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On Testing the Arbitrage Pricing Theory: Inter-Battery Factor Analysis

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ABSTRACT

This paper tests the Arbitrage Pricing Theory (APT) by estimating the factor loadings that are consistent between two industry groups of securities. One of the pitfalls in the study by Roll and Ross is that the factors estimated in one group may not be the same with the factors estimated in another group. This raises some concerns on the acceptability of their conclusions. For our study, we employ inter-battery factor analysis which enables us to estimate factor loadings by constraining the factors to be the same between two different groups. Our results show that there seem to be five or six inter-group common factors that generate daily returns for two industry groups of securities, and these inter-group common factors do not seem to depend on the size of groups. Also, based on our cross-sectional tests on the risk premia, we conclude that the APT should not be rejected.

THE ARBITRAGE PRICING THEORY (APT) introduced by Ross [35, 36] provides another model for explaining the relationship between return and risk. The APT may be conceived as an alternative to the well-known Capital Asset Pricing Model (CAPM) introduced by Sharpe [39], Lintner [27], and Mossin [29] and later modified by Black [1]. Both models assert that every asset must be compensated only according to its systematic risk. One of the major differences is that, in the CAPM, the systematic risk of an asset is defined to be the covariability of the asset with the market portfolio, whereas, in the APT, the systematic risks are defined to be the covariability with not only one factor but also possibly with several economic factors. Another difference is that the CAPM requires the economy to be in equilibrium whereas the APT requires only that the economy has no arbitrage opportunities. Note that the absence-of-arbitrage condition is necessary but not sufficient for the economy to be in equilibrium. Thus, the APT is a more fundamental relationship than the CAPM in the sense that a rejection of the APT implies the rejection of the CAPM but not vice versa.

Despite numerous attempts, the CAPM has not been satisfactorily tested and, furthermore, Roll's critique [32, 33] casts serious doubts on the testability of the CAPM itself. Roll argues that the linear relationship between return and risk will hold for any efficient portfolio. Thus, unless the market portfolio is observed, a test of the CAPM using ex post data is ambiguous; it results only in testing the

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efficiency of a proxy portfolio used in lieu of the market portfolio in estimating the systematic risk, i.e., betas, of individual assets.

The APT has attracted the attention of several empirical researchers, but the tests provided conflicting evidence with respect to the acceptability of the APT. There were studies by Roll and Ross [34], Chen [8], Hughes [19], and Brown and Weinstein [4] in support of the APT, and studies by Reinganum [31] and Dhrymes, Friend, and Gultekin [12] not in support of the APT.

All of these studies[1] were carried out in two stages. The first stage involves estimating the systematic risks for each asset using factor analysis, and the second stage involves testing the implications of the APT using cross-sectional regression analysis. Unfortunately, there are several problems in a factor analysis which are not resolved, namely, number of factors problem, nonuniqueness of factor loadings problem, and indeterminancy of factor scores problem.[2] More importantly, from a computational point of view, it is infeasible to carry out factor analysis on a set of more than 300 securities. Hence, one is forced to rely on either the portfolio approach first introduced by Chen or a group approach first introduced by Roll and Ross.

The group approach divides the entire set of securities into several groups of a manageable size; for example, Roll and Ross grouped securities into 42 groups of 30 securities each. Even though a study by Cho, Elton, and Gruber [10] shows that this methodology is rather robust in picking out the right number of factors, it still has a problem of factor comparability. That is, the estimated factor loadings are unique only up to an orthogonal transformation and thus if one were to carry out separate factor analyses for each group, it would be necessary to verify whether the factors were the same across different groups before making any generalizations over the entire sample. Brown and Weinstein made an attempt to compare factors that were obtained in two different groups. They divided a group of 60 securities into two subgroups of 30 securities each, and carried out factor analyses on each of these three groups by forcing the number of factors to be three. Two separate factor analyses on two subgroups of 30 securities do not constrain the factors to be the same, whereas a factor analysis on the group of 60 securities constrains the factors to be the same. Then, they compared the constrained residual sum of squares to the unconstrained residual sum of squares using F-statistics (see (A8) in [4]). The constrained residual sum of squares was obtained by analyzing the entire group of 60 securities and the unconstrained residual sum of squares was obtained by combining the residual sum of squares from the two subgroups. They concluded that "the three factors that best represent the observed variation in the data do not significantly differ across groups." One should note that, in Brown and Weinstein, two sets of factors were considered to be the same if one set of factors was an orthogonal transfor-

[1] Brown and Weinstein derived a set of simultaneous equations which can be solved iteratively to obtain joint estimates of factor loadings and factor scores, i.e., risks and risk premia, respectively. However, in their analyses, they carried out only one iteration which is equivalent to two stage procedures.

[2] These problems lead to misspecification of the factor structure underlying APT and ambiguity in testing nonzero risk premia. For details, see Dhrymes, Friend, and Gultekin [12].

mation of the other set of factors.[3] This is so because the residual sum of squares is invariant under the orthogonal transformation of factors. That is, by forcing only the number of factors to be the same between two subgroups of securities, they did not prohibit the factors to rotate freely.

In this study, we try to solve this factor comparability problem by employing inter-battery factor analysis rather than a traditional factor analysis. Inter-battery factor analysis was first introduced by Tucker [41] and later improved by Browne [5]. The method is very similar to the canonical correlation analysis in that it estimates the factor loadings for two groups of securities by examining only the inter-group correlation matrix. If two groups had the same set of factors then it should be reflected in the inter-group correlation matrix. Conversely, the inter-group correlation matrix should reflect only those factors that are common to two groups and not those factors that are common for only one group. Thus, this method allows us to estimate the factor loadings by constraining the factors to be the same between two groups of securities. Also, we may employ a larger set of securities than those used by a traditional factor analysis because we can concentrate only on a submatrix. Furthermore, we do not have to put any restrictions on the number of factors in the cross-sectional analyses. Unlike the previous studies which used the same number of factors for the entire sample, we can let the sample determine how many factors to use without altering the significance levels.

The remainder of this paper is structured as follows. In Section I, the APT is briefly described with some of the theoretical problems which are encountered in testing the validity of the APT. Section II discusses the methodology used in this paper along with a brief description of inter-battery factor analysis, and Section III describes the data and empirical results. Finally, some concluding remarks are made in Section IV and a more detailed discussion of inter-battery factor analysis is carried out in the Appendix.

