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Portfolio Analysis Using Single Index, Multi-Index, and Constant Correlation Models: A Unified Treatment

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ABSTRACT

In this study a simple common algorithm which is applicable to seven models is proposed for optimal portfolio selection disallowing short sales of risky securities. The models considered in the analysis consist of a single index model, four multi-index models, and two constant correlation models. Unlike the previous approach, the proposed algorithm does not require explicit ranking of securities. Therefore, it is particularly useful for two multi-index models with orthogonal indices which do not provide any ranking criterion. Also, because of its algorithmic efficiency as demonstrated in a simulation study on models with multiple groups, the approach here can enhance their usefulness in portfolio analysis.

IN A SERIES OF PUBLICATIONS, Elton and Gruber [2] and Elton, Gruber, and Padberg (hereafter EGP) [3–8] have considered extensively portfolio optimization problems using a variety of index models and constant correlation models. These models are Sharpe's [12] single index model, Cohen and Pogue's [1] multi-index model in the diagonal form, a multi-index model with orthogonal indices [8], a constant correlation model with a single group [3], and a constant correlation model with multiple groups [7] (hereafter SIM, MIM-DI, MIM-OR, CCM-S, and CCM-M, respectively). Using these models, EGP have developed some ranking procedures to reach optimal portfolios disallowing short sales of risky securities. All securities in each group are ranked according to their performance as measured by a ratio of excess return to risk. They are sequentially added to feasible portfolios following the ranking hierarchy until the optimal portfolio is reached.

For models where explicit ranking criteria can be established, the EGP approach is easy to implement and it clearly explains why a security belongs or does not belong to a portfolio. Besides the above models, it is also applicable to Cohen and Pogue's [1] multi-index model in the covariance form (hereafter MIM-CO). However, it does not work well for models lacking appropriate criteria to rank securities. In order to overcome ranking difficulties in MIM-OR, EGP [8] have introduced a market portfolio with zero residual risk to establish an alternative ranking criterion. Unfortunately, as EGP have indicated, this causes internal inconsistency in the analysis.

In the present study, a simple algorithm for optimal portfolio selection disal-

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lowing short sales is proposed. The algorithm here does not require explicit ranking of securities and takes relatively few iterative steps to reach the solution. Besides all the above models, it is also applicable to a model which has attracted considerable attention in recent years. This is the general multi-index model with orthogonal indices that provides the return generating process for securities in the arbitrage pricing theory advanced by Ross [10, 11] (hereafter MIM-G). In the arbitrage pricing theory, the indices in MIM-G are factors having influences on security returns, and therefore need not be confined to market and industrial returns as in other multi-index models. Since this general model, like MIM-OR, does not provide any criterion for explicitly ranking securities, the usefulness of the proposed algorithm becomes more obvious.

The analysis presented here starts in Section I with a summary of solutions to portfolio optimization problems for all seven models considered. The derivations of the solutions for MIM-CO and MIM-G that have not been reported previously are given in Appendices A and B, respectively. The proposed algorithm for optimal portfolio selection disallowing short sales is presented in Section II. In Section III, an illustrative example is given, and the efficiency of the proposed algorithm is examined via a simulation study. Section IV concludes this study.

I. A Summary of Optimal Solutions

EGP have shown that the optimal solutions allowing short sales of risky securities for SIM, MIM-DI, CCM-S, and CCM-M have a common expression:

$$z_i = \frac{\gamma_i}{\delta_i^2}\left(\frac{\bar{R}_i - R_f}{\gamma_i} - \psi_h\right), \tag{1}$$

for security i belonging to group h ($i \in G_h$) where $h = 1, 2, \cdots, p$. Clearly, $h = p = 1$ for SIM and CCM-S. Here, $\bar{R}_i$ is the expected return on security i, R_f is the risk-free rate of interest, values of γ_i and δ_i^2 which are defined differently for different models are input data for security i, and ψ_h is a computed value for group h securities using input data for all securities in the portfolio. In SIM γ_i is the beta coefficient, β_i, and in MIM-DI γ_i is b_i, a parameter measuring the responsiveness of security i's return to changes in I_h, an index for industrial group h where security i belongs. In both SIM and MIM-DI, δ_i^2 is the variance of the noise term. In both CCM-S and CCM-M, γ_i is the standard deviation of returns, σ_i, and $\delta_i^2 = (1 - \rho_{hh})\sigma_i^2$, ρ_{hh} being the correlation of returns for all securities in group h, with $h = 1$ for CCM-S. The variable z_i is related to the fraction of funds x_i invested in security i in the following manner. Insofar as the short seller is allowed to use the full proceeds from the transaction, $x_i = z_i/\sum_{j=1}^{n} z_j$, n being the number of securities considered. However, under Lintner's [9] definition of short sales, $x_i = z_i/\sum_{j=1}^{n} | z_j |$.

