A Simple Formula for the Expected Rate of Return of an Option over a Finite Holding Period
Author(s): Mark Rubinstein
Source: *The Journal of Finance*, Vol. 39, No. 5 (Dec., 1984), pp. 1503-1509
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2327740
Accessed: 27/11/2009 07:51

THE JOURNAL OF FINANCE • VOL. XXXIX, NO. 5 • DECEMBER 1984

A Simple Formula for the Expected Rate of Return of an Option over a Finite Holding Period

MARK RUBINSTEIN*

ABSTRACT

Under conditions consistent with the Black-Scholes formula, a simple formula is developed for the expected rate of return of an option over a finite holding period possibly less than the time to expiration of the option. Under these conditions, surprisingly, the expected future value of a European option, *even prior to expiration*, is shown equal to the current Black-Scholes value of the option, except that the expected future value of the stock at the end of the holding period replaces the current stock price in the Black-Scholes formula and the future value of a riskless invesment of the striking price replaces the striking price. An extension of this result is used to approximate moments of the distribution of returns from an option portfolio.

I. Expected Option Returns

Consider a call option with striking price K and time to expiration t on a particular underlying stock. Assume:[1]

1. Both an investor and the "market" agree that the annualized discrete interest rate will remain constant and equal to $r - 1$.
2. Both the investor and the market agree that the stock rate of return is lognormal over any holding period.
3. The investor's estimate of the annualized discrete expected rate of return and instantaneous volatility of the stock is $m - 1$ and σ, respectively, while the "market's" respective estimates are $\hat{m} - 1$ and $\hat{\sigma}$.[2]
4. The market price of the call at all times through expiration is set equal to the value it would have according to the Black-Scholes formula for European options, $C(S, K, t, r, \hat{\sigma})$.

* Graduate School of Business Administration, University of California at Berkeley. The author would like to thank the National Science Foundation for funding.

[1] We speak here as though the "market" had a mind of its own. This is a fiction for deriving inferences from security market prices. For example, the "market's" estimate of stock volatility $\hat{\sigma}$ can be inferred, given the Black-Scholes formula, from the implicit volatility contained in the current market price of an associated option.

[2] A purist might bring forth the following objection. In the context of the Black-Scholes [1] derivation, if two investors have different opinions about the volatility of the stock, then at least one will believe there is a riskless arbitrage opportunity. Even worse, if both investors agree the stock price follows geometric Brownian motion, they cannot for long disagree about the volatility since over any finite period an infinite number of observations will be available on a stationary stochastic process. Therefore, for this argument to be technically viable, we need a context for the Black-Scholes formula which admits investor disagreement about volatility. Such a context is provided in a discrete-time development of the Black-Scholes formula in Rubinstein [4].

These assumptions[3] isolate a single source of disagreement between an investor and the market about the *relative* mispricing of an option—the stock volatility. In practice there will be other sources of disagreement (interest rates, dividend yields, and the pricing formula itself), but differences in volatility estimates are likely to prove the most important.

These assumptions also embody a hypothesis about the nature of market disequilibrium or inefficiency. If a security is believed to be over- or underpriced, to measure its expected rate of return over a given time interval, we need some hypothesis about the speed with which this mispricing disappears. In the case of an option that is mispriced relative to its underlying stock, we know that any mispricing must completely disappear on its expiration date if there are to be no general arbitrage opportunities; all option pricing models and all estimators of volatility agree about the value of an option relative to the stock on the expiration date. Therefore, simply to postulate the convergence of an option price to its equilibrium price on its expiration date is of no assistance.

One possibility is to measure the mispricing by the absolute difference between value and price and suppose that this difference dwindles as a linear function of the time remaining to expiration. A more fundamental supposition would be derived from the variables that affect the price of an option. In our case, we have assumed that the source of mispricing lies with differences in estimated volatility of the stock and that both the investor and the market persist in their opinions about the volatility through the option's expiration date. To be sure, a more sophisticated model would allow for the investor to anticipate the market to revise its estimate closer and closer to the investor's as the market accumulated more and more observations of stock price changes through the expiration date.