I. The Arbitrage Pricing Theory

In this section, we briefly describe how the APT is derived. Suppose there exist k factors in the economy which generate the return of an individual security i for the time period $t = 1, 2, \cdots, T$ as follows:

$$\tilde{R}_{it} = E(\tilde{R}_i) + b_{i1}\tilde{\delta}_{1t} + b_{i2}\tilde{\delta}_{2t} + \cdots + b_{ik}\tilde{\delta}_{kt} + \tilde{\varepsilon}_{it} \tag{1}$$

where $E(\tilde{R}_i)$, $i = 1, 2, \cdots, n$, is the expected return of the security i; $\tilde{\delta}_j$, $j = 1, 2, \cdots, k$, are unobserved economy factors; b_{ij} is the sensitivity of a security i to the economy factor j; and $\tilde{\varepsilon}_i$ are the idiosyncratic risks of securities. Furthermore, assume $E(\tilde{\delta}_j) = 0$ for $j = 1, 2, \cdots, k$, $E(\tilde{\varepsilon}_i) = 0$ for $i = 1, 2, \cdots, n$, $E(\tilde{\varepsilon}_i\tilde{\varepsilon}_h) = 0$ for $i \neq h$, and $E(\tilde{\varepsilon}_1^2) = \sigma_i^2 < \infty$.

Then, one can always form a portfolio such that $\underset{\sim}{\eta}^t\underset{\sim}{e} = 0$ and $\underset{\sim}{\eta}^t\underset{\sim}{b}_j = 0$ for $j = 1, 2, \cdots, k$ where $\underset{\sim}{e}$ is an n-dimensional vector consisting of ones, $\underset{\sim}{b}_j$ is an n-dimensional vector consisting of b_{ij}, and $\underset{\sim}{\eta}$ is an n-dimensional vector consisting of portfolio weights. This follows because the number of factors (k) is assumed

[3] This had no effect on their findings because their analyses were based on the residual sum of squares.

to be very small compared to the number of securities (n).[4] Note that this portfolio requires no net investment and it contains no systematic risks. Furthermore, if this portfolio contains no idiosyncratic risk, i.e., $\underline{\eta}^t\tilde{\underline{\varepsilon}} = 0$ where $\tilde{\underline{\varepsilon}}$ is an n-dimensional vector consisting of $\tilde{\varepsilon}_i$'s, then it should earn no return in order that there be no arbitrage opportunities, i.e., $\underline{\eta}^t\tilde{\underline{R}} = 0$ where $\tilde{\underline{R}}$ is an n-dimensional vector consisting of $\tilde{R}_i$'s. Hence, one can conclude that $\underline{\eta}^t\underline{e} = 0$ and $\underline{\eta}^t\underline{b}_j = 0$ implies $\underline{\eta}^tE(\tilde{\underline{R}}) = 0$ if $\underline{\eta}^t\tilde{\underline{\varepsilon}} = 0$ where $E(\tilde{\underline{R}})$ is an n-dimensional vector consisting of $E(\tilde{R}_i)$'s. Therefore, from linear algebra, one can see that $E(\tilde{\underline{R}})$ must be a linear combination of $\underline{e}$ and $\underline{b}_j$'s.[5] That is, there exist some constants $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_k$ such that

$$E(\tilde{R}_i) = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_k b_{ik} \tag{2}$$

One should bear in mind that the requirement of $\underline{\eta}^t\tilde{\underline{\varepsilon}} = 0$ is quite crucial in deriving the above APT pricing equation (2). If there exists no portfolio with $\underline{\eta}^t\tilde{\underline{\varepsilon}} = 0$, then the APT can at best be an approximate relationship. Ross [35] and others [6, 11, 18, 20] have shown that $\underline{\eta}^t\tilde{\underline{\varepsilon}} = 0$ will hold in an economy with an infinite number of assets but the APT will not hold exactly in an economy with a finite number of assets. However, as shown by Ross [35], Dybvig [13], and Grinblatt and Titman [16], the magnitude of this mispricing is very small and can be ignored for empirical purposes. According to Grinblatt and Titman, in a finite asset economy where an asset i has a nonzero idiosyncratic risk and a positive aggregate supply, the APT pricing equation (2) underpredicts the expected return on asset i, and the magnitude of mispricing is bounded by

$$E(\tilde{R}_i) - \lambda_0 - \textstyle\sum_{j=1}^{k} \lambda_j b_{ij} \leq AX_i\sigma^2(\tilde{\varepsilon}_i) \tag{3}$$

A is a risk aversion measure which, assuming returns are normal, is the Rubinstein [37] measure of relative risk aversion $-wE(U'')/E(U')$, where w is per capita wealth, and U is a twice differentiable utility function with $U' > 0$ and $-b < U'' < 0$ for some constant b. X_i is the proportion of asset i in the economy's wealth and $\sigma^2(\tilde{\varepsilon}_i)$ is the variance of the idiosyncratic risk of asset i. Grinblatt and Titman estimated this magnitude of mispricing to be at most 0.2% per year. This relatively small magnitude of mispricing allows one to ignore the approximate nature of the APT and thus the validity of the APT can be tested. After all, the acceptability of any economic model should be based on its ability to explain the relevant economic phenomena.

II. Testing Procedures and Hypotheses

Before discussing the testing procedures and hypotheses used in this study, we first briefly describe inter-battery analysis. (For details, see the Appendix.)

[4] We have n unknown portfolio weights that have to satisfy $(k + 1)$ equations; $\underline{\eta}^t\underline{e} = 0$ and $\underline{\eta}^t\underline{b}_j = 0$. Thus, if n is larger than $k + 1$, then we have an infinite number of solutions that satisfy the above equations.

[5] Recall that there are an infinite number of solutions that satisfy $\underline{\eta}^t\underline{e} = 0$ and $\underline{\eta}^t\underline{b}_j = 0$ because $n > k + 1$. Thus, if $E(\tilde{\underline{R}})$ is not a linear combination of $\underline{e}$ and $\underline{b}_j$, then we can always find a solution of $\underline{\eta}$ such that $\underline{\eta}^t\underline{e} = 0$, $\underline{\eta}^t\underline{b}_j = 0$, and $\underline{\eta}^tE(\tilde{\underline{R}}) \neq 0$.

