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Source: *The Journal of Finance*, Vol. 39, No. 5 (Dec., 1984), pp. 1525-1539
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327742
Accessed: 27/11/2009 07:51

THE JOURNAL OF FINANCE • VOL. XXXIX, NO. 5 • DECEMBER 1984

The Valuation of Options When Asset Returns Are Generated by a Binomial Process

R. C. STAPLETON and M. G. SUBRAHMANYAM*

ABSTRACT

This paper values options on assets whose returns, over a finite interval of time, are generated by a binomial process. It shows that a simple valuation relationship, between the option and the underlying stock, obtains if investors have preference functions that belong to a particular class, even if opportunities to hedge do not exist. One particular application of the theory is in the case where the stock price over a finite interval could increase by an amount, fall by the same amount, or stay at the same level. The results in this paper may be viewed as the foundation of the preference-based approaches to obtaining a risk neutral valuation relationship.

IN THIS PAPER, we value options on assets whose returns, over a single period of time, are generated by a multi-stage binomial process of the additive and multiplicative types. The market is open only at the beginning and end of the single period and, consequently, the risks involved in holding options cannot be hedged. The case of a single-stage binomial process in such a setting has been discussed extensively by Cox, Ross, and Rubinstein [6]. In this case, the option and the underlying asset are perfectly correlated and a riskless hedge can be set up. As a consequence, the relationship between the value of the option and the underlying asset is independent of risk preferences.

In the multiplicative case, when the number of stages is very large, the return on the asset is lognormal. For this case, Rubinstein [11] has shown that, if aggregation holds and the representative investor has constant proportional risk aversion (CPRA), the Black and Scholes [2] formula applies. Again, the relationship between the value of the option and the underlying asset is independent of the parameters of investors' utility functions. Brennan [4] calls this type of relationship a risk neutral valuation relationship (RNVR), since it is consistent with universal risk neutrality. He shows that CPRA is a necessary and sufficient condition for a RNVR to hold when the asset has a lognormal distribution.[1]

In this paper, we prove that the Rubinstein-Brennan results hold for any n-stage binomial process. That is, as long as the underlying asset return is generated by the multiplicative binomial distribution, CPRA preferences allow the derivation of a RNVR. An example of the multiplicative binomial process, that has

* Manchester Business School, England, and New York University, respectively. We thank K. John for helpful discussion.

[1] Brennan [4] also proves that a RNVR obtains when asset returns are normally distributed and the representative investor has constant absolute risk aversion (CARA) preferences. In these two cases, the distributions of asset returns are not just normal and lognormal, respectively, but are joint normal and joint lognormal with aggregate wealth

some interest of its own, is the case where $n = 2$. This is illustrated in Figure 1.

In Figure 1, the stock price can take on just three values (Su^2, S, Sd^2) with probabilities (q^2, $2q(1-q)$, $(1-q)^2$). This three-state case can be modelled by a two-stage binomial process where the stock price changes by a factor $u > 1$ with probability q or falls by a factor $d < 1$ with probability $(1-q)$ at each stage.[2] In this example, $d = 1/u$, and hence, the middle state represents no change in the stock price.

For options on individual stocks, this two-stage example may be somewhat more realistic than a one-stage, two-state model. Over any short interval, it may be reasonable to expect a stock to move up, to move down, or to remain at the same level. This may be illustrated with reference to the behaviour of a stock around the date of earnings announcements. At the actual point of the announcement, the stock may rise, fall, or stay the same according to whether the news is better than, worse than, or as expected.[3] Hence, over a short interval, the price movement may be modelled by the example in Figure 1.

The two-stage, three-state binomial example may model short-run price behaviour fairly well. Figure 2 illustrates a simple case where the trinomial example is appropriate. Consider the problem of valuation of a European option with a time to maturity of $T + t_E$. At time 1, $-T < 1 < t_E$, an earnings announcement will be made which will lift the stock price to Su^2, lower it to Sd^2, or leave it unchanged. Suppose that the stock will not move in price between $-T$ and 0 (a moment before 1) and again after $t = 1$. The option is thus a claim that is contingent on an asset with a trinomial distribution.

The two- or, more generally, the multi-stage setting captures, more realistically, the hedging problems actually faced by market makers and other traders who attempt to hedge away the stock price risk in their options positions. In practice, the volatility of the underlying stock, an important determinant of the hedge ratio, is not known with certainty, which implies that the dynamic hedging strategy, based on an estimate of the volatility, no longer leads to a riskless return.[4]

The analysis in this paper is in terms of the n-stage, $(n+1)$-state case of the binomial distribution. For example, in the four-stage case, there are five possible states. In states 1 and 2, the stock moves up with probability q^4 and $4q^3(1-q)$, respectively. In state 3, the stock remains at the same level with probability $6q^2(1-q)^2$. It falls in states 4 and 5 with probability $4q(1-q)^3$ and $(1-q)^4$, respectively.

