem 1). To show $K = m$, note $K \equiv \rho(\Omega_R) \geq m$ since the rank of matrix products is less than or equal to the rank of any factor. Also, from Equation (26), $\mathrm{Var}(\tilde{r}_p) = 0$ for every spot portfolio implies $\Omega_R = \Omega_R C'[C\Omega_R C']^{-1} C\Omega_R$, and $C\Omega_R C'$ symmetric and positive definite guarantees existence of a nonsingular matrix X such that $(C\Omega_R C')^{-1} = XX'$. Thus,

$$K \equiv \rho(\Omega_R) = \rho(\Omega_R C'XX'C\Omega_R) = \rho(\Omega_R C'X) = \rho(\Omega_R C') \leq \rho(C) \leq m$$

using the product rule and nonsingularity of X to establish rank. $K \leq m$ and $K \geq m$ gives the result $K = m$.

Proof of Sufficiency: If $k = m$ and $\rho(S + PB) = K$, then $C\Omega_R C' = [S + PB]U[S + PB]'$ is a $K \times K$ matrix of rank K so the inverse exists and is given by $[C\Omega_R C']^{-1} = [S' + B'P']^{-1}U^{-1}[S + PB]^{-1}$ since all component inverses exist. Writing $C\Omega_R = [S'U + PQ' \; SQ + PV']$ and using $[Q' \; V] = B[U \; Q]$ gives

$$\Omega_R C'[C\Omega_R C']^{-1}C\Omega_R$$

$$= \begin{bmatrix} U \\ Q' \end{bmatrix}[S' + B'P'][S' + B'P']^{-1}U^{-1}[S + PB]^{-1}[S + PB][U\ Q]$$

$$= \begin{bmatrix} U & Q \\ Q' & Q'U^{-1}Q \end{bmatrix} = \begin{bmatrix} U & Q \\ Q' & V \end{bmatrix} \equiv \Omega_R,$$

since $[Q'\ V] = B[U\ Q]$ implies $Q'U^{-1}Q = V$. Therefore, $\Omega_R = \Omega_R C'[C\Omega_R C']^{-1}C\Omega_R$ and, by (26), $\text{Var}(\tilde{R}_p) = 0$ for the hedge of every spot portfolio defined by $\underset{\sim}{\alpha}'A$.

When a unique hedge exists for every spot portfolio, the variance of that hedge will be zero when the number of distinct financial futures contracts (m) equals the rank of the covariance matrix Ω_R. Furthermore, if there are K linearly independent spot rates, the rank of the covariance matrix will be K. Thus, a correspondence between the number of linearly independent spot rates and the number of distinct futures contracts required for a zero-variance hedge is established.

The theorem could be expanded using the concept of generalized inverses. For example, for K linearly independent spot rates, a set of $K + j$, $j \geq 1$, financial futures could generate a zero-variance solution but the hedged portfolio would not be unique. In this case, the set of optimal solutions is characterized by an expression containing generalized inverses.

II. Application

The futures portfolio model can be applied to problems involving minimizing the market value fluctuations of firms as well as unexpected variation in bond portfolios. It appears to be uniquely suited to applications in financial institutions since a substantial part of their future cash flows are fixed and term structure effects can be incorporated via the methodology.

While operationally suited to analysis of flows in financial institutions, a test of the model can be most rigorously accomplished by holding constant all factors other than pure interest rate risk. This study investigates bond portfolios comprised of Treasury Bills with approximately 3, 6, 9, and 12 months to expiration. The futures contracts are 3-month Treasury Bill contracts on the International Monetary Market (IMM). The nearby, the 3–6-month, and the 6–9-month set of contacts are considered.

A. Data

Treasury Bill data were obtained from *The U.S. Treasury Bulletin* beginning in January 1970 and extending through October 1982. The bid discount rate was recorded for the last trading day of each month for bills with maturity from 1 to 12 months. The n-month rate was the rate for the bill expiring closest to the end of month n. Prices for the n-month Treasury Bill (normed to one dollar) were computed by $P_n = 1 - b(n/12)$, where b is the quoted banker's discount rate. Monthly spot rates were computed by $R_{n0} = (1/P_n)^{1/n} - 1$.

