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A Risk Minimizing Strategy for Portfolio Immunization

H. GIFFORD FONG and OLDRICH A. VASICEK*

ABSTRACT

Consider a fixed-income portfolio whose duration is equal to the length of a given investment horizon. It is shown that there is a lower limit on the change in the end-of-horizon value of the portfolio resulting from any given change in the structure of interest rates. This lower limit is the product of two terms, of which one is a function of the interest rate change only, and the other depends only on the structure of the portfolio. Consequently, this second term provides a measure of immunization risk. If this measure is minimized, the exposure of the portfolio to any interest rate change is the lowest.

THE TRADITIONAL THEORY OF immunization as formalized by Fisher and Weil [6] defines the conditions under which the value of an investment in a bond portfolio is protected against changes in the level of interest rates. The specific assumptions of this theory are that the portfolio is valued at a fixed horizon date, that there are no cash inflows or outflows within the horizon, and that interest rates change only by a parallel shift in the forward rates. Under these assumptions, a portfolio is said to be immunized if its value at the end of the horizon does not fall below the target value, where the target value is defined as the portfolio value at the horizon date under the scenario of no change in the forward rates. The main result of this theory is that immunization is achieved if the duration of the portfolio is equal to the length of the horizon.

The assumption that interest rates can only change by a parallel shift (that is, by the same amount for all maturities) has been the subject of considerable concern. Bierwag [1, 2], Bierwag and Kaufman [3], Khang [7], and others have postulated alternative models of interest rate behaviors. Each of these specifications implies a different measure of duration, with immunization attained if this duration measure is equal to the horizon length. A limitation of this approach is that the portfolio is protected only against the particular type of interest rate change assumed.

In a more recent development, Cox et al. [5], Brennan and Schwartz [4], and others have investigated immunization conditions when interest rates are governed by a continuous process consistent with a market equilibrium. Depending on the specification of the interest rate process, there is a duration-like measure (possibly multidimensional, as in Brennan and Schwartz) such that the portfolio is immunized if a proper value of this measure is maintained. This assumes a continuous rebalancing of the portfolio. Again, immunization is achieved only if interest rate changes conform to the specific process assumed.

In this paper, we wish to pursue a different approach. If it turned out that the

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portfolio exposure to an *arbitrary* type of interest rate change were determined by some characteristic of the portfolio, then this characteristic could be considered a measure of immunization risk. By minimizing this risk measure, the portfolio could be structured to have as little vulnerability as possible to any interest rate movement.

It is shown in this paper that there is a lower limit on the change in the end-of-horizon value of an immunized portfolio for an arbitrary interest rate change. This lower limit is a product of two terms. One of these terms depends only on the type and magnitude of the rate change, while the other term depends solely on the structure of the portfolio. This second term provides the desired measure of immunization risk, since, when it is small, the exposure of the portfolio to any interest rate change is small.

As in Fisher and Weil, we will consider interest rate shocks of finite magnitude, rather than infinitesimal rate changes with continuous portfolio rebalancing. This appears to be a more relevant approach for applications, since in practice rates will always move by a noninfinitesimal amount before the portfolio can be restructured.

The main result is stated in the form of a theorem in the next section, followed by a discussion of the concept. The mathematical proof of the theorem is given in the Appendix.

I. Immunization Risk

Consider a portfolio at time $s_0 = 0$ to be immunized with respect to a given horizon H . Let $C_1, C_2, \dots, C_m$ be the payments on the portfolio, due at times $s_1, s_2, \dots, s_m$, and denote by I_0 the initial portfolio value,

$$I_0 = \sum_{j=1}^m C_j P_0(s_j). \quad (1)$$

Here $P_0(t)$ is the current discount function of term t . If $i(t)$ is the current forward rate of term t , $t \geq 0$, the discount function can be written as

$$P_0(t) = \exp\left(-\int_0^t i(\tau) d\tau\right).$$

Let D be the Macaulay duration of the portfolio, defined as

$$D = \sum_{j=1}^m s_j C_j P_0(s_j) / I_0. \quad (2)$$

Define the target value I_H of the investment at the horizon date as the end-of-horizon value of the portfolio if the forward rates do not change,

$$I_H = I_0 / P_0(H). \quad (3)$$

As shown by Fisher and Weil [6], if the portfolio duration D is equal to the length of the horizon, the target value I_H is a lower bound of the terminal value of the portfolio regardless of any parallel shift in interest rates. If rates of different maturities change by different amounts, the I_H is not necessarily a lower bound on the end-of-horizon investment value. In order to establish a measure of immunization risk, we shall analyze the change in the terminal value of a given portfolio for a given (nonparallel) rate change.