[2] In general, u and d need to lie on either side of r, the one-stage wealth relative obtained by investment in the risk-free asset, to avoid first-order stochastic dominance. Thus, if $r > 1$, it is possible for both u and d to be greater than unity.

[3] Parkinson [8] provides a numerical procedure for valuing American puts based on such a trinomial process.

[4] An important assumption underlying the hedging approach using a single-stage binomial process is that the magnitudes of the up and down movements at each revision point as captured by the stock volatility are assumed to be known in advance. If the actual sample volatility at each revision point differs from the estimate, the investor is forced to bear an additional risk even when a dynamic hedging strategy is followed. This is better modelled by the multi-stage approach used here. We thank the referee for highlighting this point.

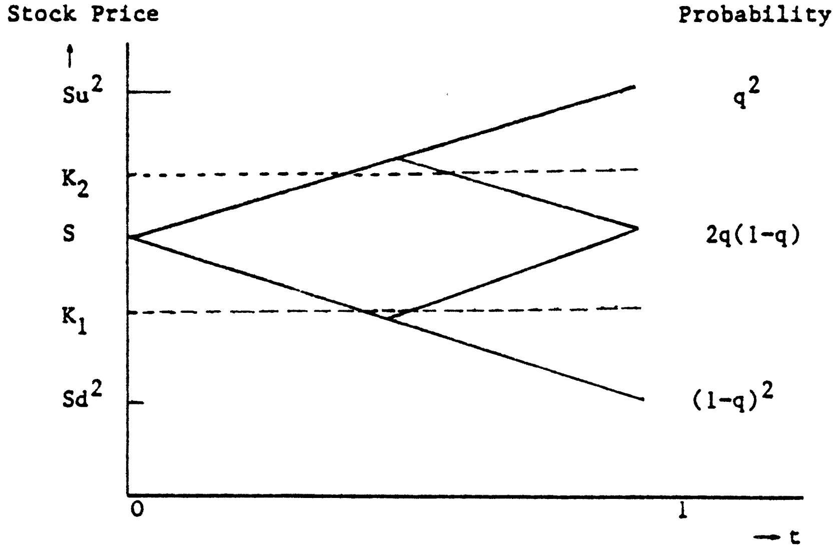

Figure 1. A Trinomial Example

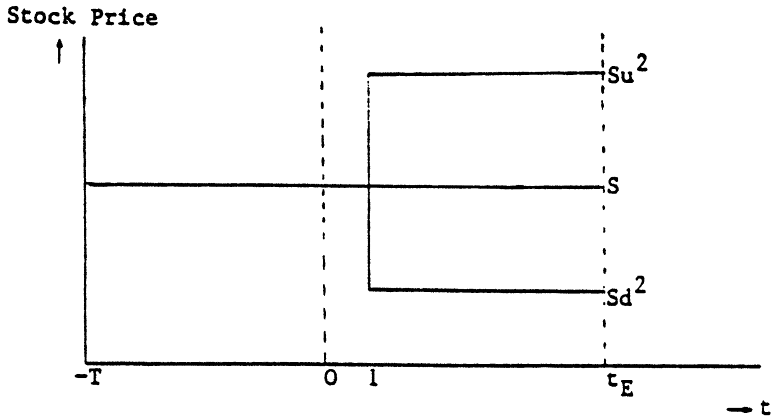

Figure 2. Stock Price Movements Around Earnings Announcements

The present paper is closely related to that of Cox, Ross, and Rubinstein [6]. One purpose of their paper was to show that options could be valued by arbitrage when the underlying asset follows a discrete-time multiplicative binomial distribution.[5] In the limit, the option prices tend to either the Black and Scholes [2] or the Cox and Ross [5] prices, depending on the limiting arguments used in each case. The basis of their argument is that investors can maintain a riskless hedge at each stage of the binomial process. In the present paper, we show that a similar result will apply, given CPRA preferences, even when investors *cannot*

[5] This approach was used independently by Rendleman and Bartter [9].

maintain such a riskless hedge. In a sense, we extend the Rubinstein [11] and Brennan [4] results to the case of a finite-stage binomial process with the continuous distribution case holding in the limit. Our prices for a finite number of stages approximate the Rubinstein [11] prices in the same way that the Cox, Ross, and Rubinstein [6] prices approximate the Black and Scholes [2] prices. In fact, just as Rubinstein's model yields the Black and Scholes [2] option prices, so our extension of his result yields the Cox, Ross, and Rubinstein [6] option prices. In both cases, RNVRs result for a rich subset of probability distributions, those generated by an n-stage (additive or multiplicative) binomial process, where n is arbitrary and does not necessarily tend to infinity.