EGP have also shown for the above four models that the Kuhn-Tucker conditions which are necessary and sufficient for optimality of the solution disallowing short sales are

$$z_i = \frac{\gamma_i}{\delta_i^2}\left(\frac{\bar{R}_i - R_f}{\gamma_i} - \psi_h\right) + \frac{\nu_i}{\delta_i^2}, \tag{2}$$

$$z_i \geq 0, \qquad \nu_i \geq 0, \qquad z_i\nu_i = 0, \tag{3}$$

for $i \in G_h$ and $h = 1, 2, \cdots, p$. The complementarity conditions in (3) require that $\nu_i = 0$ for any security i included in the optimal portfolio (i.e., $z_i > 0$) and that $\nu_i \geq 0$ for any security i excluded from it (i.e., $z_i = 0$). Thus, for all securities included in the optimal portfolio, Equation (2) becomes Equation (1).

As shown in Appendix A, the optimal solutions for the same portfolio problems using MIM-CO also have Expressions (1), (2), and (3) with γ_i and δ_i^2 as defined in MIM-DI. For all these five models, since securities in each group can be ranked according to their excess return to risk ratios, optimal portfolios with short sales disallowed can be formed via the EGP approach by sequentially adding securities to feasible portfolios following the ranking hierarchy.

For MIM-OR, however, the optimal solution allowing short sales obtained by EGP [8] has the form

$$z_i = \frac{1}{\delta_i^2}\,(\bar{R}_i - R_f - \psi\beta_i - \psi_h b_i) \tag{4}$$

for $i \in G_h$ and $h = 1, 2, \cdots, p$. Here, β_i and b_i are parameters measuring the responsiveness of security i's return to changes in the market index I_m and the index I_h for industrial group h, respectively, δ_i^2 is the variance of the noise term of the model, and ψ and ψ_h are computed values using input data for all securities in the portfolio. In the no short sales case, the first Kuhn-Tucker condition for optimality has a ν_i/δ_i^2 term added to the right-hand side of Equation (4) and the complementarity conditions are as given by (3). As Equation (4) indicates, there are three terms, $\bar{R}_i - R_f$, β_i, and b_i (i.e., the excess return and two factor loadings) in the solution for z_i. Unlike the other five models where Equations (1) and (2) apply, an explicit ranking criterion of securities is difficult to establish for MIM-OR, and to implement the EGP approach here becomes difficult.

Likewise, the same difficulty will also be encountered when the EGP approach is applied to MIM-G where the return on security i, R_i, is described to depend linearly on p orthogonal indices I_h, $h = 1, 2, \cdots, p$, i.e.,

$$R_i = b_{i0} + \textstyle\sum_{h=1}^{p} b_{ih}I_h + e_i, \tag{5}$$

for $i = 1, 2, \cdots, n$. Here, b_{i0} is a parameter capturing the part of security i's return which is uncorrelated with the p indices, each b_{ih} is a parameter measuring the responsiveness of security i's return to changes in index I_h, and e_i is random noise with a zero mean and a finite nonzero variance δ_i^2. The p indices have been made orthogonal, i.e., $\text{cov}(I_h, I_k) = 0$, for $h, k = 1, 2, \cdots, p$ and $h \neq k$, via a simple procedure as described by Elton and Gruber in [2, Ch. 6]. Also, it is assumed that $\text{cov}(e_i, I_h) = 0$ and $\text{cov}(e_i, e_j) = 0$ for $h = 1, 2, \cdots, p$ and $i, j = 1, 2, \cdots, n$ with $i \neq j$.

As shown in Appendix B, the optimal solution allowing short sales for the above model is

$$z_i = \frac{1}{\delta_i^2}\,(\bar{R}_i - R_f - \textstyle\sum_{h=1}^{p} \psi_h b_{ih}), \tag{6}$$

for all i, where each ψ_h is a computed value for all securities included in the portfolio. In the no short sales case, the first Kuhn-Tucker condition is Equation (6) with a ν_i/δ_i^2 term added to its right-hand side, and the complementarity conditions are as given by (3). Because of the presence of $b_{i1}, b_{i2}, \cdots, b_{ip}$ in the expression for z_i, it is difficult to rank securities here. Then, like the case of MIM-OR, the EGP approach cannot be used to select portfolios for MIM-G. Therefore, an algorithm which can bypass the requirement of explicitly ranking securities is particularly useful for these two models.

II. A Common Algorithm for Optimal Portfolio Selection

In the following, a common algorithm for optimal portfolio selection disallowing short sales which is applicable to all seven models is proposed. Justifications for the same algorithm as applied to different models are presented separately.