A. Inter-Battery Factor Analysis

Suppose that we have two groups of securities and k common factors such that the return of each security is generated according to the following equation

$$\begin{bmatrix}\tilde{\underline{R}}_{1t}\\ \tilde{\underline{R}}_{2t}\end{bmatrix} = \begin{bmatrix}E(\tilde{\underline{R}}_1)\\ E(\tilde{\underline{R}}_2)\end{bmatrix} + \begin{bmatrix}\underline{B}_1\\ \underline{B}_2\end{bmatrix}\tilde{\underline{\delta}}_t + \begin{bmatrix}\tilde{\underline{\varepsilon}}_{1t}\\ \tilde{\underline{\varepsilon}}_{2t}\end{bmatrix} \tag{4}$$

where 1 and 2 represent two different groups of securities. $\underline{B}_h$ are $n_h \times k$ factor loading matrices, n_h is the number of securities in a group h, $\tilde{\underline{\delta}}$ is a $k \times 1$ column vector of common factors, and $\tilde{\underline{\varepsilon}}_h$ are $n_h \times 1$ column vectors of idiosyncratic risks for $h = 1$ and 2. We assume that $\tilde{\underline{\delta}}_t$, $\tilde{\underline{\varepsilon}}_{1t}$, and $\tilde{\underline{\varepsilon}}_{2t}$ have zero means and $\tilde{\underline{\varepsilon}}$'s are orthogonal to $\tilde{\underline{\delta}}$. For convenience, we assume that the covariance matrix of $\tilde{\underline{\delta}}_t$ is an identity matrix. The covariance matrix of $[\tilde{\underline{\varepsilon}}_{1t}, \tilde{\underline{\varepsilon}}_{2t}]^t$ is assumed to be block diagonal with the covariance matrices of $\tilde{\underline{\varepsilon}}_{ht}$ being ψ_h for $h = 1$ and 2. Note that we allow more than k common factors within each group by not assuming ψ_h to be a diagonal matrix.

Then, a traditional factor analysis would find estimates $\hat{B}_1$, $\hat{B}_2$, $\hat{\psi}_1$, and $\hat{\psi}_2$ such that

$$\hat{\Sigma} = \begin{bmatrix}\hat{\underline{B}}_1\hat{\underline{B}}_1^t + \hat{\underline{\psi}}_1 & \hat{\underline{B}}_1\hat{\underline{B}}_2^t\\ \hat{\underline{B}}_2\hat{\underline{B}}_1^t & \hat{\underline{B}}_2\hat{\underline{B}}_2^t + \hat{\underline{\psi}}_2\end{bmatrix} \tag{5}$$

is close to the sample covariance matrix

$$S = \begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix} \tag{6}$$

Observe that $\hat{\underline{B}}_1\hat{\underline{B}}_2^t$ should be close to S_{12}. Therefore, one may estimate loadings $\hat{\underline{B}}_1$ and $\hat{\underline{B}}_2$ by examining only S_{12} rather than the entire sample covariance matrix S. This is exactly what the inter-battery factor analysis is designed to do in estimating loadings.

One advantage of inter-battery factor analysis is that we do not have to estimate unique variances; this eliminates estimation problems caused by the Heywood cases. However, an even better advantage for our purpose here is that we can estimate the factor loadings of two groups by focusing on a smaller dimensioned inter-group covariance matrix, i.e., S_{12}. Furthermore, since we can estimate factor loadings that are common only to two groups, we can eliminate the effects of group-specific local factors and have a more powerful test in the investigation of the relationship among assets. It is essential to isolate common factors from group-specific local factors and idiosyncratic risks because, in pricing an asset, we can diversify away the effects of local factors but not those of common factors. This ability to estimate common factor loadings between two groups also enables us to employ the general linear hypothesis test—that is, the Chow test—which is very efficient in testing linear constraints.

In estimating inter-group common factor loadings, Tucker [41] used the unweighted least squares methodology that can only be solved iteratively. Recently, assuming the Wishart distribution on a sample covariance matrix, Browne

[5] derived the maximum-likelihood estimates in a closed form[6] along with asymptotic statistical properties which may be used in determining the number of common factors. It turns out that maximum likelihood inter-battery factor analysis is very similar to canonical correlation analysis in that estimates of inter-battery factor loadings may be computed by rescaling correlation coefficients between the original variables and the canonical variables obtained in the canonical correlation analysis. The difference is that inter-battery factor analysis attempts to explain the correlation coefficients among variables using a single set of unobservable factor variables whereas the canonical correlation analysis attempts to explain the correlation coefficients among variables using two sets of observable linear combinations of variables, i.e., canonical variables.

B. Testing Procedures and Hypotheses

In order to estimate common factor loadings $\underline{B}_1$ and $\underline{B}_2$ in Equation (4), we used the correlation matrix of the combined securities in groups 1 and 2. The correlation matrix is used because it is known to be more stable across time than the covariance matrix. Note that, since these estimates are based on the correlation matrix, we must rescale them by multiplying by the corresponding standard deviations to make them consistent with Equation (4).

Once we have estimates of common factor loadings, we can perform cross-sectional tests in order to verify the validity of the APT. Cross-sectional tests are carried out as in Brown and Weinstein by comparing the risk-free rate and the risk premia between two different groups using the Chow test. We consider five hypotheses: the first is that the risk-free rate is the same between two groups; the second is that the risk premia are the same between two groups; the third is that both the risk-free rate and the risk premia are the same between two groups; the fourth is that the risk-free rate is different from zero while the risk premia are constrained to be the same between two groups; and the fifth is that the risk-free rate is different from zero while the risk premia are allowed to differ between two groups.