The 3-month Treasury Bill future is traded on the IMM, a division of the Chicago Merchantile Exchange. There are, at all times, eight contracts corresponding to four settlement dates for the next 2 years. The settlement dates are on the third business day following auction of 13-week bills in the third week of March, June, September, and December. Since 13-week bills are generally auctioned on Mondays, this means the futures settlement dates are typically on Thursdays. At any date, the next four contracts outstanding can be referred to as the nearby contract (nearest settlement date), the 3–6-month contract (2nd settlement date), the 6–9-month contract (3rd settlement date), and the 9–12-month contract (4th settlement date).

Futures data were obtained from *The Wall Street Journal* beginning on the last trading day in December 1977 and extending through October 1982. Prices were recorded at the close on the last trading day of each month so as to be essentially synchronous with Treasury Bill data. Monthly price yields for the nearby, the 3–6-month, and the 6–9-month contracts were computed as $f_1 = (F_1 - F_0)/F_0$, where F_0 and F_1 are normed prices of futures contracts computed as $F = 1 - (1 - F_Q)/4$, where F_Q is the *WSJ* quoted settlement price.

B. The Covariance Matrix

The initial computation involved estimation of the covariance matrix of unexpected spot rate changes for bills with maturities from 1 through 12 months. The 12×12 covariance matrix of changes was first estimated using the entire data set. Changes in spot rates from their expected value were computed via the pure expectations theory, i.e., forward rates = expected rates, and the random walk (RW) model of rate changes, i.e., unexpected change is measured by deviation from last month's rate. Under both models, the covariance matrix behaved as expected with correlation between n- and m-month bills declining with increases in $|n - m|$ and short-term bills having larger variances than longer term bills. However, the random walk estimates produced unexpected rate changes ($\underline{dR}$) with smaller variances (and covariances) than the forward rate model. Because of apparent greater efficiency, the RW model is the spot estimator used in this paper. (See Pesando [23] for a discussion of this model applied to interest rates.) Both the RW covariance and correlation matrices are given in the Appendix.

To test the model on out-of-sample data, a covariance matrix was also estimated from Treasury Bill data beginning in January 1970 and extending through December 1977. This matrix is used to test the effectiveness of the model on bill/futures portfolios formed following 1977.

C. Unexpected Yields

Experimental validation of the model required that unexpected one-period bond yields ($\underline{r}_u$) be computed. These were obtained by subtracting expected monthly yields from realized holding period yields. The expected one-period yield for any n-period bill was taken to be the one-period spot rate, R_{10}. The realized holding period yield was $(\tilde{P}_{n-1,1} - P_{n0})/P_{n0}$. The unexpected yield was computed as

$$(\tilde{P}_{n-1,1} - P_{n0})/P_{n0} - R_{10} = (\tilde{P}_{n-1,1} - (1 + R_{10})P_{n0})/P_{n0}.$$

D. Research Design

To avoid term structure complications inherent in long-term rates (coupon effects, tax effects, etc.) the model was evaluated in a short-term hedging environment. The variance of unexpected yields of a random and preselected set of Treasury Bill portfolios was hedged by adding a portfolio of futures contracts whose magnitudes were chosen according to the minimum variance formula given in Equation (25). The Treasury Bill portfolios consisted of bills with 3, 6, 9, and 12 months to expiration. The bill portfolio is uniquely defined by the $\underset{\sim}{\alpha}$ array. For example, $\underset{\sim}{\alpha}' = (\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4) = (0.25\ 0.25\ 0.25\ 0.25)$ corresponds to a portfolio with equal investments in each of the four bills. $\underset{\sim}{\alpha}' = (0\ 0\ 0\ 1)$ implies the entire investment is in the 12-month bill. The random portfolio is defined by $\alpha_i = X_i/\sum_{i=1}^{4} X_i$, where X_i is a random number in the (0, 1) interval.

In this study, the nearby, 3–6 month, and 6–9 month Treasury Bill futures are used to develop hedging portfolios. The composition of the futures portfolio is denoted by the array $\gamma' = (\gamma_1\ \gamma_2\ \gamma_3)$ where γ_j is the ratio of the settlement price of futures contracts j to total investment in the bond portfolio. For example, a 10 million dollar bond portfolio hedged by $\gamma' = (-1.2\ -0.8\ 0.0)$ implies a $(-1.2)(10) = -12$ million dollar (short) position in the nearby contract and a $(-0.8)(10) = -8$ million dollar (short) position in the 3–6 month contract.