A. *Single Index Model (SIM) and Constant Correlation Model with a Single Group (CCM-S)*

In SIM and CCM-S, securities can be ranked and labelled according to their excess return to γ ratios in the same manner as described by EGP [2, 3, 6]. Explicitly, if all γ's are restricted to be positive and $(\bar{R}_i - R_f)/\gamma_i \geq (\bar{R}_{i+1} - R_f)/\gamma_{i+1}$, security i is ranked no lower than security $i + 1$. With this ranking arrangement, if security $i + 1$ belongs to a portfolio, so does security i. In SIM the restriction of $\gamma_i > 0$ for all i means that the beta coefficients of all securities considered in the analysis must be positive. As estimated beta coefficients for most securities are positive, and the few negative beta coefficients observed are often caused by measurement errors, this restriction is not troublesome at all. In CCM-S, the requirement of $\gamma_i > 0$ for all i is always satisfied because γ_i is the standard deviation of returns of security i.

With all γ's being positive, ψ_1 as a function of the lowest ranking security k in the portfolio, labelled as $\psi_1(k)$ for $k = 1, 2, \cdots, n$, has a single maximum which defines the cut-off rate for the securities to be included in the optimal portfolio.[1] This property can easily be recognized in several numerical examples for these two models in the EGP studies [2, 3, 6]. Based on this property, a simple algorithm can be established to reach optimal portfolios disallowing short sales. Suppose that among the n securities, m top ranking securities actually form the optimal portfolio. For $i = 1, 2, \cdots, m$, $(\bar{R}_i - R_f)/\gamma_i$ must exceed $\psi_1(m)$. Now, if a portfolio is formed using all n securities, the calculations using Equation (1) must produce no more than $n - m$ negative z's. These negative z's do not correspond to the m top ranking securities because, for $i = 1, 2, \cdots, m$, $(\bar{R}_i - R_f)/\gamma_i > \psi_1(m) > \psi_1(n)$, given that $\psi_1(m)$ is the maximum of the function. Since securities with negative z's do not belong to the optimal portfolio, they are removed from the initial portfolio containing all n securities. Equation (1) is used again to compute the z's for the portfolio with a reduced number of securities. Likewise, securities with negative z's are removed from such a portfolio. The same procedure is repeated until all z's are nonnegative. As securities are sequentially removed from the initial portfolio, ψ_1 increases until it reaches its

[1] The proof which is omitted here is available from the author upon request.

maximum, $\psi_1(m)$. The final portfolio is the optimal one containing the m top ranking securities.[2]

If not all of the n securities considered for the analysis using SIM have positive beta coefficients, the optimal portfolio may no longer correspond to ψ_1 at its maximum. Then, the algorithm described above may cause the removal of some securities belonging to the optimal portfolio during the iterations. But these securities can easily be recognized via the Kuhn-Tucker conditions. For any security i in the final portfolio above, Equation (1) is Equation (2) with $\nu_i = 0$, and thus the complementarity condition that $\nu_i = 0$ when $z_i > 0$ is always satisfied. To ensure that $z_i = 0$ for each removed security i, a usually nonzero value is given to ν_i. A negative ν_i obtained from Equation (2) with $z_i = 0$ indicates that Equation (1) must produce a positive z_i. This suggests that security i, which was removed during the iterative procedure, may belong to the optimal portfolio. Therefore, all removed securities with negative ν's are brought back to the portfolio, and the same iterative procedure described above is repeated. When all securities in the portfolio have positive z's, the complementarity conditions for all n securities are examined again. If $\nu_i = 0$ when $z_i > 0$ and $\nu_i \geq 0$ when $z_i = 0$ for all i, the optimal portfolio is obtained. Otherwise, the same procedure is repeated until optimality is reached.

The above algorithm is consistent with EGP's ranking procedure of sequentially adding securities to feasible portfolios. The portfolio with all positive z's reached by sequentially removing securities with negative z's can be treated here as a feasible portfolio obtained during EGP's ranking procedure. The securities in this feasible portfolio must have higher rankings than those outside with the same sign of the beta coefficients. Also, following EGP, if both securities i and j, with β_i and β_j having the same sign, are not in the above feasible portfolio, the one that enters the portfolio first must be of higher ranking. Suppose that according to Equation (2) $z_i = z_j = 0$, $\nu_i < 0$, and $\nu_j > 0$. If both β_i and β_j are positive (negative), $(\bar{R}_i - R_f)/\beta_i$ is greater (less) than $(\bar{R}_j - R_f)/\beta_j$. This indicates that security i has a higher ranking than security j and it deserves to enter the feasible portfolio before security j does. In the algorithm here, security i with $\nu_i < 0$, but not security j with $\nu_j > 0$, is brought back to the feasible portfolio for subsequent iterations. Therefore, using the signs of ν's, to identify which of the removed securities from earlier iterations deserve to reenter the portfolio, is consistent with EGP's ranking procedure.

B. Multi-Index Model in the Diagonal Form (MIM-DI)

For all models with multiple groups where Equation (1) applies, securities in each group with the same sign of γ's can be ranked and labelled according to

[2] The solution thus obtained satisfies the Kuhn-Tucker conditions for optimality in (2) and (3). Clearly, for $i = 1, 2, \cdots, m$, $(\bar{R}_i - R_f)/\gamma_i > \psi_1(m)$, $z_i > 0$, and substituting the value of each z_i into Equation (2) always yields $\nu_i = 0$. For any security i not belonging to the optimal portfolio ($i = m + 1, m + 2, \cdots, n$), $(\bar{R}_i - R_f)/\gamma_i < \psi_1(m)$, and to make $z_i = 0$ in Equation (2), ν_i must be nonnegative. Therefore, the complementarity conditions are satisfied for all n securities considered for the portfolio.