Below, we discuss the statistics used for these tests by focusing on the first test. If the APT holds, then the expected returns of securities must be linear combinations of factor loadings. In other words,

$$E(\underline{\tilde{R}}_h) = [\underline{e}:\underline{B}_h]\begin{bmatrix}\lambda_{0h} \\ \underline{\Lambda}_h\end{bmatrix} \quad \text{for} \quad h = 1 \text{ and } 2 \tag{7}$$

where λ_{0h} represents the risk-free rate and $\underline{\Lambda}_h$ represents a column vector of k risk premia corresponding to k common factors for the group h. Hence, by substituting (7) into (4) and rearranging the terms, we get

$$\begin{bmatrix}\underline{\tilde{R}}_{1t} \\ \underline{\tilde{R}}_{2t}\end{bmatrix} = \begin{bmatrix}\underline{e}:\underline{B}_1: & 0 \\ 0 & :\underline{e}:\underline{B}_2\end{bmatrix}\begin{bmatrix}\lambda_{01} \\ \underline{\Lambda}_1 \\ \lambda_{02} \\ \underline{\Lambda}_2\end{bmatrix} + \begin{bmatrix}\underline{\tilde{U}}_{1t} \\ \underline{\tilde{U}}_{2t}\end{bmatrix} \tag{8}$$

[6] Traditional factor analyses are based on iterative schemes, which do not guarantee the global optimum. On the other hand, this closed form solution always yields the global optimum.

where $\tilde{\underline{U}}_{ht} = \underline{B}_h \tilde{\underline{\delta}}_t + \tilde{\underline{\varepsilon}}_{ht}$ for $h = 1$ and 2. This relationship must hold for all t, which implies that, by taking averages across time, the following relationship must also hold:

$$\begin{bmatrix} \underline{\bar{R}}_1 \\ \underline{\bar{R}}_2 \end{bmatrix} = \begin{bmatrix} \underline{e}:\underline{B}_1: & 0 \\ 0 & :\underline{e}:\underline{B}_2 \end{bmatrix} \begin{bmatrix} \lambda_{01} \\ \underline{\Lambda}_1 \\ \lambda_{02} \\ \underline{\Lambda}_2 \end{bmatrix} + \begin{bmatrix} \underline{\bar{U}}_1 \\ \underline{\bar{U}}_2 \end{bmatrix} \tag{9}$$

where $\underline{\bar{R}}_h = \sum_{t=1}^{T} \underline{R}_{ht}/T$ and $\underline{\bar{U}}_h = \sum_{t=1}^{T} \underline{U}_{ht}/T$ for $h = 1$ and 2. This is the unconstrained regression equation.

If we now force the risk-free rate to be the same between the two groups 1 and 2, then Equation (9) reduces to

$$\begin{bmatrix} \underline{\bar{R}}_1 \\ \underline{\bar{R}}_2 \end{bmatrix} = \begin{bmatrix} \underline{e}:\underline{B}_1: & 0 \\ \underline{e}:0 & :\underline{B}_2 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \underline{\Lambda}_1 \\ \underline{\Lambda}_2 \end{bmatrix} + \begin{bmatrix} \underline{\bar{V}}_1 \\ \underline{\bar{V}}_2 \end{bmatrix} \tag{10}$$

where λ_0 is the risk-free rate that is constrained to be the same between two groups. The Chow test implies that if the hypothesis of equal risk-free rates were true, then the constrained residual sum of squares from Equation (10), SSE_c, should be close to the unconstrained residual sum of squares from Equation (9), SSE_u. More precisely

$$F = \frac{(\text{SSE}_c - \text{SSE}_u)/(\text{df}_c - \text{df}_u)}{\text{SSE}_u/\text{df}_u} \tag{11}$$

has an F-distribution with $(\text{df}_c - \text{df}_u)$ and df_u degrees of freedom where df_c and df_u are degrees of freedom for the constrained and unconstrained regressions, respectively. For the above regressions, $\text{df}_u = n - (2k + 2)$ and $\text{df}_c = n - (2k + 1)$, where n is the total number of securities in the combined group and k is the number of common factors. Note that in order to carry out cross-sectional generalized least square regressions, we have to know the covariance matrix of $\underline{\bar{U}}$. However, this is not known. Thus, we estimate it by $\hat{\Sigma}/T$ where $\hat{\Sigma}$ is the covariance matrix estimated according to Equation (5) using estimated values of loadings and residual covariances, and where T is the total number of time periods.[7] This would then imply that SSE's for Equations (9) and (10) are $\underline{\bar{U}}^t \Omega^{-1} \underline{\bar{U}}$ and $\underline{\bar{V}}^t \Omega^{-1} \underline{\bar{V}}$, respectively, where $\underline{\bar{U}}^t = [\underline{\bar{U}}_1^t, \underline{\bar{U}}_2^t]$, $\underline{\bar{V}}^t = [\underline{\bar{V}}_1^t, \underline{\bar{V}}_2^t]$, and $\Omega = \hat{\Sigma}/T$.

The statistics for the other four tests can be derived similarly by changing the pattern of the factor loading matrix in the constrained equation (10). For example, the constrained equation for the fifth test is

$$\begin{bmatrix} \underline{\bar{R}}_1 \\ \underline{\bar{R}}_2 \end{bmatrix} = \begin{bmatrix} \underline{B}_1 & 0 \\ 0 & \underline{B}_2 \end{bmatrix} \begin{bmatrix} \underline{\Lambda}_1 \\ \underline{\Lambda}_2 \end{bmatrix} + \begin{bmatrix} \underline{\bar{V}}_1 \\ \underline{\bar{V}}_2 \end{bmatrix} \tag{12}$$

with $\text{df}_c = n - 2k$.

[7] We assume that $\hat{\Sigma}$ is stationary as others have done. Note that $\tilde{\underline{U}}_{ht} = \underline{B}_h \tilde{\underline{\delta}}_t + \tilde{\underline{\varepsilon}}_{ht}$ from Equation (8). Thus, the covariance matrix of $\tilde{\underline{U}}$ can be estimated by Equation (5).

III. Data and Empirical Results

In order to compare our results with other studies, we decided to investigate those stocks that are listed on the New York or American Stock Exchanges on both July 3, 1962 and December 31, 1972 using the 1982 version of the CRSP Daily Returns file. In total, there were 1286 stocks and 2619 trading days to be considered. However, we did some additional screening in selecting our samples. As Sinclair [40] points out, the number of factors tends to depend inversely on the number of observations. Thus, in order to have concurrent trading days for the entire sample, we decided to eliminate those stocks which had more than 10 missing observations. This left us with 1171 stocks which had 1719 concurrent trading days. Note that this also lessens problems that may be caused by infrequent trading. Finally, since our primary objective in this study was to estimate common factors between two different groups, we wanted our groups to be as different as possible so that we could estimate factors that approximate the marketwide common factors. Note that if two groups were really distinct in a sense that there are no group-specific local factors, then inter-battery factor analysis would extract only those factors that were common across the entire securities in the market, i.e., marketwide common factors. Otherwise, we would estimate not only marketwide common factors but also some group-specific local factors that are common to two groups in a sample. Hence, we decided to group these 1171 stocks according to their industry classifications using the first two-digit SIC codes. In so doing, we also made sure that each industry contained at least 20 stocks and not more than 80 stocks. This was necessary in order to have large enough sample sizes in the second stage cross-sectional regressions and to ease the computations in calculating correlation matrices of combined groups. This gave us 22 industry groups and 930 stocks to be investigated. These industry groups are summarized in Table I. Note that there were two industries which had more than 80 stocks. These industries were further divided into two groups each according to their three-digit SIC codes. They are group numbers 15, 16, 18, and 19.