Under these conditions, we can derive a surprisingly simple formula for the expected future market price of the call $E(\tilde{C}\mid h)$ at the end of holding period $h < t$, measured as a fraction of a year:

> $E(\tilde{C}\mid h)$ is equal to the Black-Scholes European option value $C(S, K, t, r, \sigma)$ except that C is evaluated at
>
> $$C(Sm^h, Kr^h, t, r, \sqrt{a\sigma^2 + (1-a)\hat{\sigma}^2}), \qquad \text{where } a \equiv \frac{h}{t}$$

In other words, the expected future market price of an option at the end of holding period h is equal to the price the option would have today if the current stock price were Sm^h, the striking price Kr^h, the time to expiration t, and the market's estimate of the volatility $\sqrt{a\sigma^2 + (1-a)\hat{\sigma}^2}$, where a is the fraction of the option's time to expiration taken up by the holding period.

When the holding period is very short relative to t, the volatility in this formula is almost entirely determined by the market since $a \simeq 0$ and

$$\sqrt{a\sigma^2 + (1-a)\hat{\sigma}^2} \simeq \hat{\sigma}$$

[3] An unstated but implicit assumption requires that the investor knows the market and that he agree about r, knows that the market uses the Black-Scholes formula to price options, and knows the market's estimate $\hat{\sigma}$. However, it is not required that the investor know the market's estimate $\hat{m}$.

On the other hand, if the option is to be held to expiration, then the volatility is completely determined by the investor since $a \simeq 1$ and

$$\sqrt{a\sigma^2 + (1-a)\hat{\sigma}^2} \simeq \sigma$$

The call's annualized expected rate of return is thus

$$\left[\frac{E(\tilde{C} \mid h)}{C}\right]^{1/h} - 1$$

where C is its current market price (using $\hat{\sigma}$).

Table I shows that the well-known instantaneous approximation[4] tends to be adequate when $\sigma = \hat{\sigma}$. However, when the investor believes a call is over- or undervalued relative to its underlying stock (because $\sigma \neq \hat{\sigma}$), then the call expected return can be quite sensitive to the holding period. However, Table I also suggests that if the stock volatility used to calculate the instantaneous approximation is set equal to the weighted average $\sqrt{a\sigma^2 + (1-a)\hat{\sigma}^2}$, then the instantaneous approximation will be very accurate even for over- or undervalued calls.

Proof: Our problem is to integrate

$$E(\tilde{C} \mid h) = \int_{-\infty}^{\infty} [Se^y N(x) - Kr^{h-t} N(x - \hat{\sigma}\sqrt{t-h})] \frac{1}{\sigma\sqrt{2\pi h}} e^{-(y-\mu h)^2/2\sigma^2 h} \, dy$$

N is the standard normal distribution function, where y is the natural logarithm of the return of the stock over holding period h,

$$\mu \equiv \frac{1}{h} E(\log y) = m - \frac{1}{2}\sigma^2$$

and

$$x \equiv \frac{\log(Se^y / Kr^{h-t})}{\hat{\sigma}\sqrt{t-h}} + \frac{1}{2}\hat{\sigma}\sqrt{t-h}$$

Observe that the expression in brackets is the Black-Scholes formula for the value of a call with time $t - h$ left to expiration, volatility $\hat{\sigma}$, and stock price Se^y. Since e^y is lognormal by assumption, y will be normal, which explains the density function multiplying the term in brackets and the limits of the integral.

To solve this integral, first write it as the difference between two integrals, one involving $Se^y N(\tilde{x})$ and the other $N(\tilde{x} - \hat{\sigma}\sqrt{t-h})$. The first integral can be shown equal to

$$Sm^h \int_{-\infty}^{\infty} N(A + Bz) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

and the second integral can be shown equal to

$$Kr^{h-t} \int_{-\infty}^{\infty} N(A' + B'_z) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

[4] See Galai and Masulis [3], Equation 13, page 60.

Table I

Representative Black-Scholes Call Expected Rates of Return[a] ($m = 1.10$, $\sigma = 0.3$, $S = 40$, $r = 1.05$)