To provide maximum uniformity, 1-month holding period yields were computed for portfolios formed on the last trading day of December, March, June, and September. The observations began in December 1977, and continued through September 1982, giving a total of 20 observations. At the beginning of each hedge observation, approximately the same amount of time remained before each of the settlement dates (on average, about 2.7, 5.7, and 8.7 months).[9]

The optimal hedge developed by the covariance methodology (hereafter "full information" hedge) was compared to alternate hedging portfolios constructed using the price sensitivity methodology. Specifically, alternate hedges were developed via price sensitivity using a) equal weights on futures contracts, b) random weights on futures contracts, and c) the second nearest futures contract (the 3–6-month contract).

E. Price Sensitivity Models

To develop the price sensitivity models, let W_B, the value of the Treasury Bill portfolio, be defined as

$$W_B = \textstyle\sum_{i=1}^{N} n_i P_i = B \sum_{i=1}^{N} n_i/(1 + R_{3i,0})^{3i}, \tag{27}$$

where $N = 4$, P_i is the price of bill i, B is face value, and $i = 1$ corresponds to 3-month bills, $i = 2$, corresponds to 6-month bills, etc. The portfolio contains n_i type i bills with $3i$ months to expiration. Portfolio duration is $D_A = \sum_{i=1}^{N} \alpha_i D_i$, where $\alpha_i = n_i P_i / W_B$ is the portfolio weight and $D_i = 3i$ is the duration of a type i bill.

[9] Months-to-settlement dates fall between the monthly integer trading dates. To determine the appropriate spot rate, polynomial interpolation and rounding schemes were tested. Since the rounding schemes produced smaller hedged portfolio variances, term-to-settlement rates were taken from rounded monthly settlement rates, e.g., the rate for $s = 2.7$ was taken to be the 3-month rate.

The futures contract price with s months to settlement, as given by the forward rate model, is

$$W_F = F\cdot(1 + R_{s0})^s/(1 + R_{s+k,0})^{s+k}, \tag{28}$$

where F is cash flow at the maturity of the underlying Treasury Bill, s is months to settlement, and k is months to cash flow after settlement. The duration of the instrument underlying the futures contract is $D_F = k$.

The instantaneous change in the value of the hedged portfolio with y futures contracts is

$$\begin{aligned} d\tilde{W} &= d\tilde{W}_B + ydW_F \\ &= -W_B \textstyle\sum_{i=1}^N \dfrac{\alpha_i D_i}{(1 + R_{3i,0})}\, d\tilde{R} + yW_F\left[\dfrac{s}{1 + R_{s0}} - \dfrac{(s + D_F)}{(1 + R_{s+k,0})}\right] d\tilde{R}, \end{aligned} \tag{29}$$

where all rates are assumed to change by a uniform amount $d\tilde{R}$. Setting $d\tilde{W} = 0$ gives the hedge ratio

$$yW_F/W_B \equiv \gamma_{PS} = -\textstyle\sum_{i=1}^N \dfrac{\alpha_i D_i}{(1 + R_{3i,0})} \Big/ \left[\dfrac{s + D_F}{(1 + R_{s+k,0})} - \dfrac{s}{(1 + R_{s0})}\right], \tag{30}$$

where γ_{PS} is the price sensitivity hedge ratio, and where, for the 3-month Treasury Bill contract, $k = 3$. Equation (30) is formally equivalent to the Kolb and Chiang (KC) hedge ratio (see [20]) where $1 + R_{s+k,0} = 1 + R_{s0} = \bar{R}_j$ and $1 + R_{3i,0} = \bar{R}_i$, for all i, where $\bar{R}_i$ and $\bar{R}_j$ are asset and futures returns in KC.

Random and equal portfolio weights were computed as $\gamma_i = (X_i/\sum_{i=1}^n X_i)\gamma_{PS}$ and $\gamma_i = \gamma_{PS}/3$, $i = 1, 2, 3$, where X_i is a random number in the (0, 1) interval. The 3–6-month hedge ratio is $\gamma_2 = \gamma_{PS}$ with $s = 6$ in Equation (30).

III. Results

A unique minimum variance hedge was found for each spot portfolio, consistent with Theorem 1 and $\rho(S + PB) = m = 3$ for the futures contracts considered here. It can also be verified that the covariance matrix Ω_R is of full rank ($K = 12$), so that a zero variance hedge does not exist for an arbitrarily selected spot portfolio (Theorem 2).