Unlike other studies, most of our samples were constructed by combining two adjacent groups such that one group is overlapped between two adjacent samples. For example, group 3 is in sample 2 and also in sample 3. This allows us to generate more samples and to investigate whether risk premia are the same across all groups rather than just for the two groups used in each sample. However, the industry groups 20, 21, and 22 were arbitrarily combined with group 9 in order to ease computations. The groups 20, 21, and 22 had at least 70 stocks each. In total, we had 21 samples. Results of inter-battery factor analyses on these 21 samples are summarized in Table II. The number of common factors used in our subsequent analyses are reported in Column 3 with their corresponding p-levels in Column 4. We also report p-levels for other numbers of factors; namely, for the numbers that are either one above or one below the number of factors in Column 3. These are reported in Columns 5 through 8.

In determining the number of common factors, we used the significance level of 0.1 rather than 0.5 used by Roll and Ross, and others. In an exploratory factor

Table I

Industry Groups

Group Order	Number of Stocks	SIC* Codes	Industry Names***
1	36	10	Metal Mining, Copper Ores, Lead & Zinc Ores, Gold Ores, Misc. Metal Ores
2	36	13	Crude Petroleum & Natural Gas, Drilling Oil and Gas Wells
3	72	20	Food and Kindred Products, Meat Products, Dairy Products, Bottled-Canned Soft Drinks
4	23	22	Textile Mill Products, Knitting Mills
5	21	23	Apparel and Other Finished Products
6	20	54	Retail-Grocery Stores
7	21	30	Rubber & Misc. Plastic Products, Fabricated Rubber Products, Misc. Plastic Products
8	25	26	Paper and Allied Products
9	28	53	Retail-Department Stores, Retail-Variety Stores
10	31	38	Engineer Lab & Research Equipment, Industrial Measurement Inst., Surgical & Medical Inst.
11	31	29	Petroleum Refining
12	37	32	Glass, Cement, Concrete
13	37	67	Real Estate Investment Trust, Misc. Investing
14	41	34	Metal Cans, Heating Equipment, Hardware, Metal Work, Fabricated Metal Products
15**	47	361–365	Electric & Electronic Machinery Equip. & Suppl. Household Appliance, Radio-TV Receiving Sets
16**	47	366–369	Telephone & Telegraph Apparatus, Electronic Components, Radio-TV Transmitting Equipments
17	50	33	Blast Furnaces and Steel Works
18**	50	491	Electric Services
19**	55	492–494	Natural Gas Transmission, Natural Gas Dist., Electric, Gas & Other Services Combined
20	70	37	Motor Vehicles, Motor Vehicle Parts, Aircraft and Parts, Ship-Boat, Railroad Equipment
21	74	28	Chemicals, Drugs, Perfumes, Paints, Cosmetics
22	78	35	Engines, Farm Machinery, Construction Machinery, Oil Field Machinery, Electronic Computing Equip.

* We used the first two digits of the SIC code except for the groups denoted by **

** We used the three-digit SIC code to further subdivide these groups.

*** We list some industry names that belong to these groups using the COMPUSTAT manual.

Table II

Results of Inter-Battery Factor Analysis

Sample	Group	Factors	p-Level	Factors	p-Level	Factors	p-Level
1	1, 2	6	0.355	5	0.053	7	0.715
2	2, 3	5	0.102	4	0.007	6	0.372
3	3, 4	4	0.329	3	0.053	5	0.737
4	4, 5	3	0.262	2	0.017	4	0.669
5	5, 6	4	0.303	3	0.079	5	0.608
6	6, 7	2	0.389	1	0.091	3	0.731
7	7, 8	2	0.186	1	0.016	3	0.460
8	8, 9	5	0.179	4	0.029	6	0.536
9	9, 10	5	0.388	4	0.069	6	0.729
10	10, 11	4	0.294	3	0.038	5	0.663
11	11, 12	4	0.362	3	0.038	5	0.779
12	12, 13	5	0.226	4	0.031	6	0.663
13	13, 14	4	0.101	3	0.004	5	0.431
14	14, 15	6	0.349	5	0.077	7	0.739
15	15, 16	9	0.134	8	0.018	10	0.414
16	16, 17	4	0.240	3	0.038	5	0.587
17	17, 18	2	0.100	1	0.003	3	0.544
18	18, 19	3	0.103	2	0.009	4	0.375
19*	9, 20	7	0.166	6	0.028	8	0.470
20*	9, 21	5	0.376	4	0.062	6	0.727
21*	9, 22	8	0.193	7	0.037	9	0.453

Note: The number of factors in Column 3 is used for subsequent analyses.

* These samples did not combine two adjacent groups because of the large number of stocks in groups 20, 21, and 22.

analysis, it is not quite clear how one can make the proper choice of the significance level. As Jöreskog points out in his two papers [21, 22]

> "In most exploratory studies, it is not possible to specify a hypothesis on the number of factors. In such a situation the statistical problem is not one of testing a given hypothesis, but rather one of determining an appropriate number of common factors. One usually wants to determine the smallest number of common factors such that the model fits the observed data."
>
> "If the value of χ^2 is obtained, which is large compared to the number of degrees of freedom, this is an indication that more information can be extracted from the data. This can be done by relaxing some restrictions on the common factor space or by introducing additional factors or both. If on the other hand, a value of χ^2 is obtained which is close to the number of degrees of freedom, this is an indication that the model 'fits too well.' Such a model is not likely to remain stable in future samples and all parameters may not have real meaning. When to stop fitting additional parameters can not be decided on a purely statistical basis. This is largely a matter of the experimenter's interpretations of the data based on substantive theoretical and conceptual considerations. Ultimately the criteria for goodness of the model depend on the usefulness of it and the results it produces."