I. Option Pricing Models Given Aggregation in Capital Markets

It is appropriate to begin the derivation of our option pricing model with a general valuation statement. Following Rubinstein [11] and Brennan [4], the value of a claim, contingent on a cash flow X, can be written as[6]

$$V[g(X)] = (1/R)E[g(X)z(X)] \tag{1}$$

where $g(X)$ is the payoff function, $z(X) = E[U'(W \mid X)]/E[U'(W)]$ is the conditional expected relative marginal utility function of the representative investor, and R is one plus the risk-free rate of interest over the time interval $0 - 1$. Equation (1) is derived given:

1. Perfect and complete markets
2. Aggregation

The first assumption allows us to write the value of any contingent claim as the sum of its state-contingent payoffs multiplied by the appropriate state-contingent prices. The second assumption permits the valuation of state-contingent claims in terms of the utility function of a representative investor.

Equation (1) has little empirical content because the prices depend upon the unknown parameters of $z(X)$, the preference-specific pricing function. Option pricing theory has been directed, therefore, towards looking for relationships between $V[g(X)]$, the value of the state contingent claim, and the value of the underlying asset,

$$V(X) = (1/R)E[Xz(X)] \tag{2}$$

which are not dependent on the parameters of $z(X)$, i.e., the derivation of RNVRs. One approach, used by Black and Scholes [2], is to show that, when X follows a diffusion process that is lognormally distributed over any finite time interval,

$$V[g(X)] = 1/R \int_0^\infty g(X)\hat{f}(X)\, dX \tag{3}$$

[6] See Rubinstein [11] and Brennan [4] for details of the derivation. Basically, Equation (1) follows from the first-order conditions of the portfolio optimization problem of the representative investor.

where $\hat{f}(X)$ differs from the probability density function $f(X)$ by a shift of the mean. Cox, Ross, and Rubinstein [6] generalized and clarified this result to show that if X were generated by an n-state multiplicative binomial process, investors could once again hedge the risk of a claim contingent on X on a stage-by-stage basis. As a result, it is possible to value an option independently of the parameters of $z(X)$. In this case

$$V[g(X)] = 1/R \sum_{\text{all}\,X} g(X)\hat{p}(X) \tag{4}$$

where $\hat{p}(X)$ differs from the binomial probabilities in the following way: if q is the probability of the stock rising by a factor $u > R$ and $(1 - q)$ is the probability of its falling by a factor $d < R$, then $p = (R - d)/(u - d)$, and $\hat{p}(X)$ is the binomial density function based on p, given n-stages.

The alternative approach to option pricing used by Rubinstein [11] and Brennan [4] involves specifying a given class of utility functions and the return distribution which imply a particular $z(X)$. The difference between this approach and the hedging approach is that even when investors cannot hedge the risk of the option, Equation (3) holds. Specifically, for the case of CPRA and a lognormal distribution of stock returns, a RNVR, as in (3), holds. For example, if the contingent claim is a call option with an exercise price K then

$$g(X) = \max[X - K, 0]$$

and

$$V[g(X)] = 1/R \int_K^\infty (X - K)z(X)f(X)\,dX = 1/R \int_K^\infty (X - K)\hat{f}(X)\,dX \tag{5}$$

where $f(X)$ is the density function of the lognormal distribution and $\hat{f}(x)$ is the density function after the shift of the mean.

At the most general level, options can be valued as "bundles" of state-contingent claims. This is, in fact, similar to the approach of Banz and Miller [1] and Breeden and Litzenberger [3] who argue that by finely partitioning the range of exercise prices of options, state prices can be derived from observed option prices and vice versa. If riskless hedging is possible, arbitrage considerations provide more structure and lead to the Black and Scholes [2] and Cox, Ross, and Rubinstein [6] models.[7] Finally, if hedging is *not* possible but CPRA preferences are assumed, Rubinstein [11] and Brennan [4] have shown that the Black and Scholes [2] valuation relationship holds with lognormal returns. We will now proceed to prove that CPRA is also sufficient to establish a RNVR for options on assets following a finite-stage multiplicative binomial process, again, when hedging is not possible. In a sense, we will be explaining why CPRA works in the case of lognormal returns, just as Cox, Ross, and Rubinstein [6] illustrated the basic intuition of the Black and Scholes [2] results.

LEMMA. *If the distribution of the future stock price is binomial, a RNVR exists if*

[7] A general approach to risk neutral valuation using arbitrage arguments is presented in Ross [10].

there exists a p such that

$$\begin{aligned} p^{n-1} &= q^{n-1}z(X(1)) \\ p^{n-2}(1-p) &= q^{n-2}(1-q)z(X(2)) \\ &\vdots \\ p(1-p)^{n-2} &= q(1-q)^{n-2}z(X(n-1)) \\ (1-p)^{n-1} &= (1-q)^{n-1}z(X(n)) \end{aligned} \tag{6}$$

where $z(X(j))$ is the preference function, conditional on the stock price in state j.