		Time to Expiration											
		One Month				Four Months				Seven Months			
		Holding Period (Days)											
$\hat{\sigma}$	K	1	15	30		1	15	60	120	1	15	60	120
0.25	35	0.63	0.64	0.64	—	0.58	0.57	0.55	0.53	0.47	0.47	0.45	0.42
	40	25.89	19.39	15.11	—	1.79	1.72	1.55	1.38	0.95	0.93	0.89	0.77
	45	*	*	*	—	6.84	6.35	5.14	4.11	1.98	1.93	1.77	1.42
0.3	35	0.47	0.47	0.47	—	0.34	0.34	0.34	0.34	0.29	0.29	0.29	0.28
	40	1.08	1.07	1.06	—	0.48	0.48	0.48	0.48	0.37	0.37	0.37	0.36
	45	2.38	2.37	2.36	—	0.66	0.66	0.65	0.65	0.45	0.45	0.45	0.44
0.35	35	0.21	0.22	0.24	—	0.15	0.15	0.15	0.16	0.15	0.15	0.15	0.16
	40	−0.58	−0.61	−0.64	—	−0.01	−0.01	−0.01	−0.02	0.09	0.09	0.09	0.08
	45	−0.98	−0.99	−0.99	—	−0.24	−0.25	−0.27	−0.30	−0.01	−0.01	−0.02	−0.03

r, m, σ, $\hat{\sigma}$, and the option expected rates of return are expressed in annual terms. Option rates of return are *not* in percent so that 0.55 means at 55% rate of return.

* Greater than 10,000.

[a] No adjustment for dividends.

where A, B, A', and B' are constants depending on the nonstochastic parameters in the original integral. Now, using the fact that

$$\int_{-\infty}^{\infty} N(A + Bz) \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = N\left(\frac{A}{\sqrt{1 + B^2}}\right)$$

the result follows.

II. Extensions

If there are to be no general arbitrage opportunities, at all times during their lives, otherwise identical puts and calls must obey the put-call parity relationship

$$P = C - S + Kr^{-t}$$

Thus, at time h into the future:

$$E(\tilde{P} \mid h) = E(\tilde{C} \mid h) - Sm^h + Kr^{-(t-h)}$$

Since we now know a simple way to compute $E(\tilde{C} \mid h)$, we also have a simple way to compute the expected future value of a put or its annualized expected rate of return

$$\left[\frac{E(\tilde{P} \mid h)}{P}\right]^{1/h} - 1$$

Since, knowing $m - 1$, the expected rate of return of a stock, we can calculate the expected rates of return of its associated options, we clearly have a convenient way to calculate the annualized expected rate of return of an arbitrary portfolio of options. Let $V_j(J)$ be the value of option j associated with the underlying stock J, so that V_j is either the value of a call, a put, or perhaps the underlying stock itself. If $n_j(J)$ represents the number of units of option j associated with the underlying stock J in a portfolio, then the annualized portfolio expected rate of return is

$$\left[\textstyle\sum_J \sum_{j \in J} n_j(J) \dfrac{E(\tilde{V}_j(J) \mid h)}{V_j(J)}\right]^{1/h} - 1$$

The notation $j \in J$ means that the interior summation is taken over all items in the portfolio associated with stock J. Since different options have different lives, an important limitation of this formula is that h must be less than or equal to the life of the option with the shortest time to expiration. Otherwise, we would be forced to prespecify what we intend to do with the proceeds of options on their expiration dates.

A more interesting extension is motivated by a recent article by Bookstaber and Clarke [2]. A difficult problem faced by many option investors is to estimate the probability distribution of portfolio returns. They may construct complex option positions involving several different underlying stocks and options which have returns complicated by the unique sensitivity of each option to its underlying stock price and the time remaining to its expiration. Using our approach for calculating option expected returns, we may be able to approximate the probability distribution of returns of possibly complex option portfolios.

To see how this might be done, first suppose each underlying stock annualized discrete rate of return $r_J - 1$ is related to the corresponding rate of return on the market portfolio $r_M - 1$ according to:

$$\ln r_J - \ln r = \beta_J[\ln r_m - \ln r] + \ln \varepsilon_J$$

where

$$\rho(\ln \varepsilon_J, \ln r_M) = 0 \quad \text{and} \quad \rho(\ln \varepsilon_J, \ln \varepsilon_{K \neq J}) = 0$$