We thought that using a significance level of 0.5 would result in overfitting the model, considering the rather wide departures of samples from the assumptions underlying the inter-battery factor analysis. For example, it is rather well documented that the daily returns are not normally distributed.

In Table II, we first note that the number of stocks used in a sample does not seem to have much effect on the number of factors. The number of stocks in samples 6 through 18 was monotonically increasing but the number of common factors showed no pattern. This is contrary to what Dhrymes, Friend, and Gultekin [12] found when they applied a traditional factor analysis. They found that, for a given significance level, the number of factors increases with the number of stocks used in a sample. The differences in results seem to indicate that a traditional factor analysis extracts not only common factors but also group-specific local factors, but inter-battery factor analysis extracts only common factors.

However, the results in Table II show strong sampling fluctuations in the number of common factors. The number of factors ranges from 2 to 9 and it depends on which two industry groups are paired. This is a little disturbing but, unless one can devise a scheme which can estimate common factors across the entire samples in one step, one must expect these kinds of sampling fluctuations. These sampling fluctuations suggest that more consideration should have been given to designing the sample. Industry classifications using the two-digit SIC code might have been too crude and consequently might have resulted in extracting not only marketwide common factors but also group-specific local factors. For example, there are nine common factors between groups 15 and 16, whereas there are six common factors between groups 14 and 15 and four common factors between groups 14 and 16.[8] Recall that groups 15 and 16 had the same two-digit SIC code. Also, groups 18 and 19 had the same two-digit SIC code and there are three common factors between groups 18 and 19, whereas there are two common factors between groups 17 and 18 and three common factors between groups 17 and 19.[9] The fact that there are more factors between groups 15 and 16 and between groups 18 and 19 implies that we are picking up not only the marketwide common factors but also the group-specific local factors—in these cases, industry factors.

But, why do we seem to have more industry factors between groups 15 and 16 than between groups 18 and 19? In order to answer this, we examined how our groups were constituted using the three-digit SIC code. These are reported in Table III. It shows how many stocks are in each of the three-digit SIC codes. Note that groups 14, 15, 16, 17, 18, and 19 contain 8, 5, 3, 5, 1, and 3 different industries, respectively, according to the three-digit SIC code. Thus, it seems that the extent to which we extract more industry factors depends on the homogeneity of each group.

In order to see how many factors are generating daily returns, we present p-level distributions for our samples using five and six factors along with that of Roll and Ross in Table IV. Roll and Ross concluded that five factors were sufficient in explaining daily returns because about 75% of their samples indicated that five factors were sufficient at a 0.5 significance level. Note the similarities between our results using six factors with the results of Roll and Ross. Thus, if we use a 0.5 significance level, then we would conclude that there

[8] The results for groups 14 and 16 are not reported in Table II.

[9] The results for groups 17 and 19 are not reported in Table II.

Table III

Number of Stocks According to the Third Digit of the SIC Code

Group	First Two Digit SIC	Third Digit SIC/Number of Stocks 1	2	3	4	5	6	7	8	9	Total Number
1	10	2	8	4	17	0	1	0	1	3	36
2	13	29	1	0	0	0	0	0	6	0	36
3	20	9	5	9	10	4	12	3	17	3	72
4	22	10	2	0	1	3	1	2	2	2	23
5	23	1	7	7	4	0	0	0	1	1	21
6	54	20	0	0	0	0	0	0	0	0	20
7	30	9	0	1	1	0	3	7	0	0	21
8	26	0	11	5	4	3	2	0	0	0	25
9	53	23	0	4	0	0	0	0	0	1	28
10	38	6	8	0	5	0	8	4	0	0	31
11	29	25	0	0	0	2	0	0	0	4	31
12	32	3	5	3	9	1	0	9	0	7	37
13	67	14	18	0	0	0	0	0	0	5	37
14	34	4	5	6	8	5	3	0	1	9	41
15	36	7	12	12	5	11	0	0	0	0	47
16	36	0	0	0	0	0	19	20	0	8	47
17	33	25	4	5	0	13	3	0	0	0	50
18	49	50	0	0	0	0	0	0	0	0	50
19	49	0	25	27	3	0	0	0	0	0	55
20	37	34	22	7	4	1	2	0	0	0	70
21	28	18	6	21	11	5	5	2	0	6	74
22	35	5	3	15	9	12	14	13	5	2	78

Table IV

p-Level Distribution (%)

p-Level	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
Five factors	14.3	4.8	14.3	9.5	4.8	4.8	9.5	4.8	9.5	23.8
Cumulative	14.3	19.1	33.4	42.9	47.7	52.5	62.0	66.8	76.3	100.1
Six factors	38.1	19.0	9.5	4.8	4.8	0.0	9.5	0.0	0.0	14.3
Cumulative	38.1	57.1	66.6	71.4	76.2	76.2	85.7	85.7	85.7	100.0
RR	38.1	16.7	7.1	2.4	11.9	2.4	4.8	4.8	9.5	2.4
Cumulative	38.1	54.8	61.9	64.3	76.2	78.6	83.4	88.2	97.7	100.1

RR: Results of Roll and Ross using five factors.

Cumulative: % of samples which had higher than the corresponding p-levels.

are six factors rather than five factors that are common between two groups of stocks. However, if we use a 0.1 significance level, then we would conclude that there are five common factors that are sufficient in explaining daily returns for those stocks in two groups.

Now, we investigate the validity of the APT. For this purpose, we ran tests of the five hypotheses mentioned in Section II.*B* for each sample. In carrying out these tests, we did not force the number of factors to be the same across the entire sample. Instead, we decided to have a similar significance level across the