Proof: Substituting Condition (6) in the general valuation equation (1) yields

$$V[g(X)] = 1/R \textstyle\sum_{\text{all}X} g(X)\hat{p}(X) \tag{4}$$

where the probabilities are binomial.

In words, the lemma simply states that a RNVR exists if the utility weighted probabilities can be replaced by a probability measure generated by the binomial expansion. If we eliminate one of the n equations in (6), by successive division and rearrangement of terms we have,

$$\begin{aligned} p^{n-1} &= q^{n-1}z(X(1)) \\ z(X(1))z(X(3)) &= [z(X(2))]^2 \\ z(X(2))z(X(4)) &= [z(X(3))]^2 \\ &\vdots \\ z(X(n-2))z(X(n)) &= [z(X(n-1))]^2 \end{aligned} \tag{7}$$

The first equation in (7) fixes the scale of p while the remaining $(n-2)$ equations restrict the preference function. We can now state the conditions for a RNVR to exist.

THEOREM I. *A sufficient condition for a RNVR to exist when the future stock price, X, and aggregate wealth, W, are joint multiplicative binomial is that the utility of wealth of the representative investor is of the CPRA class.*

Proof: We will first show that if $z(X)$ is a power function, Condition (7) is satisfied. If X is multiplicative binomial, we can write, for example,

$$\begin{aligned} X(1) &= Su^{n-1} \\ X(2) &= Su^{n-2}d \\ &\vdots \\ X(n-1) &= Sud^{n-2} \\ X(n) &= Sd^{n-1} \end{aligned} \tag{8}$$

where S is the initial stock price and u and d are the factors by which the price moves at each stage of the $(n-1)$-stage process.

If $z(X)$ is a power function, i.e., $z(X) = X^{\gamma}$,

$$\begin{aligned} z(X(j-1))z(X(j+1)) &= [Su^{n-(j-1)}d^{j-2}]^{\gamma}[Su^{n-(j+1)}d^{j}]^{\gamma} \\ &= [Su^{n-j}d^{j-1}]^{2\gamma} \\ &= [z(X(j))]^{2} \end{aligned} \tag{9}$$

as in Equations (7).

In the Appendix, we show that the conditional relative marginal utility of wealth function $z(X)$ (defined in Equation (1)) is a power function if and only if the utility function $U(W)$ is a power function. This follows from the fact that for the joint multiplicative binomial case, the logarithm of X is linearly related to the logarithm of W.

It is, of course, not necessary to assume CPRA preferences in order to obtain a RNVR for the multiplicative binomial case. For example, for the two-stage, three-state case, a quadratic $U'(W)$ (i.e., cubic utility of wealth function) can be fitted through the three points defined by the relevant Equations (6). Clearly, this will involve specific coefficients which depend on q, u, and d and hence does not qualify as a RNVR in the strict sense. Furthermore, if the example is changed to include one more stage, the coefficients of the polynomial utility function will change. Thus, if it is required that the RNVR holds for an arbitrary number of stages, it follows by inspection of Equations (7) and (8) taken together, that it is necessary for $z(X)$ to be a power function.

We have concentrated so far on multiplicative binomial processes since these relate closely to the dominant models in option pricing theory. However, just as Brennan [4] extended Rubinstein's [11] results to the case of joint normality and constant absolute risk aversion preferences (CARA), the analysis above is easily extended to the additive binomial case (for example, if $n = 2$, X may rise from S to $S + 2u'$, fall to $S + 2d'$, or, if $u' = d'$, it may stay the same). Here, the resulting distribution is symmetric and limits to the normal distribution which may be a realistic assumption in some cases.

THEOREM II. *A sufficient condition for a RNVR to exist when X and W are joint additive binomial is that the utility of wealth of the representative investor is of the CARA class.*

Proof: If X is additive binomially distributed, we can write

$$\begin{aligned} X(1) &= S + (n-1)u' \\ X(2) &= S + (n-2)u' + d' \\ &\vdots \\ X(n-1) &= S + u' + (n-2)d' \\ X(n) &= S + (n-1)d' \end{aligned} \tag{10}$$

If $z(X)$ is of the CARA class, it is exponential, i.e., $z(X) = \exp \alpha X$. Therefore,

$$\begin{aligned} z(X(j-1))z(X(j+1)) &= \exp \alpha[S + \{n - (j-1)\}u' + (j-2)d'] \\ &\quad \cdot \exp \alpha[S + \{n - (j+1)\}u' + jd'] \\ &= \exp 2\alpha[S + (n-j)u' + (j-1)d'] \\ &= [z(X(j))]^2 \end{aligned} \tag{11}$$

as required in Equations (7).