Notice also that the purpose of explicitly ranking securities above is to facilitate a description of the functional form of ψ_1 and to demonstrate why the algorithm works. When implementing the algorithm, however, there is no need to label securities according to their rankings. Regardless of how they are labelled, securities with inferior performance are automatically identified.

their excess return to γ ratios in the same manner as described by EGP [7, 8]. When all γ's are positive, the n securities can be labelled in such a way that, for $i, i+1 \in G_h$, $(\bar{R}_i - R_f)/\gamma_i \geq (\bar{R}_{i+1} - R_f)/\gamma_{i+1}$, and security i is ranked no lower than security $i+1$. With this ranking arrangement, if security $i+1$ is in a portfolio, so is security i. Then each ψ_h can be described as a function of the p lowest ranking securities in the portfolio, one from each group.

For MIM-DI γ_i is b_i, a measure of the responsiveness of security i's return, R_i, to changes in index I_h for industrial group h where security i belongs. Since I_h depends linearly on the market index I_m with slope b_{n+h} in this model, R_i can be described as linearly dependent on I_m with slope $b_i b_{n+h}$ which is security i's beta coefficient. It can be shown that under the restriction of $b_i > 0$, for $i = 1, 2, \cdots, n+p$, the optimal portfolio disallowing short sales corresponds to every ψ_h, for $h = 1, 2, \cdots, p$, at its single maximum which serves as the cut-off rate for group h securities to be included in the portfolio.[3] Then, for the same reason as given for SIM and CCM-S, the algorithm of sequentially removing securities with negative z's can also be applied to MIM-DI. For each group h security i belonging to the optimal portfolio, $(\bar{R}_i - R_f)/b_i$ is above the maximum of the function ψ_h. For such a security, $(\bar{R}_i - R_f)/b_i$ always exceeds the value of ψ_h calculated for each iteration, and thus it cannot have any negative value for z_i and cannot be removed during the iterations.

The above requirements on input data should not be troublesome. As most securities have positive beta coefficients, it is reasonable to have $b_i b_{n+h} > 0$. This occurs when both b_i and b_{n+h} are positive, or both are negative. The latter case is unlikely as it suggests inappropriate grouping of securities. In the event that some b's are zero or negative, the optimal portfolio may correspond to ψ_h below its maximum. Then, some low ranking securities belonging to the optimal portfolio are also susceptible to be removed during the iterations. But, for the same reason as described earlier for SIM, the Kuhn-Tucker conditions can be used to identify those securities which were unintentionally removed. Securities with negative ν's were brought back to the portfolio. The same iterative procedure is repeated until $\nu_i = 0$ when $z_i > 0$, and $\nu_i \geq 0$ when $z_i = 0$, for all i.

To bring back securities with negative ν's to the feasible portfolio (where all z's are positive) is consistent with EGP's ranking procedure. To see this, suppose that both securities i and j from the same group, with b_i and b_j having the same sign, are not in the feasible portfolio. Also, according to Equation (2) $z_i = z_j = 0$, $\nu_i < 0$, and $\nu_j > 0$. If both b_i and b_j are positive (negative), $(\bar{R}_i - R_f)/b_i$ is greater (less) than $(\bar{R}_j - R_f)/b_j$. Following EGP, security i has a higher ranking than security j, and deserves to enter the feasible portfolio before security j does.

Intuitively, the same algorithm is expected to produce optimal portfolios regardless of the sign of each b_i or b_{n+h}. However, without a formal proof, one cannot rule out the possibility that the same securities leave and reenter the portfolio alternately during the iterations and the solution is never reached. Fortunately, this does not appear to be a potential problem for MIM-DI, as well as the remaining models with multiple groups, judging from the results of a simulation study reported in Section III.

[3] The proof is also available from the author upon request.

C. Multi-Index Model in the Covariance Form (MIM-CO) and Constant Correlation Model with Multiple Groups (CCM-M)

In view of the fact that SIM, CCM-S, MIM-DI, MIM-CO, and CCM-M all share Expressions (1)–(3) as their solutions to portfolio optimization problems, it seems likely that under some restrictions on input data, the optimal portfolios for MIM-CO and CCM-M also correspond to the situation where each ψ_h, for $h = 1, 2, \cdots, p$, is at its single maximum. However, unlike SIM, CCM-S, and MIM-DI, where ψ_1 or each ψ_h can be expressed explicitly in terms of input data, values of the p cut-off rates, $\psi_1, \psi_2, \cdots, \psi_p$, in MIM-CO and CCM-M can only be obtained by solving simultaneous linear equations. While it is straightforward to compute each ψ_h for these two models given input data for the analysis, it is very cumbersome to express the algebraic form of each ψ_h explicitly when there are many industrial groups in the analysis. Because of this problem, the requirements on input data for these two models yielding the above functional forms of cut-off rates will not be sought analytically.