The amount by which the terminal value of the portfolio may be short of the target, as a result of an interest rate change, will depend on the character and magnitude of the change as well as on the structure of the portfolio. The immunity of a portfolio to parallel rate changes is attained as a consequence of balancing the effect of changes in reinvestment rates on payments received during the horizon against the change in capital value of the portion of the portfolio still outstanding at the end of the horizon. For a nonparallel change in interest rates such balancing may not take place. Consider the case when the change in short rates is algebraically less than the change in long rates (for example, short rates decline while long rates go up). Such a scenario, characterized by an increase in the slope of the interest rate structure, will result in a decline of the terminal portfolio value below the target. The larger the magnitude of such a twist of the yield curve, the bigger the resulting shortfall will be for a given portfolio.

Not all portfolios, however, will be affected by a given change equally. Consider a "barbell" portfolio composed of very short and very long bonds, and a "bullet" portfolio consisting of low-coupon securities with maturities close to the horizon date. Assume that both portfolios have durations equal to the horizon length. Should short rates go down and long rates go up, both portfolios will realize a decline in the end-of-horizon value, since they experience a capital loss in addition to lower reinvestment rates. The decline, however, would be substantially higher for the barbell portfolio for two reasons. First, the lower reinvestment rates are experienced on the barbell portfolio for longer time intervals than on the bullet portfolio, so that the opportunity loss is much greater. Second, the portion of the barbell portfolio still outstanding at the horizon date is much longer than that of the bullet portfolio, which means that the rate increase would result in a much steeper capital loss.

To characterize these arguments quantitatively, suppose that the forward rates change instantaneously from $i(t)$ to $i'(t) = i(t) + \Delta i(t)$, where $\Delta i(t)$ is an arbitrary function of the term t . Consider a portfolio whose duration is equal to the horizon length, and denote by ΔI_H the corresponding change in the portfolio value as of the horizon date. We will state the following theorem, a proof of which is given in the Appendix:

THEOREM. *Let K be an arbitrary constant. If $d\Delta i(t)/dt \leq K$ for all $t \geq 0$, then*

$$\Delta I_H/I_H \geq -\frac{1}{2}K \cdot M^2, \quad (4)$$

where

$$M^2 = \sum_{j=1}^M (s_j - H)^2 C_j P_0(s_j)/I_0. \quad (5)$$

The inequality in (4) provides a lower bound on the change in the terminal value of the portfolio. The theorem states that this lower bound is the product of two terms. The first term, $-\frac{1}{2}K$, is a function of the interest rate change only, while the second term, M^2 , depends solely on the structure of the investment portfolio.

The quantity K can be interpreted as an upper bound, over maturity, on the change in the slope of the term structure. Note that the derivative $d\Delta i(t)/dt$ is with respect to the term, not time. In effect, this quantity characterizes the twist

of the yield curve. Since the rate change can be arbitrary, the maximum slope change is an uncertain variable, outside the control of the investor.

The investor, however, can determine the portfolio composition and therefore the quantity M^2 . This term, which is a multiplier of the unknown rate change, is a *measure of risk for an immunized portfolio*, since it determines the exposure of the portfolio to rate changes. The lower bound for the change in the end-of-horizon value of the investment as a result of an arbitrary rate change is proportional to M^2 . Note that to justify M^2 as a measure of risk, no assumptions are necessary about the nature or dimensionality of the stochastic process that governs the behavior of the term structure of interest rates.

To gain insight into the meaning of the risk measure M^2 , note the similarity in form of M^2 in Equation (5) to the definition of duration in Equation (2). While duration is a weighted *average* of time to payments on the portfolio, the weights being the present value of the payments, M^2 is a similarly weighted *variance* of time to payments around the horizon date. If the portfolio payments occur close to the end of the horizon, as with a portfolio of deep-discount bonds maturing close to the horizon date, M^2 is low. If the payments are widely dispersed in time, as with a portfolio consisting of very short bonds and very long bonds, M^2 is high. The theorem states that a low M^2 portfolio has less exposure to whatever the change in the interest rate structure may be than a high M^2 portfolio. An optimally immunized portfolio that has the minimum exposure to interest rate changes is obtained by minimizing the risk measure M^2 , subject to the duration condition $D = H$ and any portfolio constraints. It may be noted that M^2 is linear in the portfolio weights, so that minimizing M^2 can be written as a linear program.