Although it is not proved in detail, an argument similar to that used in the multiplicative case in the Appendix may be used to show that $z(X)$ is exponential if and only if $U(W)$ is. Again, as in the multiplicative exponential binomial case, it is also necessary that $z(X)$ is an exponential function.

II. Option and Stock Prices Using the Risk Neutral Valuation Relationship

We have shown that if the stock price and aggregate wealth follow a multiplicative bivariate binomial process, a risk neutral valuation relationship holds. This means that the *relationship* between the value of the underlying asset and options written on it is independent of risk parameters. Alternatively, the probability measure q of the underlying process can be shifted to produce a risk-adjusted probability measure p, as in Cox, Ross, and Rubinstein [6] and Harrison and Kreps [7]. Given the risk-adjusted probability measure p, all assets, including the stock and contingent claims written on it, can be priced by the binomial probability density function implied by p.

Specifically, the value of the stock is given by

$$S = 1/r^n\left[\sum_{j=0}^{n} z(X(j+1)) \frac{n!}{(n-j)!j!} q^{n-j}(1-q)^j X(j+1)\right] \tag{12}$$

where $R = r^n$, r being one plus the risk-free rate in each stage. Substituting for $X(j) = Su^{n-j}d^j$ and using the risk neutral valuation relationship in (6), we can write

$$\sum_{j=0}^{n} \frac{n!}{(n-j)!j!} p^{n-j}(1-p)^j u^{n-j} d^j = r^n \tag{13}$$

which implies

$$\{pu + (1-p)d\} = r \tag{14}$$

The value of any contingent claim whose payoffs are dependent on the stock price X can be written from (1) and (6). In particular, the value of a call option on the stock whose payoffs are given by $g(X) = \max[(X - K), 0]$ can be written as

$$C = 1/r^n \sum_{j=0}^{n} \max[\{Su^{n-j}d^j - K\}, 0] \frac{n!}{(n-j)!j!} p^{n-j}(1-p)^j \tag{15}$$

Equation (15) above is identical to Equation (6) in Cox, Ross, and Rubinstein [6]. Both results are based implicitly on a risk neutral valuation relationship holding. However, there is an important distinction between them. While the earlier derivation of this relationship is based purely on arbitrage considerations, the analogous result here follows from preference restrictions—that the utility of wealth of the representative investor exhibits constant proportional risk aversion.

Further, the arbitrage approach yields only *relative* prices for contingent claims in terms of the underlying asset, the price of the stock being taken as given. In contrast, the preference-based approach yields an *absolute* price for the stock and obtains values for contingent claims relative to the stock value through a relative valuation relationship. Thus, implicit in the preference-based approach is a capital asset pricing model (CAPM) for valuing the stock. For instance, in the Rubinstein [11] and Brennan [4] approaches, the logarithmic version of the standard CAPM holds and is used to shift the mean of the underlying distribution so that the resultant valuation of the contingent claim involves only the risk-free rate and not the expected return on the stock, since the latter would, in general, depend on the particular parameters of the utility functions of the investors. For the CARA-normally distributed stock price case, similarly, the standard mean-variance CAPM holds. In our case, since we derive prices for each state by shifting the probability measure, it is not necessary to derive a CAPM explicitly. It is sufficient to replace the utility-weighted probabilities by the risk-adjusted probabilities.

As in Cox, Ross, and Rubinstein [6], if a is the minimum number of moves the stock must make over the next n stages to finish in the money, i.e., $X(j+1) = u^a d^{n-a} S \geq K$, we can write

$$\operatorname{Max}[\{Su^{n-j}d^1 - K\}, 0] = \begin{cases} 0, & j < a \\ Su^{n-j}d^j - K, & j \geq a \end{cases} \tag{16}$$

It follows that

$$\begin{aligned} C = {} & S \textstyle\sum_{j=a}^{n} \dfrac{n!}{(n-j)!j!} p^{n-j}(1-p)^j \dfrac{u^{n-j}d^j}{r^n} \\ & - Kr^{-n} \textstyle\sum_{j=a}^{n} \dfrac{n!}{(n-j)!j!} p^{n-j}(1-p)^j \end{aligned} \tag{17}$$

From Equation (14), we can write $\dfrac{pu}{r} = p'$ and $\dfrac{(1-p)d}{r} = 1 - p'$, so that

$$\begin{aligned} C = {} & S \textstyle\sum_{j=a}^{n} \dfrac{n!}{n-j!j!} p'^{n-j}(1-p')^j \\ & - Kr^{-n} \textstyle\sum_{j=a}^{n} \dfrac{n!}{n-j!j!} p^{n-j}(1-p)^j \\ = {} & S\Phi[a; n, p'] - Kr^{-n}\Phi[a; n, p] \end{aligned} \tag{18}$$

where Φ is the complementary binomial function.