Portfolio composition results are given in Table I. For the random spot portfolio of Treasury Bills, the full information hedge requires a short position in each of the three futures contracts, i.e., $\gamma_i = -1.137$ (nearby), $\gamma_2 = -0.5852$ (3–6-month), and $\gamma_3 = -0.2522$ (6–9-month). The corresponding random, equal, and 3–6-month hedge weights are, respectively, $-(0.7655\ 0.9063\ 0.8939)$, -0.8553, and -2.565. The hedge portfolio compositions are, by and large, intuitive. For example, spot portfolios with longer maturities, e.g., portfolios 7, 8, 11, and 12, require larger hedge positions in absolute value terms. Note also that the net short position of the full information strategy is consistently smaller than that of the price sensitivity strategy, e.g., for portfolio one, $\sum_{i=1}^3 \gamma_i = -1.9744$ versus $\gamma_{PS} = -2.565$.

Portfolio risk is quantified in Table II. The measures computed are sample monthly variances of unexpected yields and probability intervals centered on expected yields (see footnote 5). For each of the twelve portfolios, the full

Table I

Portfolio Composition

	Spot Portfolio				Futures Portfolio[a]							
					Full Information Hedge			Random Weights[b]			Equal Weights[b]	3–6-Month[b,c]
	(α_1	α_2	α_3	α_4)	(γ_1	γ_2	γ_3)	(γ_1	γ_2	γ_3)	$\gamma_1 = \gamma_2 = \gamma_3$	γ_2
1	Random Weights[d]				−(1.137	0.5852	0.2522)	−(0.7655	0.9063	0.8939)	−0.8553	−2.565
2	0.25	0.25	0.25	0.25	−(1.101	0.6016	0.2139)	−(0.7382	0.8929	0.8692)	−0.8335	−2.500
3	0.50	0.50	0	0	−(0.8902	0.1385	−0.0190)	−(0.5131	0.5482	0.4391)	−0.5002	−1.500
4	0.50	0	0.50	0	−(0.8713	0.6099	−0.0187)	−(0.7256	0.6255	0.6492)	−0.6669	−2.000
5	0.50	0	0	0.50	−(0.8532	0.5952	0.4462)	−(1.0260	0.6313	0.8434)	−0.8335	−2.500
6	0	0.50	0.50	0	−(1.348	0.6080	−0.01846)	−(0.9512	0.7639	0.7853)	−0.8335	−2.500
7	0	0.50	0	0.50	−(1.330	0.5933	0.4465)	−(0.9614	0.9903	1.0480)	−1.0000	−3.000
8	0	0	0.50	0.50	−(1.311	1.065	0.4467)	−(1.3520	1.0530	1.0960)	−1.1670	−3.500
9	1	0	0	0	−(0.4133	0.1404	−0.01925)	−(0.3100	0.3462	0.3442)	−0.3335	−1.000
10	0	1	0	0	−(1.367	0.1365	−0.01872)	−(0.5738	0.7006	0.7259)	−0.6668	−2.000
11	0	0	1	0	−(1.329	1.079	−0.0182)	−(1.060	1.0360	0.9049)	−1.0000	−3.000
12	0	0	0	1	−(1.293	1.050	0.9117)	−(1.5170	1.2850	1.1980)	−1.3340	−4.000

[a] These are futures weights averaged over 20 observations. The variation in weights among observations is negligible.

[b] Random, equal, and 3–6-month weights (γ_2) were chosen using the price sensitivity formula.

[c] Spot portfolio hedged by 90-day Treasury Bill futures with settlement date 3 to 6 months away.

[d] Weights for the random spot portfolio were chosen as $\alpha_i = X_i/\sum^4 X_i$, where X_i is a random number on (0, 1).

Table II

Portfolio Risk[a]