Furthermore, assume all stocks under examination have returns which are jointly lognormally distributed with each other and with r_M. Technically, except for an uninteresting degenerate case, this cannot be true for all securities in the economy, so we must presume we are only looking at a subset of these securities. Since we assume β_J is a constant, (ε, r_M) must be jointly lognormal, so that $(\ln \varepsilon, \ln r_M)$ must be jointly normal. Using the formulas for the conditional moments of jointly normal random variables and since $\rho(\ln \varepsilon, \ln r_M) = 0$:

$$E(\ln \varepsilon \mid \ln r_M) = \alpha + \rho\left(\frac{\omega}{\sigma_M}\right)(\ln r_M - \mu_M) = \alpha$$

$$\mathrm{Var}(\ln \varepsilon \mid \ln r_M) = (1 - \rho^2)\omega^2 = \omega^2$$

where by definition

$$\alpha \equiv E(\ln \varepsilon) \qquad \text{and} \qquad \omega^2 \equiv \mathrm{Var}(\ln \varepsilon)$$

Armed with this analysis, we are prepared to derive the expected future value of a call after time h conditional on the realized market portfolio rate of return through time h being r_M:

> $E(\tilde{C} \mid r_M, h)$ is equal to the Black-Scholes European option value $C(S, K, t, r, \sigma)$ except that C is evaluated at
>
> $$C\left(Sr^{h(1-\beta)}e^{\alpha h}e^{(1/2)\omega^2 h}\, r_M^{h\beta},\ Kr^h,\ t,\ r,\ \sqrt{a\omega^2 + (1-a)\hat{\sigma}^2}\right),$$
>
> $$\text{where } a \equiv \frac{h}{t}$$

Proof: Restating the stock return generating process in multiplicative form, $r_J = r^{1-\beta} r_M^{\beta} \varepsilon$. As in the previous unconditional case:

$$E(\tilde{C} \mid h) = \int_{-\infty}^{\infty} \tilde{C}(Sr_J^h) f(r_J^h)\, dr_J = \int_{-\infty}^{\infty} E(\tilde{C} \mid r_M^h) g(r_M^h)\, dr_M$$

The conditional expected future value of a call at the end of the holding period is:

$$E(\tilde{C} \mid r_M, h) = \int_{-\infty}^{\infty} \tilde{C}(Sr^{h(1-\beta)} r_M^{h\beta} \varepsilon^h) k(\varepsilon \mid r_M)\, d\varepsilon$$

$$= \int_{-\infty}^{\infty} \left[Sr^{h(1-\beta)} r_M^{h\beta} e^y N(\tilde{x}) - Kr^{h-t} N(\tilde{x} - \hat{\sigma}\sqrt{t-h}) \right]$$

$$\cdot \frac{1}{\omega\sqrt{2\pi h}} e^{-(y-\alpha h)^2/2\omega^2 h} \, dy$$

This integral is formally equivalent to the integral in the unconditional case except S is replaced by $Sr^{h(1-\beta)} r_M^{h\beta}$, m is replaced by $e^{\alpha + (1/2)\omega^2}$, σ is replaced by ω, and μ is replaced by α.

This result can now be used to approximate the probability distribution of returns of option portfolios. Armed with $E(\tilde{C} \mid r_M, h)$ and following our previous discussion, it is a simple matter to calculate conditional expected rates of return for the portfolio $E(\tilde{r}_P \mid r_M, h)$ after holding period h, where the portfolio P possibly contains puts and calls on a variety of stocks, as well as the stocks themselves. The unconditional expected rate of return of the portfolio is exactly

$$E_h \equiv \int_0^{\infty} E(\tilde{r}_P \mid \tilde{r}_M, h) f(\tilde{r}_M) \, dr_M$$

where $f(r_M)$ is the density function describing the lognormal distribution of returns for r_M. Following Bookstaber and Clarke [2], if the portfolio P is sufficiently "well-diversified" then the variance of r_P can be approximated by

$$\int_{-\infty}^{\infty} [E(\tilde{r}_P \mid \tilde{r}_M, h) - E_h]^2 f(\tilde{r}_M) \, dr_M$$

Other moments of the portfolio return distribution can be calculated in a similar manner.

REFERENCES

1. Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May/June 1973), 637–59.
2. Richard Bookstaber and Roger Clarke. "Options Can Alter Portfolio Return Distributions." *Journal of Portfolio Management* (Spring 1981), 63–70.
3. Dan Galai and Ronald W. Masulis. "The Option Pricing Model and the Risk Factor of Stock." *Journal of Financial Economics* 3 (January/March 1976), 53–81.
4. Mark Rubinstein. "The Valuation of Uncertain Income Streams and the Pricing of Options." *Bell Journal of Economics* 7 (Autumn 1976), 407–25.