After this, the analysis is identical to Cox, Ross, and Rubinstein [6] in determining the solution when the number of stages becomes very large, keeping in mind that the price changes in each stage become correspondingly smaller. With continuous trading, the solution converges to the Black and Scholes [2] valuation formula.[8] Note that the multiplicative binomial formulation assumed here tends to the lognormal distribution in the limit. This, of course, is identical to the result obtained by Rubinstein [11] and Brennan [4] for the case of constant proportional risk aversion utility and a lognormal distribution of stock prices. Just as the limting case of the Cox, Ross, and Rubinstein [6] formulation with continuous trading is the Black and Scholes [2] model, the result here converges to the Rubinstein [11] and Brennan [4] model as the number of stages becomes very large.

It can be seen that, in the additive binomial case, the value of the call option (analogous to Equation (17)) can be written as

$$C = S \sum_{j=a}^{n} \frac{n!}{n-j!j!} p^{n-j}(1-p)^{j} \frac{(n-j)u' + jd'}{r^{n}}$$

$$- Kr^{-n} \sum_{j=a}^{n} \frac{n!}{n-j!j!} p^{n-j}(1-p)^{j} \tag{19}$$

where a is the number of up-moves to just bring the call into the money and is given by

$$X(j+1) = au' + (n-a)d'S \geq K$$

In this case, the limiting distribution of stock prices is the normal distribution. Since the preferences are CARA, this result is identical to that obtained by Brennan [4] who showed that for joint normally distributed stock prices and aggregate wealth, CARA utility is both necessary and sufficient for a RNVR to hold.

III. The Multivariate Case

The results of the previous sections extend readily to the case of complex contingent claims whose payoffs depend on two or more stochastic variables. A simple example of such a claim is a call option on a dividend-paying stock.[9] In the case of such multivariate contingent claims, the general valuation function presented in Equation (1) has to be modified in two important respects. First, the payoff function $g(X)$ is now dependent on two or more stochastic variables. For instance, in the case of the call option on a dividend paying stock, the cum-dividend stock price could move up or down and the dividend yield could be high or low, following a bivariate binomial process as indicated in Figure 3, where uS and dS are defined as before with associated probabilities q and $(1-q)$, respectively, except that they refer to the cum-dividend stock price. However, in each state, the dividend yield could be high, b_1, or low, b_2, with associated

[8] If the limits are taken so that the stock price moves in a smooth deterministic way but with occasional sudden jumps, we can derive the Cox and Ross [5] jump valuation model.

[9] See Stapleton and Subrahmanyam [12] for other examples of such complex multivariate contingent claims.

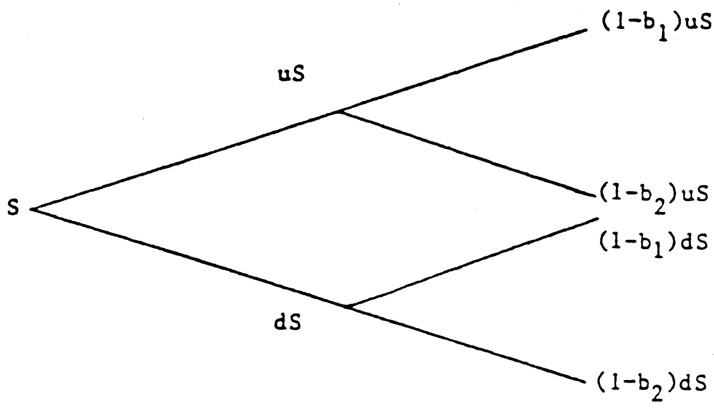

Figure 3. Call Option on a Dividend-Paying Stock

probabilities q_2 and $(1 - q_2)$, respectively. Thus, the ex-dividend stock prices, in the up-state are $(1 - b_1)uS$ and $(1 - b_2)uS$, respectively. The payoffs on the call option are thus dependent on both the end-of-period stock price, X, and the dividend yield b. In general, $g(X)$ is a function of a vector of the underlying stochastic variables.