	Sample Variance ($\times 10^{-5}$)					0.99 Probability Interval (Basis Points)[c]				
	Hedged Portfolios					Hedged Portfolios				
Spot Portfolio[b]	Full Information	Random Weights	Equal Weights	3–6-Month	Unhedged	Full Information	Random Weights	Equal Weights	3–6-Month	Unhedged
1[d]	0.6618	0.8834	0.9124	1.056	7.050	147.2	170.1	172.8	185.9	480.4
2	0.5675	0.8564	0.8044	0.9078	6.956	136.3	167.4	162.3	172.4	477.2
3	0.2264	0.5005	0.4813	0.5219	2.091	86.1	128.0	125.5	130.7	261.6
4	0.3879	0.6121	0.6516	0.6862	4.200	112.7	141.6	142.1	149.9	370.8
5	0.5730	0.9623	0.8750	0.9811	6.497	137.0	177.5	169.3	179.2	461.2
6	0.5819	0.7418	0.7696	0.8703	7.451	138.0	155.8	158.7	168.8	493.9
7	0.8204	1.099	1.063	1.207	10.45	163.9	189.7	186.6	198.8	585.0
8	1.224	1.493	1.406	1.601	14.87	200.2	221.1	214.5	228.9	697.9
9	0.1697	0.4908	0.4631	0.4848	0.6550	74.6	126.8	123.1	126.0	146.4
10	0.3615	0.5979	0.5848	0.6498	4.493	108.8	133.9	138.4	145.9	383.5
11	0.9222	1.020	1.056	1.201	11.230	173.8	182.8	186.0	198.3	606.4
12	1.633	2.338	1.883	2.136	19.10	231.3	276.7	248.3	285.1	790.9

[a] Entries are the monthly variance and probability interval for the "unexpected" component of holding period yields.

[b] Weights defining spot portfolios are given in Table I.

[c] The width of a 99 percent probability interval centered at the expected yield (interval = $2.861\sqrt{\mathrm{Var}(2)}$ (10^4 basis points)).

[d] The random spot portfolio.

information hedge produces superior results. For the random spot portfolio (#1), the sample variances range from 0.6618×10^{-5} for the full information hedge to 1.056×10^{-5} for the hedge with the 3–6 month Treasury Bill. For the same spot portfolio, the width of a 0.99 probability interval for 1-month yields is 147.2 basis points for the full information hedge and 170.1 basis points for the random weight portfolio, the next best hedging portfolio.

Table III displays sample means and variances of spot, futures, and hedged portfolios. In every case, the hedged portfolio sample mean exceeds that of the spot portfolio. There is no significant difference in the hedged portfolio yield, however, since the final column in Table III indicates that sample mean futures yield, although positive (for a short position), is not significantly different from zero. The t-statistics in the final column are interdependent since they are generated by the same set of futures contracts. They differ slightly because of different weights in portfolios one through twelve.

To simulate both asset and liability positions in the spot portfolio, outputs were obtained for both positive and negative weights. Table IV gives the futures portfolio composition for the full information and price sensitivity hedges. The sign and magnitude of weights in the futures portfolio are as expected. The response of longer term spot positions to interest rate risk is exemplified by portfolios three, four, and five. In these portfolios, short positions in longer term portfolios result in long positions in the 6–9-month futures contract. In fact, in each of these portfolios, the futures portfolio contains both long and short positions. This result, which is consistent with the popular notion of gap management in the financial institutions literature, demonstrates different hedging strategies for asset/liability gaps in the maturity dimension. Portfolios nine, ten, and eleven are hedging solutions for a pure liability spot portfolio and are symmetric with the pure asset spot portfolios displayed in Table I.

Variances corresponding to the hedged asset/liability portfolios are given in Table V. The superiority of full information hedging is even more pronounced here since the price sensitivity hedge cannot produce a hedge with both long and short positions. The sample variances produced by full information hedges in portfolios four and five (with long and short positions) are approximately one-half the size of sample variances produced by the pure price sensitivity model (3–6-month futures hedge).

IV. Summary

The analysis developed here extends the price sensitivity futures hedge into a multivariate model via Taylor Series approximations and standard portfolio theory. In every application tested, the model was superior to competing hedge models. It is likely that the degree of superiority will frequently be greater than shown here, however, due to the atypical nature of the test period. Specifically, the test period (1978–1982) was one of relatively greater volatility than the pre-test period (1970–1977) used in estimating the covariance matrix for the hedge. Variances estimated from the pre-test matrix and Equation (26) are much smaller than the ex post variance estimates given in the tables. For example, portfolio two, Table II, has variance 0.5675×10^{-5} while the variance computed using

Table III

Portfolio Means[a]