Table V

Results of Cross-Sectional Generalized Regression Analysis

	Test 1		Test 2		Test 3		Test 4		Test 5	
Sample	F	p-Level	F	p-Level	F	p-Level	F	p-Level	F	p-Level
1	1.168	0.284	1.299	0.272	1.180	0.329	1.466	0.190	1.823	0.171
2	0.816	0.369	1.590	0.170	1.621	0.150	2.665	0.015*	5.603	0.005*
3	0.145	0.704	0.777	0.543	0.713	0.615	2.637	0.022*	2.074	0.132
4	2.439	0.127	2.373	0.086	1.781	0.154	2.562	0.044*	3.048	0.060*
5	3.543	0.069	0.808	0.530	1.396	0.253	1.198	0.334	2.469	0.101
6	0.110	0.743	0.605	0.552	1.982	0.135	1.486	0.227	0.485	0.620
7	0.002	0.969	0.069	0.933	0.100	0.959	0.211	0.931	0.317	0.730
8	3.004	0.091	1.887	0.118	1.777	0.128	2.513	0.030*	3.918	0.028*
9	8.807	0.005*	2.398	0.051	2.280	0.052	4.114	0.001*	10.456	0.000*
10	9.134	0.004*	2.627	0.045*	3.069	0.017*	5.231	0.000*	12.520	0.000*
11	0.322	0.573	0.890	0.476	2.123	0.075	2.868	0.016*	3.696	0.031*
12	0.636	0.428	1.478	0.210	3.332	0.007*	5.719	0.000*	6.976	0.002*
13	2.115	0.150	1.787	0.142	1.443	0.220	2.946	0.013*	4.336	0.017*
14	2.887	0.093	0.665	0.678	0.794	0.595	1.013	0.434	2.259	0.112
15	0.210	0.648	1.100	0.373	1.914	0.056	3.069	0.002*	4.539	0.014*
16	1.633	0.205	0.558	0.694	0.522	0.759	1.219	0.304	2.540	0.085
17	5.717	0.019*	5.847	0.004*	4.350	0.006*	4.435	0.003*	5.050	0.008*
18	2.891	0.092	0.990	0.401	1.065	0.378	2.064	0.077	3.889	0.024*
19	3.289	0.073	3.153	0.005*	2.887	0.007*	3.734	0.001*	6.467	0.002*
20	0.002	0.966	0.723	0.608	0.744	0.615	3.578	0.002*	10.994	0.000*
21	1.749	0.189	1.655	0.121	1.472	0.171	3.308	0.001*	9.194	0.000*

* p-Levels that are smaller than the significance level of 0.05.

Test 1: Results of testing the hypothesis that risk-free rates are the same between two groups in a sample.

Test 2: Results of testing the hypothesis that risk premia are the same between two groups in a sample.

Test 3: Results of testing the hypothesis that risk-free rates and risk premia are the same between two groups in a sample.

Test 4: Results of testing the hypothesis that the risk-free rate in a sample is equal to zero while the risk premia are constrained to be the same between two groups in a sample.

Test 5: Results of testing the hypothesis that the risk-free rate in a sample is equal to zero while the risk premia are allowed to differ between two groups in a sample.

entire sample.[10] Again, the reason is that we did not want to overfit the model by having more than the necessary number of factors.

Table V summarizes our findings. The columns under the heading of Test 1 report the F-statistics and p-levels for the hypothesis that the risk-free rates are the same between two groups. At a 0.05 significance level, there are three cases out of 21 possible cases in which we reject the hypothesis. The columns under the heading of Test 2 report the F-statistics and p-levels for the hypothesis that the risk premia are the same between two groups. Again, at a 0.05 significance level, there are three cases out of 21 possible cases in which we reject the

[10] This has the effect of not having the same number of factors in all samples. As was pointed out by the referee, this may bias the tests in favor of the APT since the APT requires the same number of factors. However, the APT does not say what this number of factors should be. Furthermore, different rotations may yield a different number of factors. Thus, it seems more important to have the same level of significance than to have the same number of factors.

hypothesis. The columns under the heading of Test 3 report the F-statistics and p-levels for the hypothesis that both the risk-free rates and the risk premia are the same between two groups. At a 0.05 significance level, there are four cases out of 21 possible cases in which we reject the hypothesis. These three tests seem to suggest that the APT may not be valid. If our samples were independent, we would have observed only 5% of our samples rejecting the hypothesis. On the contrary, about 15% of our samples reject the hypothesis. However, note that our samples were not independent because a group of securities was contained in two different samples. In addition, by using daily data, we might have violated the normality assumptions when we estimated the factor loadings. Note that the test statistic is based on the normality assumption. These considerations led us to conclude that our data seem to support the APT.

Tests 4 and 5 were carried out to see whether the risk-free rates are different from zero or not. According to Dhrymes, Friend, and Gultekin, the risk-free rate over this period was positive. They estimated the mean weekly rate on Treasury bills over this period to be 0.00084166 with the standard deviation 0.00023444 and the mean daily rate computed as the seventh root of one plus the weekly rate minus one to be 0.0001204. Note that Equation (12) constrains both risk-free rates of two groups in a sample to equal zero, which is more stringent than necessary. We should have tested this using only one group rather than two groups. However, due to sampling fluctuations, the factor loadings estimated for a group of stocks in two adjacent samples were different, and thus there was no unique way in which we could have tested this on a group basis. Results are reported under the headings Test 4 and Test 5. If we exclude those samples which reject the hypothesis of equal risk-free rates, there remain 18 samples and, among these, 11 samples reject the hypothesis that the risk-free rates are equal to zero. If our samples were independent we would expect only 5% of our samples to reject the hypothesis of zero risk-free rates at a 0.05 significance level. Hence, the fact that about 60% of our samples reject the hypothesis strongly indicates that the risk-free rate over this period was nonzero. Furthermore, if we were to use a 20% significance level, then 19 out of 21 samples would indicate nonzero risk-free rates. These findings further support the validity of the APT.

IV. Conclusions

One of the major difficulties in testing the validity of the APT is the problem of comparing factors that are estimated from separate factor analyses in different groups. In this study, we made an attempt to solve this problem by identifying the factors that are common between two groups of stocks. For this purpose, we used the maximum likelihood inter-battery factor analysis which estimates the common factor loadings of two different groups of stocks by examining only the inter-group sample covariance matrix rather than the full sample covariance matrix of two groups combined. In addition, i) we used only those stocks that had the concurrent trading days; ii) we used industry groups; iii) we constructed our samples by combining two adjacent groups with one group being overlapped between two adjacent samples; iv) we used the significance level of 0.1 in estimating the factor loadings; and v) we did not fix the number of factors to be the same for the second stage cross-sectional regression analyses.

Our results indicate that there are five or six inter-group common factors that generate daily returns for two groups and that these inter-group common factors do not depend on the size of groups. Also, the APT could not be rejected in the sense that the risk-free rate and the risk premia are the same across groups and that the risk-free rate is different from zero. We should note that our results could have been further improved if we could estimate marketwide factor loadings that are common across all groups. We believe that the problem of extracting marketwide common factors across all groups can be handled by extending the methodology used in this study to more than two groups. This is left for future research.