The other change is that the pricing function $z(X)$ has to be modified to take into account the additional stochastic variables. To avoid complexity in the notation, let us consider the case of two stochastic variables. For the bivariate binomial case, where there are two possible values associated with each variable, Equations (6) have to be changed as follows for the $(n - 1)$-stage case:

$$\begin{aligned} p_1^{n-1}p_2^{n-1} &= q_1^{n-1}q_2^{n-1}z(X_1(1), X_2(1)) \\ p_1^{n-1}p_2^{n-2}(1 - p_2) &= q_1^{n-1}q_2^{n-2}(1 - q_2)z(X_1(1), X_2(2)) \\ &\vdots \\ (1 - p_1)^{n-1}p_2(1 - p_2)^{n-2} &= (1 - q_1)^{n-1}q_2(1 - q_2)^{n-2}z(X_1(n)X_2(n - 1)) \\ (1 - p_1)^{n-1}(1 - p_2)^{n-1} &= (1 - q_1)^{n-1}(1 - q_2)^{n-1}z(X_1(n), X_2(n)) \end{aligned} \tag{20}$$

where p_1 and p_2 are the risk-adjusted probabilities associated with q_1 and q_2, respectively. Equations (20) can be reduced to equations analogous to (7).

$$\begin{aligned} p_1^{n-1}p_2^{n-1} &= q_1^{n-1}q_2^{n-1}z(X_1(1), X_2(1)) \\ z(X_1(1), X_2(1))z(X_1(2), X_2(2)) &= z(X_1(1), X_2(2))z(X_1(2), X_2(1)) \\ &\vdots \\ z(X_1(n), X_2(n))z(X_1(n - 1), X_2(n - 1)) & \\ &= z(X_1(n), X_2(n - 1))z(X_1(n - 1), X_2(n)) \end{aligned} \tag{21}$$

By using arguments similar to those used in Theorem I, we can prove that the sufficient condition for a RNVR to exist when the underlying stochastic variables,

X_1 and X_2, and aggregate wealth, W, are joint multiplicative binomial is that the utility of wealth function of the representative investor is of the CPRA class. Obviously, there is nothing in the analysis that restricts the result to two underlying variables, and hence it holds for the general case where there are n stochastic variables—the necessary and sufficient condition for a RNVR is that the representative investor has CPRA preferences.

As the number of stages becomes very large, taking into account the resulting effect on the magnitude of the changes in each stage, the joint multiplicative binomial formulation tends to the joint lognormal distribution in the limit. This is the same as the result in Stapleton and Subrahmanyam [12], where CPRA preferences are necessary and sufficient for a RNVR in this case.

As before, for the case where the n stochastic variables and wealth, W, are joint additively binomially distributed, CARA preferences are necessary and sufficient for a RNVR. Again, in the limit, this leads to the result in Stapleton and Subrahmanyam [12] for multivariate joint normally distributed X's and W.

IV. Conclusion

Most of the literature on the valuation of contingent claims, such as options, has used the approach pioneered by Black and Scholes [2] that involves setting up a riskless hedge involving the underlying asset and the contingent claim. In contrast, the preference-based approach of Rubinstein [11], Brennan [4], and Stapleton and Subrahmanyam [12] involves the establishment of risk neutral valuation relationships (RNVRs) which are not based on arbitrage considerations. In this paper, we explore the case when continuous trading and hedging are not possible and the underlying stock has a distribution generated by an n-stage binomial process and derive conditions when RNVRs hold. We thus demonstrate the preference-based analogue of the Cox, Ross, and Rubinstein [6] binomial process model which is based on arbitrage considerations. We show that RNVRs hold if the representative investor's utility function belongs to the CARA class (for additive processes) and the CPRA class (for multiplicative processes). Hence, the discrete time RNVRs derived earlier for the cases of the lognormal and normal distributions can be extended to the whole range of finite-order binomial distributions, with the continuous distribution results holding in the limit. Also, multivariate contingent claims, whose payoffs depend on the outcomes of two or more stochastic variables can also be valued using RNVRs derived under similar circumstances. These may be seen as the foundation of the results derived in Stapleton and Subrahmanyam [12].

Appendix

Necessary and Sufficient Conditions for the Utility Function

PROPOSITION I. *If the logarithms of the stock price and aggregate wealth are bivariate binomially distributed, the necessary and sufficient condition for $z(X)$ to be a power function is that the utility function also is a power function.*

Proof of Sufficiency: Suppose the utility function is a power function, the marginal utility function can be written as

$$U'(W) = W^{\gamma} \tag{A1}$$

Consider the first stage of the binomial process

Outcome	Probability	Outcome	Probability
$X_1^1 = uS$	q	$X_1^1 = uS,\ W = W_{11}^1$	qt
		$X_1^1 = uS,\ W = W_{10}^1$	$q(1-t)$
$X_0^1 = dS$	$1-q$	$X_0^1 = dS,\ W = W_{01}^1$	$(1-q)t$
		$X_0^1 = dS,\ W = W_{00}^1$	$(1-q)(1-t)$