Suppose that, under some restrictions on input data, the optimal portfolio actually corresponds to every ψ_h at its single maximum. Then, the algorithm developed above for SIM, CCM-S, and MIM-DI will also work here. If such a correspondence is not present, some relatively low ranking securities belonging to the optimal portfolio may produce negative z's and be removed during the iterations. This is not troublesome at all because such securities can easily be identified via the Kuhn-Tucker conditions and brought back to the feasible portfolio with all positive z's. Again, as securities can be ranked according to their excess return to γ ratios in MIM-CO and CCM-M, to use the signs of ν's to identify which of the removed securities deserve to reenter the feasible portfolio is consistent with EGP's ranking procedure.

D. Multi-Index Models with Orthogonal Indices (MIM-OR and MIM-G)

It is worth noting that MIM-OR actually encompasses MIM-DI as a special case; by imposing parameters β_i and b_i in MIM-OR to have the same ratio b_{n+h}, for $i \in G_h$, the model becomes MIM-DI.[4] This restriction on the parameters has made it possible to rank securities in MIM-DI. However, since explicit ranking of securities is not required in the algorithm proposed above, the removal of such a restriction is not expected to cause difficulties in the implementation of the same algorithm. Intuitively, given the functional dependence of z_i on $\bar{R}_i - R_f$, β_i, and b_i for MIM-OR as shown in Equation (4), securities with negative z's must implicitly have low rankings according to $(\bar{R}_i - R_f)/\beta_i$, $(\bar{R}_i - R_f)/b_i$, or both. These securities tend to be the ones not belonging to the optimal portfolio, and they can easily be identified and removed during the iterations. It must be recognized, however, that securities actually belonging to the optimal portfolio are also susceptible to be removed during the iterations. Nevertheless, the use of the Kuhn-Tucker conditions to identify all unintentionally removed securities can ensure the optimality of the solution.

While MIM-DI is a special case of MIM-OR, the latter model in turn is a

[4] See Elton and Gruber [2], Ch. 6, Appendix C.

special case of MIM-G. Suppose that in MIM-G, I_1 is restricted to be the market index I_m, and each I_h, for $h = 2, 3, \cdots, p$, is restricted to be the index for industrial group h which has been made orthogonal to each other and to I_m. Suppose also that each of the n securities belongs to only one of the $p - 1$ groups. Then, by imposing $b_{ik} = 0$, for each $i \in G_h$, $k = 2, 3, \cdots, p$ and $k \neq h$, the model becomes MIM-OR. Under such restrictions on input parameters, the solution for MIM-G in Equation (6) reduces to that for MIM-OR as shown in Equation (4). While Equation (4) is somewhat simpler than Equation (6), the functional dependence of z_i on input parameters still exhibits similar features in the two cases. Thus, if the proposed algorithm for optimal portfolio selection works for MIM-OR, it is reasonable to expect the same for MIM-G. However, the usefulness of an algorithm depends not only on the fact that it can produce optimal solutions, but also on its efficiency in reaching the solutions. This is considered next.

III. An Illustrative Example and a Simulation Study

An illustrative example showing the results of all iterative steps and the optimal solution is presented in Table I. Since the algorithm is common for all seven models, there is no need to show one example for each model. The example in

Table I

An Illustrative Example for the General Multi-Index Model with Orthogonal Indices (MIM-G) Showing Results of Iterative Steps and the Optimal Solution under the Condition of No Short Sales of Risky Securities (Input Data: $\text{var}(I_1) = 40$, $\text{var}(I_2) = 44$, $\text{var}(I_3) = 30$, $\text{var}(I_4) = 23$, $R_f = 8$)