Portfolio	Means ($\times 10^{-2}$)			Unexpected Means ($\times 10^{-2}$)			Variances ($\times 10^{-4}$)[c]			t-statistic[d]
	Spot Portfolio	Futures Portfolio	Hedged Portfolio[b]	Spot Portfolio	Futures Portfolio	Hedged Portfolio	Spot Portfolio	Futures Portfolio	Hedged Portfolio	Futures Yield
1	0.9103	0.04178	0.9521	0.06710	0.04178	0.1089	0.7050	0.5697	0.06618	0.247
2	0.9304	0.02816	0.9585	0.08713	0.02816	0.1153	0.6956	0.5570	0.05675	0.169
3	0.9393	0.01502	0.9543	0.09607	0.01502	0.1111	0.2091	0.1686	0.02264	0.164
4	0.9429	0.02090	0.9638	0.09968	0.02090	0.1206	0.4200	0.3346	0.03879	0.162
5	0.9098	0.02775	0.9376	0.06658	0.02775	0.09433	0.6497	0.5192	0.05730	0.172
6	0.9509	0.02857	0.9795	0.1077	0.02857	0.1363	0.7451	0.5968	0.05819	0.177
7	0.9178	0.03542	0.9532	0.07459	0.03542	0.1100	1.045	0.8359	0.08204	0.173
8	0.9214	0.04131	0.9627	0.07820	0.04131	0.1195	1.487	1.172	0.1224	0.171
9	0.9313	0.007348	0.9386	0.08806	0.007348	0.09541	0.0655	0.04681	0.01697	0.152
10	0.9473	0.02269	0.9700	0.1041	0.02269	0.1268	0.4493	0.3658	0.03615	0.168
11	0.9545	0.03446	0.9890	0.1113	0.03446	0.1458	1.123	0.8848	0.9222	0.164
12	0.8883	0.04815	0.9365	0.4509	0.04815	0.09325	1.910	1.501	0.1633	0.176

[a] Monthly holding period yields, 20 observations.
[b] Portfolio hedged using the covariance methodology.
[c] Variance of the unexpected component of yield.
[d] Computed as t = mean futures yield/$\sqrt{\text{variance}/20}$.

Table IV

Portfolio Composition: Long and Short Spot Positions[a]

	Spot Portfolio				Futures Portfolio							
					Full Information Hedge			Random Weights			Equal Weights	3–6-Month
	$(\alpha_1$	α_2	α_3	$\alpha_4)$	$(\gamma_1$	γ_2	$\gamma_3)$	$(\gamma_1$	γ_2	$\gamma_3)$	$\gamma_1 = \gamma_2 = \gamma_3$	γ_2
1	0	−0.50	0.50	0.50	−(0.6276	0.9964	0.4561)	−(0.7381	0.8928	0.8691)	−0.8334	−2.50
2	0	0.50	−0.50	0.50	−(0.6655	0.05358	0.4556)	−(0.5131	0.5482	0.4391)	−0.5001	−1.50
3	0	0.50	0.50	−0.50	−(0.7017	0.08296	−0.4743)	−(0.1814	0.1564	0.1623)	−0.1667	−0.50
4	0	1.50	0	−0.50	−(1.404	−0.3202	−0.4839)	−(0.4103	0.2526	0.3374)	−0.3335	−1.00
5	0.75	0.75	−0.25	−0.25	−(0.6798	−0.3246	−0.2518)	−(0.1904	0.1529	0.1572)	−0.1668	−0.50
6	−0.25	−0.25	0.75	0.75	−(1.522	1.528	0.6796)	−(1.442	1.485	1.573)	−1.500	−4.50
7	1.50	0	0	−0.50	−(−0.0265	−0.3144	−0.4847)	(0.1929	0.1502	0.1564)	0.1665	0.50
8	−0.50	0	0	1.50	−(1.733	1.505	1.377)	−(1.705	1.903	1.893)	−1.834	−5.50
9	−0.25	−0.25	−0.25	−0.25	−(−1.101	−0.6016	−0.2139)	(0.7172	0.8758	0.9073)	0.8335	−2.50
10	−1	0	0	0	−(−0.4133	−0.1404	−0.01925)	(0.3533	0.3454	0.3018)	0.3335	1.00
11	0	0	0	−1	−(−1.293	−1.050	−0.9117)	(1.517	1.285	1.198)	1.334	4.00

[a] See notes for Table I.