where X_i^k is the value of the stock with i up-movements in k stages and W_{ij}^k is the aggregate wealth after i up-movements in the stock and j up-movements in aggregate wealth in k stages and q and t are the probabilities of up-movements in the stock price and wealth, respectively. By definition,

$$z(X) = \frac{E[U'(W \mid X)]}{E[U'(W)]}$$

so that

$$z(uS) = \frac{tW_{11}^{1\gamma} + (1-t)W_{10}^{1\gamma}}{qtW_{11}^{1\gamma} + q(1-t)W_{10}^{1\gamma} + (1-q)tW_{01}^{1\gamma} + (1-q)(1-t)W_{00}^{1\gamma}}$$

$$z(dS) = \frac{tW_{01}^{1\gamma} + (1-t)W_{00}^{1\gamma}}{qtW_{11}^{1\gamma} + q(1-t)W_{10}^{1\gamma} + (1-q)tW_{01}^{1\gamma} + (1-q)(1-t)W_{00}^{1\gamma}} \tag{A2}$$

Since the logarithms of the stock price and aggregate wealth are bivariate binomial we can write a linear relationship between the two variables as follows.

$$W_{11}^1 = (uS)\varepsilon_1 \qquad W_{10}^1 = (uS)\varepsilon_2$$

$$W_{01}^1 = (dS)\varepsilon_3 \qquad W_{00}^1 = (dS)\varepsilon_4 \tag{A3}$$

Substituting the Equations (A3) in (A2) and simplifying yields

$$z(uS) = \frac{(uS)^{\gamma}[t\varepsilon_1^{\gamma} + (1-t)\varepsilon_2^{\gamma}]}{E[U(W)]}$$

$$= C_1(uS)^{\gamma}$$

and

$$z(dS) = \frac{(dS)^{\gamma}[t\varepsilon_3^{\gamma} + (1-t)\varepsilon_4^{\gamma}]}{E[U(W)]}$$

$$= C_2(dS)^{\gamma} \tag{A4}$$

Hence, $z(X)$ is a power function.

It is easy to see that the same argument applies even if there is more than one stage for the binomial process. The only difference is that in Equations (A4), we would have $(n + 1)$ terms, where n is the number of subsequent stages, rather than 2 terms. For example, if j is the number of up-movements,

$$z(u^j d^{n-j} S) = \frac{[u^j d^{n-j} S]^\gamma [\sum_{i=0}^{n} t^{n-i}(1 - t)^i \varepsilon_{i+1}]}{E[U(W)]}$$

$$= C_{n-j+1}(u^j d^{n-j} S)^\gamma \quad \text{(A5)}$$

and $z(X)$ would still be a power function.

Proof of Necessity: Suppose $z(X)$ is of the power type, i.e.,

$$z(X) = k' X^\gamma \quad \text{(A6)}$$

Since

$$z(X) = \frac{E[U'(W \mid X)]}{E[U'(W)]}$$

where the denominator does not depend on X, this implies that

$$E[U'(W \mid X)] = k'' X^\gamma \quad \text{(A7)}$$

Since the $z(X)$ function is, in general, a power function, it should be so for the particular case when log X and log W are perfectly correlated, i.e., when there is no error

$$W_{11}^1 = buS \qquad W_{10}^1 = buS$$

$$W_{01}^1 = bdS \qquad W_{00}^1 = bdS \quad \text{(A8)}$$

where b is a constant.

When log X and log W are perfectly correlated, it follows that the relationship between W and X is one-to-one and, hence,

$$E\ [U'(W \mid X)] = U'(W \mid X) \quad \text{(A9)}$$

Therefore,

$$U'(W \mid X) = k'' X^\gamma = k''(W/b)^\gamma = k'''(W)^\gamma \quad \text{(A10)}$$

Since $U'(W \mid X)$ does not depend on X in this case, it follows that

$$U'(W) = k''' W^\gamma \quad \text{(A11)}$$

which is a power function.

PROPOSITION II. *If the stock price and aggregate wealth are bivariate binomially distributed, the necessary and sufficient condition for $z(X)$ to be an exponential function is that the utility function also is an exponential function.*

Proof of Sufficiency and Necessity: The details of the proof are similar to those in Proposition I except that

$$U'(W) = \exp(\alpha W) \quad \text{(A12)}$$

and

$$X_j^n = S + (n - j)u' + jd', \qquad j = 0, \cdots, n$$

Also, if the stock price and aggregate wealth are bivariate binomial, we can write

$$W_{11}^1 = buS + \varepsilon_1 \qquad W_{10}^1 = buS + \varepsilon_2$$

$$W_{01}^1 = bdS + \varepsilon_3 \qquad W_{00}^1 = bdS + \varepsilon_4 \tag{A13}$$

The other steps in the proof are the same as for the previous case.

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