Security	Input Data						First Iteration	Second Iteration	Third Iteration		Fourth Iteration		
i	$\bar{R}_i$	b_{i1}	b_{i2}	b_{i3}	b_{i4}	δ_i^2	z_i	z_i	z_i	v_i	z_i	v_i	x_i
1	22	0.7	−0.4	−0.2	0.3	42	0.127	0.035	0.029	0	0.032	0	0.017
2	14	0.1	−0.3	0.2	−0.4	27	0.108	0.124	0.122	0	0.121	0	0.064
3	19	0	0.3	−0.5	−0.2	38	0.345	0.307	0.306	0	0.307	0	0.163
4	18	0.4	0	0.3	−0.3	39	0.127	0.085	0.085	0	0.083	0	0.044
5	20	0.2	0.2	−0.4	0.4	41	0.288	0.229	0.228	0	0.230	0	0.122
6	18	0.1	−0.3	−0.4	−0.1	49	0.168	0.152	0.148	0	0.149	0	0.079
7	20	0.3	0	0.5	−0.3	42	0.182	0.166	0.168	0	0.166	0	0.088
8	9	0.4	0.4	0.4	0.5	51	−0.036	—	0	4.044	0	4.065	0
9	12	1.0	0.2	0	−0.1	22	−0.267	—	0	12.082	0	12.065	0
10	14	0.9	0	−0.5	−0.5	52	−0.059	—	0	9.407	0	9.350	0
11	24	0.7	−0.4	0.4	−0.1	28	0.212	0.120	0.116	0	0.115	0	0.061
12	10	0.7	0.3	−0.2	0.4	48	−0.072	—	0	8.611	0	8.539	0
13	23	0.3	−0.4	−0.3	−0.4	58	0.177	0.151	0.146	0	0.147	0	0.078
14	22	0.4	−0.5	0.1	0.2	34	0.220	0.180	0.176	0	0.177	0	0.094
15	7	−0.1	0.1	0.3	−0.2	29	−0.006	—	0	−0.977	0.031	0	0.017
16	24	1.0	0.5	0	0	20	0.377	0.045	0.044	0	0.044	0	0.023
17	15	0	0.2	0.5	0.4	28	0.260	0.278	0.285	0	0.284	0	0.150
18	14	0.4	−0.4	−0.4	−0.1	44	0.014	−0.041	0	2.064	0	1.976	0
19	8	0.7	0	−0.3	0	21	−0.330	—	0	11.758	0	11.689	0
20	15	0.9	0.1	0.2	0.4	30	−0.075	—	0	7.398	0	7.359	0

Table I, which is based on MIM-G, the most general model among the seven models, considers 20 securities whose returns are influenced by four orthogonal indices. The initial portfolio for the first iteration contains all 20 securities. According to Equation (6), securities 8–10, 12, 15, 19, and 20 have negative z's and are therefore removed from the portfolio, leaving behind 13 securities for the subsequent iteration. This time only z_{18} is negative, and consequently security 18 is removed from the portfolio. In the third iteration, all 12 z's are positive. However, the Kuhn-Tucker conditions, that $\nu_i = 0$ when $z_i > 0$ and $\nu_i \geq 0$ when $z_i = 0$, are violated by security 15 which was removed in the first iteration. With this security brought back to the portfolio for the fourth iteration, all 13 securities included have positive z's. Also, the Kuhn-Tucker conditions are completely satisfied by the 20 securities. The composition of the optimal portfolio is shown in the last column of Table I.

The above example has been used to illustrate how the proposed algorithm works. This example alone, however, should not be considered as convincing evidence that the algorithm is efficient. To examine its efficiency, a simulation study has been performed on the five models with multiple groups. For all five models, the numbers of securities considered are 24, 48, and 96 which with the exception of MIM-G are divided equally among 3, 4, and 6 groups, respectively. For each model and for each given value of n, optimal portfolios are formed repeatedly using input data which are random draws from uniform distributions within some specified ranges of values similar to those used in illustrative examples in various EGP studies [2, 3, 6, 7]. In each case, the distribution of numbers of iterations required to reach optimal portfolios is summarized in Table II. In the 1,500 simulation runs (300 runs for each model), optimal portfolios have always been obtained within 8 or 9 iterations. For each of CCM-M, MIM-CO, and MIM-DI, the medians of the distributions are 3, 4, and 5 iterations for $n = 24$, 48, and 96, respectively, and for MIM-OR or MIM-G the medians are 4, 5, and 6 iterations for $n = 24$, 48, and 96, respectively. This indicates that, when the number of securities is doubled, only one extra iteration is required to reach the optimal solution on average.

The algorithm to reach optimal solutions developed here can be divided into two parts. In part one, securities with negative z's are sequentially removed from the initial portfolio containing n securities until all securities remaining in the portfolio have positive z's. Part two contains all remaining iterative steps to reach the optimal portfolio. In CCM-M, MIM-CO, and MIM-DI, as the randomly generated input data used in the simulation study do not always satisfy the requirements that lead to optimal portfolios corresponding to each ψ_h at its single maximum, it is inevitable that the portfolios obtained from part one of the iterative procedure are sometimes suboptimal.[5] Table II also presents for all five

[5] A simulation study has been performed on CCM-M and MIM-CO to examine if some particular restrictions on input data can lead to the correspondence between optimal portfolios and every ψ_h at its single maximum. In the case of CCM-M, the restriction considered is $\rho_{hk} > 0$ for all $h, k = 1, 2, \cdots, p$. The same data as described in Table II for this model are used except that the range of values of ρ_{hk} for $h \neq k$ is from 0 to 0.3. In 300 simulation runs (with 100 runs for each value of $n = 24$, 48, and 96), there are 282 cases where each ψ_h increases during the iterations and reaches its single maximum when the corresponding portfolio is optimal. Of the remaining 18 cases, there are only

models the distribution of numbers of additional iterations required to reach optimality for cases where part one of the iterative procedure produces suboptimal portfolios. For each of the five models, the number of cases requiring additional iterations beyond part one increases as more securities are considered for the analysis. In most cases, if optimal solutions are not reached in part one of the iterative procedure, they can be reached in one additional iteration. In all cases, the number of additional iterations required is no more than three.