Table V

Portfolio Risk: Long and Short Spot Positions[a]

Spot Portfolio	Sample Variance ($\times 10^{-5}$): Hedged Portfolios: Full Information	Random Weights	Equal Weights	3–6-Month	Unhedged	0.99 Probability Interval (Basis Points): Hedged Portfolios: Full Information	Random Weights	Equal Weights	3–6-Month	Unhedged
1	0.7314	0.7923	0.7468	0.8454	7.925	154.8	161.1	156.4	166.4	509.4
2	0.2053	0.3140	0.2969	0.3325	2.448	81.98	101.4	98.59	104.3	283.1
3	0.0837	0.1231	0.1234	0.1277	0.3833	52.35	63.48	63.55	64.66	112.0
4	0.2566	0.4737	0.4486	0.4657	1.227	91.65	124.5	121.2	123.5	200.4
5	0.2005	0.4422	0.4367	0.4431	0.2792	81.03	120.3	119.6	120.4	95.61
6	2.195	2.390	2.286	2.601	25.85	268.1	279.8	273.6	291.8	919.9
7	0.4235	0.6524	0.6476	0.6472	1.579	117.8	146.2	145.6	145.6	227.4
8	3.351	3.409	3.488	3.949	38.48	331.2	334.1	337.9	359.6	1122.0
9	0.5675	0.8078	0.8044	0.9078	6.956	136.3	162.6	162.3	172.4	477.2
10	0.1697	0.4629	0.4631	0.4848	0.6550	74.55	123.1	123.1	126.0	146.4
11	1.633	2.338	1.883	2.136	19.10	231.3	276.7	248.3	264.5	790.9

[a] See notes for table II.

Equation (26) and the pre-test matrix as Ω_R is 0.25×10^{-6}, approximately 23 times smaller than the ex post estimate.

The covariance estimates from the full (1970–1982) and pre-test data sets, reproduced as Tables A1 and A3, show the magnitude of pre-test and full period covariance estimates to be significantly different (full period estimates are two to four times larger than the pre-test estimates) although the basic structure remains the same, e.g., short-term rates tend to fluctuate more and covariances generally diminish with increasing $| m - n |$ for m- and n-period bills. The impact of this instability on portfolio weights appears minimal. A comparison of solutions for portfolio two, defined by $\underline{\alpha}' = (0.25\ 0.25\ 0.25\ 0.25)$, illustrates this point. The covariance matrix for the full data set (Table A1) produced a futures portfolio $\underline{\gamma}' = -(1.022\ 0.4974\ 0.3987)$. The out-of-sample (pre-test) covariance matrix (Table A3) produced a futures portfolio $\underline{\gamma}' = -(1.101\ 0.6016\ 0.2139)$. The net position of the former portfolio is $\sum_{i=1}^{3} \gamma_i = -1.9181$ while that of the latter is -1.9165. Thus, optimal hedging ratios were relatively insensitive to covariance instability of the type experienced in the 1970–1982 period. This result is consistent with instability generated by a scalar multiple of rate changes (Corollary 1.2).

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Appendix: Table A1

Monthly Covariances ($\times 10^{-6}$): All Data[a]

0.75937	0.60651	0.53315	0.52955	0.50908	0.49838	0.47898	0.44805	0.46729	0.45823	0.46368	0.45556
0.60651	0.58408	0.53285	0.53564	0.51861	0.50806	0.49171	0.46864	0.47794	0.46975	0.46658	0.45814
0.53315	0.53285	0.54984	0.54132	0.52126	0.51948	0.49474	0.47608	0.48074	0.47264	0.46421	0.46059
0.52955	0.53564	0.54132	0.56859	0.54826	0.53636	0.52283	0.49912	0.50603	0.49782	0.48581	0.48028
0.50908	0.51861	0.52126	0.54826	0.54629	0.53539	0.52423	0.50417	0.51402	0.50861	0.49912	0.49352
0.49838	0.50806	0.51948	0.53636	0.53539	0.53656	0.51979	0.50123	0.51051	0.50706	0.49840	0.49570
0.47898	0.49171	0.49474	0.52283	0.52423	0.51979	0.52344	0.50194	0.51259	0.50872	0.49849	0.49603
0.44805	0.46864	0.47608	0.49912	0.50417	0.50123	0.50194	0.49092	0.49916	0.49685	0.48660	0.48515
0.46729	0.47794	0.48074	0.50603	0.51402	0.51051	0.51259	0.49916	0.51711	0.51114	0.50202	0.50064
0.45823	0.46975	0.47264	0.49782	0.50861	0.50706	0.50872	0.49685	0.51114	0.51237	0.50300	0.50081
0.46368	0.46658	0.46421	0.48581	0.49912	0.49840	0.49849	0.48660	0.50202	0.50300	0.49926	0.49519
0.45556	0.45814	0.46059	0.48028	0.49352	0.49570	0.49603	0.48515	0.50064	0.50081	0.49519	0.49693

[a] Covariances among unexpected changes in spot rates. Data taken from time period January 1970 through October 1982.