The risk measure M^2 is always nonnegative. It attains its lowest possible value of zero if and only if the portfolio consists of a single discount bond with maturity equal to the length of the horizon. This is indeed the perfectly immunized portfolio, since no interest rate change affects its end-of-horizon value. Any other portfolio is to some extent vulnerable to an adverse interest rate movement. The immunization risk M^2 in effect measures how much a given portfolio differs from this ideally immunized portfolio consisting of the single discount bond.

It may be pointed out that the above theorem contains, as a special case, the main result of classical immunization theory. Indeed, for parallel shifts, we have

$$\frac{d}{dt} \Delta i(t) = 0$$

for all $t \geq 0$, which then implies $\Delta I_H \geq 0$ for any portfolio whose duration is equal to the horizon length.

Appendix

Proof of the Theorem

Suppose that forward rates change from $i(t)$ to $i'(t) = i(t) + \Delta i(t)$. The discount function then becomes

$$P'_0(t) = \exp\left(-\int_0^t i'(\tau) d\tau\right)$$

$$= P_0(t) \exp\left(- \int_0^t \Delta i(\tau) d\tau\right).$$

The change ΔI_H in the end-of-horizon value of the portfolio due to the change $\Delta i(t)$ in the forward rates is

$$\begin{aligned} \Delta I_H &= \sum_{j=1}^m C_j P'_0(s_j)/P'_0(H) \\ &\quad - \sum_{j=1}^m C_j P_0(s_j)/P_0(H) \\ &= \sum_{j=1}^m C_j \exp\left(\int_{s_j}^H \Delta i(\tau) d\tau\right) P_0(s_j)/P_0(H) \\ &\quad - \sum_{j=1}^m C_j P_0(s_j)/P_0(H), \end{aligned}$$

or

$$\Delta I_H = \sum_{j=1}^m f(s_j) C_j P_0(s_j)/P_0(H) - I_0/P_0(H), \quad (\text{A1})$$

where

$$f(t) = \exp\left(\int_t^H \Delta i(\tau) d\tau\right).$$

Let

$$g(t) = \frac{d}{dt} \Delta i(t).$$

Then

$$\Delta i(H) - \Delta i(\tau) = \int_{\tau}^H g(u) du$$

and

$$\begin{aligned} \int_t^H \Delta i(\tau) d\tau &= \Delta i(H) \cdot (H - t) - \int_t^H d\tau \int_{\tau}^H g(u) du \\ &= \Delta i(H) \cdot (H - t) - \int_t^H (u - t) g(u) du. \end{aligned}$$

Assume that $g(t) \leq K$ for all $t \geq 0$. If $t \leq H$, then

$$\int_t^H (u - t) g(u) du \leq K \int_t^H (u - t) du = \frac{1}{2} K (H - t)^2.$$

If $t > H$, then

$$\int_t^H (u - t) g(u) du = \int_H^t (t - u) g(u) du \leq K \int_H^t (t - u) du = \frac{1}{2} K (H - t)^2.$$

Therefore,

$$\int_t^H (u - t) g(u) du \leq \frac{1}{2} K (H - t)^2$$

for all $t \geq 0$, and, consequently,

$$\int_t^H \Delta i(\tau) d\tau \geq \Delta i(H) \cdot (H - t) - \frac{1}{2} K(H - t)^2.$$

Since $e^x \geq 1 + x$, we have

$$\begin{aligned} f(t) &= \exp\left(\int_t^H \Delta i(\tau) d\tau\right) \\ &\geq 1 + \int_t^H \Delta i(\tau) d\tau \\ &\geq 1 + \Delta i(H) \cdot (H - t) - \frac{1}{2} K(H - t)^2. \end{aligned}$$

From Equation (A1), we then have

$$\begin{aligned} \Delta I_H &\geq \sum_{j=1}^m [1 + \Delta i(H) \cdot (H - s_j) - \frac{1}{2} K(H - s_j)^2] C_j P_0(s_j) / P_0(H) \\ &\quad - I_0 / P_0(H) \\ &= -\frac{1}{2} K \sum_{j=1}^m (H - s_j)^2 C_j P_0(s_j) / P_0(H) \\ &= -\frac{1}{2} K M^2 I_H, \end{aligned}$$

which completes the proof.

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