IV. Conclusion

In this study, a unified portfolio analysis for seven models is presented. The models considered here are Sharpe's single index model, Cohen and Pogue's multi-index models in diagonal and covariance forms, two multi-index models with orthogonal indices, and two constant correlation models.

Under the condition of no short sales of risky securities, a simple common algorithm of optimal portfolio selection is developed for the seven models. The algorithm here successfully bypasses the requirement of explicitly ranking securities which is essential in previous studies on the topic. Because of this feature, the algorithm is particularly useful for two multi-index models with orthogonal indices where there are difficulties in establishing a ranking criterion.

It is also demonstrated in a simulation study that the procedure involved in the search for optimality requires only small numbers of simple iterative steps. Thus, the approach here can enhance the usefulness of these index models and constant correlation models in portfolio analysis.

Appendix A

MIM-CO can be described as

$$R_i = a_i + b_i I_h + e_i, \qquad \text{for} \qquad i \in G_h, \qquad h = 1, 2, \cdots, p, \tag{A1}$$

where R_i is the rate of return on security i, I_h is an index for group h, a_i and b_i are parameters, and e_i is random noise. This model imposes on each e_i a zero mean and a finite nonzero variance δ_i^2. It is further assumed that $\text{cov}(e_i, e_j) = 0$ and $\text{cov}(I_h, e_i) = 0$ for $i, j = 1, 2, \cdots, n$, $i \neq j$, and $h = 1, 2, \cdots, p$.

To describe the variance-covariance structure of this model, the n securities are first labelled in such a way that securities $1, 2, \cdots, n_1 \in G_1$; $n_1 + 1, n_1 + 2, \cdots, n_2 \in G_2$; $\cdots$; $n_{p-1} + 1, n_{p-1} + 2, \cdots, n_p \in G_p$. Each G_h, for $h = 1, 2,$

three cases where the final portfolios with all positive z's obtained by sequentially removing securities with negative z's are suboptimal.

The restrictions considered for MIM-CO are $b_i > 0$ for all $i = 1, 2, \cdots, n$ and $\text{cov}(I_h, I_k) > 0$ for all $h, k, = 1, 2, \cdots, p$. The same data as described in Table II for this model are used except that the ranges of values of b_i and $\text{cov}(I_h, I_k)/[\text{var}(I_h)\text{var}(I_k)]^{1/2}$ are from 0 to 2 and from 0 to 0.5, respectively. In 300 simulation runs, there are 289 cases where each ψ_h increases during the iterations and reaches its maximum when the optimal portfolio is obtained. In all of the remaining 11 cases where the above correspondence is absent, the final portfolio with all positive z's obtained by sequentially removing securities with negative z's are still optimal.

Table II

Input Data and Results of Simulation Runs

Model		CCM-M			MIM-CO			MIM-DI			MIM-OR			MIM-G		
Number of Groups, p		3	4	6	3	4	6	3	4	6	3	4	6	3	4	6
Number of Securities, n		24	48	96	24	48	96	24	48	96	24	48	96	24	48	96
		Distribution of Numbers of Observations														
Total Number of Iterations Required to Reach Optimality	2	0	0	0	6	0	0	4	0	0	2	0	0	2	0	0
	3	63	13	1	64	22	4	49	15	2	28	5	1	31	4	0
	4	36	70	34	29	54	21	35	49	24	51	30	9	42	34	1
	5	1	16	47	1	23	53	10	29	50	18	35	38	21	43	29
	6	0	1	13	0	1	17	2	4	17	1	26	34	2	14	44
	7	0	0	4	0	0	4	0	2	6	0	4	15	1	4	19
	8	0	0	1	0	0	1	0	1	1	0	0	3	1	1	2
	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5
	Total	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
Number of Additional Iterations Required to Reach Optimality When the Final Portfolio Obtained by Sequentially Removing Securities with Negative z's Is Suboptimal	1	1	2	10	13	22	51	15	37	50	23	42	51	13	32	40
	2	0	1	5	0	1	3	0	3	3	2	5	10	4	6	9
	3	0	0	1	0	0	1	0	0	0	0	1	2	1	1	6
	Total	1	3	16	13	23	55	15	40	53	25	48	63	18	39	55
Input Data: Random Draws from Uniform Distributions Within Specified Ranges of Values		$R_f = 8$ $\bar{R}_i = 5$ to 25 $\sigma_i = 3$ to 10 $\rho_{hh} = 0.3$ to 0.6 $\rho_{hk} = -0.15$ to 0.3 for $i = 1, 2, \cdots, n$ $h, k = 1, 2, \cdots, p$ $h \neq k$			$R_f = 8$ $\bar{R}_i = 5$ to 25 $b_i = -0.5$ to 2.0 $\text{var}(I_h)$ and $\delta_i^2 = 10$ to 50 $\frac{\text{cov}(I_h, I_k)}{[\text{var}(I_h)\text{var}(I_k)]^{1/2}} = -0.2$ to 0.5 for $i = 1, 2, \cdots, n$ $h, k = 1, 2, \cdots, p$ $h \neq k$			$R_f = 8$ $\bar{R}_i = 5$ to 25 b_i and $b_{n+h} = -0.5$ to 2.0 $\text{var}(I_m)$, $\text{var}(I_h)$, and $\delta_i^2 = 10$ to 50 for $i = 1, 2, \cdots, n$ $h = 1, 2, \cdots, p$			$R_f = 8$ $\bar{R}_i = 5$ to 25 β_i and $b_i = -0.5$ to 2.0 $\text{var}(I_m)$, $\text{var}(I_h)$, and $\delta_i^2 = 10$ to 50 for $i = 1, 2, \cdots, n$ $h = 1, 2, \cdots, p$			$R_f = 8$ $\bar{R}_i = 5$ to 25 $b_{i1} = -0.5$ to 2.0 b_{i2} to $b_{ip} = -1$ to 1 $\text{var}(I_h)$ and $\delta_i^2 = 10$ to 50 for $i = 1, 2, \cdots, n$ $h = 1, 2, \cdots, p$		