Appendix: Table A2

Monthly Correlations: All Data[a]

1.00000	0.91070	0.82509	0.80590	0.79041	0.78088	0.75973	0.73383	0.74571	0.73463	0.75305	0.74161
0.91070	1.00000	0.94026	0.92948	0.91811	0.90755	0.88927	0.87518	0.86966	0.85870	0.86403	0.85038
0.82509	0.94026	1.00000	0.96813	0.95110	0.95640	0.92219	0.91635	0.90157	0.89048	0.88599	0.88114
0.80590	0.92948	0.96813	1.00000	0.98374	0.97107	0.95836	0.94471	0.93322	0.92233	0.91180	0.90354
0.79041	0.91811	0.95110	0.98374	1.00000	0.98890	0.98034	0.97356	0.97612	0.96136	0.95571	0.94721
0.78077	0.90755	0.95640	0.97107	0.98890	1.00000	0.98082	0.97661	0.96918	0.96708	0.96295	0.95999
0.75973	0.88927	0.92219	0.95836	0.98034	0.98082	1.00000	0.99018	0.98524	0.98232	0.97512	0.97259
0.73383	0.87518	0.91635	0.94471	0.97356	0.97661	0.99018	1.00000	0.99071	0.99067	0.98289	0.98225
0.74571	0.86966	0.90157	0.93322	0.96712	0.96918	0.98524	0.99071	1.00000	0.99302	0.98803	0.98762
0.73463	0.85870	0.89048	0.92233	0.96136	0.96708	0.98232	0.99067	0.99302	1.00000	0.99451	0.99251
0.75305	0.86403	0.88599	0.91180	0.95571	0.96295	0.97512	0.98289	0.98803	0.99451	1.00000	0.99416
0.74161	0.85038	0.88114	0.90354	0.94721	0.95999	0.97259	0.98225	0.98762	0.99251	0.99416	1.00000

[a] Correlations among unexpected changes in spot rates. Data taken from time period January 1970 through October 1982.

Appendix: Table A3

Monthly Covariances ($\times 10^{-6}$): Out-of-Sample Data[a]

0.23110	0.21059	0.21823	0.20074	0.19570	0.20091	0.17061	0.17295	0.16957	0.16959	0.17082	0.16884
0.21059	0.21174	0.20966	0.19654	0.19726	0.19961	0.17562	0.17981	0.17689	0.17659	0.17672	0.17558
0.21823	0.20966	0.24897	0.21726	0.20806	0.21849	0.17962	0.17902	0.17544	0.17471	0.17400	0.17260
0.20074	0.19654	0.21726	0.20866	0.20534	0.20728	0.17894	0.18071	0.17886	0.18017	0.17950	0.17595
0.19570	0.19726	0.20806	0.20534	0.21467	0.21256	0.18953	0.19298	0.19222	0.19425	0.19455	0.18960
0.20091	0.19961	0.21849	0.20728	0.21256	0.21910	0.19103	0.19285	0.19124	0.19364	0.19352	0.19095
0.17061	0.17562	0.17962	0.17894	0.18953	0.19103	0.18526	0.18666	0.18518	0.18770	0.18613	0.18617
0.17295	0.17981	0.17902	0.18071	0.19298	0.19285	0.18666	0.19300	0.19132	0.19372	0.19277	0.19190
0.16957	0.17689	0.17544	0.17886	0.19222	0.19124	0.18518	0.19132	0.19733	0.19472	0.19356	0.19379
0.16959	0.17659	0.17471	0.18017	0.19425	0.19364	0.18770	0.19372	0.19472	0.19958	0.19830	0.19646
0.17082	0.17672	0.17400	0.17950	0.19455	0.19352	0.18613	0.19277	0.19356	0.19830	0.20071	0.19683
0.16884	0.17558	0.17260	0.17595	0.18960	0.19095	0.18617	0.19190	0.19379	0.19646	0.19683	0.19840

[a] Covariances among unexpected changes in spot rates. Data taken from time period January 1970 through December 1977.