Appendix

Inter-Battery Factor Analysis

Let $\tilde{R}_1$ be a vector of n_1 variables in group 1 and let $\tilde{R}_2$ be a vector of n_2 variables in group 2. If there are k factors $\tilde{\delta}$ that are common across two groups, then allowing the possible existence of group-specific local factors we have the following factor structure equations:

$$\begin{aligned} \tilde{R}_1 &= E(\tilde{R}_1) + B^1\tilde{\delta} + L^1\tilde{\eta}^1 + \tilde{\varepsilon}^1 \\ \tilde{R}_2 &= E(\tilde{R}_2) + B^2\tilde{\delta} + L^2\tilde{\eta}^2 + \tilde{\varepsilon}^2 \end{aligned} \tag{A.1}$$

where $\tilde{\eta}^1$ and $\tilde{\eta}^2$ are group-specific local factors with loadings L^1 and L^2 and $\tilde{\varepsilon}^1$ and $\tilde{\varepsilon}^2$ are idiosyncratic risks. We assume that $\tilde{\delta}$, $\tilde{\eta}^1$, $\tilde{\eta}^2$, $\tilde{\varepsilon}^1$, and $\tilde{\varepsilon}^2$ all have multivariate normal distributions with zero means and are orthogonal to each other. For convenience, we assume that the covariance matrix of $\tilde{\delta}$ is an identity matrix. Covariance matrices, ϕ^{hh}, of $\tilde{\eta}^h$ may be any semipositive definite matrices whereas covariance matrices, θ^{hh}, of $\tilde{\varepsilon}^h$ are diagonal matrices.

Equation (A.1), then, implies that the covariance matrix of $\tilde{R}_1$ and $\tilde{R}_2$ is

$$\Sigma = \begin{bmatrix} B^1B^{1^t} & B^1B^{2^t} \\ B^2B^{1^t} & B^2B^{2^t} \end{bmatrix} + \begin{bmatrix} \Psi^1 & 0 \\ 0 & \Psi^2 \end{bmatrix} \tag{A.2}$$

where

$$\Psi^h = L^h\phi^{hh}L^{h^t} + \theta^{hh} \quad \text{for} \quad h = 1, 2$$

Note that the residual covariance matrix

$$\Psi = \begin{bmatrix} \Psi^1 & 0 \\ 0 & \Psi^2 \end{bmatrix}$$

is no longer a diagonal matrix but a block diagonal matrix.

Let the sample covariance matrix S be an estimate of Σ obtained from a sample of $(t + 1)$ independent observations on $\tilde{R}_1$ and $\tilde{R}_2$. Then tS has the Wishart distribution with t degrees of freedom and parameter Σ. Just as in the maximum likelihood factor analysis, we now minimize

$$F(B, \Sigma) = \ln|\Sigma| - \ln|S| + \mathrm{tr}(S\Sigma^{-1}) - n \tag{A.3}$$

where $n = n_1 + n_2$. First derivatives of F with respect to B and Σ are

$$\partial F/\partial B = 2\Sigma^{-1}(S - \Sigma)\Sigma^{-1}B \tag{A.4}$$

$$\partial F/\partial \Psi = \mathrm{Bdiag}[\Sigma^{-1}(S - \Sigma)\Sigma^{-1}] \tag{A.5}$$

where Bdiag(A) represents a block diagonal matrix formed from the principal submatrices of A which correspond to Ψ^1 and Ψ^2 in (A.2). Setting these derivatives equal to zero yields, after some algebraic manipulations, the following relationships among estimates:

$$\hat{B}_1 = S_{12}S_{22}^{-1}\hat{B}_2(\hat{B}_2^t s_{22}^{-1}\hat{B}_2)^{-1} \tag{A.6}$$

$$\hat{B}_2 = S_{21}S_{11}^{-1}\hat{B}_1(\hat{B}_1^t S_{11}^{-1}\hat{B}_1)^{-1} \tag{A.7}$$

Now, substituting (A.7) into (A.6) yields

$$S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}\hat{B}_1 = \hat{B}_1(\hat{B}_2^t S_{22}^{-1}\hat{B}_2)(\hat{B}_1^t S_{11}^{-1}\hat{B}_1) \tag{A.8}$$

Equations (A.7) and (A.8) along with $\Psi = \mathrm{Bdiag}(S - \hat{B}\hat{B}^t)$, which is obtained by setting (A.5) equal to zero, constitute the likelihood equations for $\hat{B}_1$, $\hat{B}_2$, and $\hat{\Psi}$.

In order to solve these equations, we now impose the following conditions:

$$\hat{B}_1^t S_{11}^{-1}\hat{B}_1 = \hat{B}_2^t S_{22}^{-1}\hat{B}_2 = \hat{D}_\rho \tag{A.9}$$

where $\hat{D}_\rho$ is a diagonal matrix. This is possible because B's can be estimated only up to an orthogonal transformation and we can always find an orthogonal matrix that will satisfy (A.9). Then, substituting (A.9) into (A.8), we get

$$S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}\hat{B}_1 = \hat{B}_1\hat{D}_\rho^2 \tag{A.10}$$

Thus, we can see that the diagonal elements of $\hat{D}_\rho^2$ are k of the n_1 eigenvalues of $S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}$ and the columns of $\hat{B}_1$ are the corresponding eigenvectors. Estimates of B_2 are obtained by substituting (A.9) into (A.7); i.e.,

$$\hat{B}_2 = S_{21}S_{11}^{-1}\hat{B}_1\hat{D}_\rho^{-1} \tag{A.11}$$

Furthermore, it can be shown that with these estimates the objective function value F becomes

$$F(\hat{B}, \hat{\Psi}) = -\ln \prod_{i=k+1}^{n_1} (1 - \hat{\rho}_i^2) \tag{A.12}$$

where $\hat{\rho}_i^2$ are the eigenvalues of $S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}$. Consequently, $F(\hat{B}, \hat{\Psi})$ will be minimized if we take the k largest eigenvalues.

In order to test the fit of the model, we use $tF(\hat{B}, \hat{\Psi})$ which can be shown to be minus two times the logarithm of the likelihood ratio. Thus, $tF(\hat{B}, \hat{\Psi})$ has a chi-square distribution and its degrees of freedom are $(n_1 - k)(n_2 - k)$.

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