$\cdots, p$, has $n_h - n_{h-1}$ securities (where $0 = n_0 < n_1 < \cdots < n_p = n$). Then, the variance-covariance matrix $\underline{V}$ can be partitioned as

$$\begin{bmatrix} \underline{V}_{11} & \cdots & \underline{V}_{1p} \\ \vdots & & \vdots \\ \underline{V}_{p1} & \cdots & \underline{V}_{pp} \end{bmatrix}$$

with submatrices

$$\left.\begin{aligned} \underline{V}_{hh} &= s_{hh}\underline{C}_h\underline{C}'_h + \underline{D}_h, \qquad \text{and} \\ \underline{V}_{hk} &= s_{hk}\underline{C}_h\underline{C}'_k, \qquad \text{for} \quad h, k = 1, 2, \cdots, p, \qquad \text{and} \quad h \neq k, \end{aligned}\right\} \tag{A2}$$

where

$$s_{hh} = \operatorname{var}(I_h), \qquad s_{hk} = \operatorname{cov}(I_h, I_k), \qquad \underline{C}_h = [b_{n_{h-1}+1} \cdots b_{n_h}]',$$

and

$$\underline{D}_h = \operatorname{diag}[\delta^2_{n_{h-1}+1} \cdots \delta^2_{n_h}].$$

Given risk-free lending and borrowing at the same rate R_f, the solution to the portfolio optimization problem allowing short sales can be obtained via

$$\underline{V}\underline{Z} = \underline{\bar{R}} - R_f\underline{1} \tag{A3}$$

where $\underline{Z} = [z_1 \cdots z_n]'$, $\underline{\bar{R}} = [\bar{R}_1 \cdots \bar{R}_n]'$, and $\underline{1}$ is an n-dimensional column vector of ones. In terms of submatrices, Equation (A3) can be expressed as

$$\textstyle\sum_{i=1}^{p} \underline{V}_{hi}\underline{Z}_i = \underline{\bar{R}}_h - R_f\underline{1}_h, \qquad \text{for} \quad h = 1, 2, \cdots, p, \tag{A4}$$

where $\underline{Z}_i = [z_{n_{h+1}+1} \cdots z_{n_h}]'$, $\underline{\bar{R}}_h = [\bar{R}_{n_{h+1}+1} \cdots \bar{R}_{n_h}]'$, and $\underline{1}_h$ is an $(n_h - n_{n-1})$-dimensional column vector of ones. Combining Equations (A2) and (A4) yields

$$\underline{Z}_h = \underline{D}_h^{-1}(\underline{\bar{R}}_h - R_f\underline{1}_h - \psi_h\underline{C}_h) \tag{A5}$$

which is equivalent to Equation (1). Here, $\psi_h = \sum_{i=1}^{p} s_{hi}\underline{C}'_i\underline{Z}_i$.

To determine $\underline{\psi} = [\psi_1 \cdots \psi_p]'$, Equation (A5) is premultiplied by $s_{kh}\underline{C}'_h$ and then summed over h from 1 to p, yielding

$$\begin{aligned} \psi_k = {}& \textstyle\sum_{h=1}^{p} s_{kh}\underline{C}'_h\underline{D}_h^{-1}(\underline{\bar{R}}_h - R_f\underline{1}_h) \\ & - \textstyle\sum_{h=1}^{p} s_{kh}\underline{C}'_h\underline{D}_h^{-1}\underline{C}_h\psi_h, \qquad \text{for} \quad k = 1, 2, \cdots, p, \end{aligned} \tag{A6}$$

which is a system of p simultaneous linear equations for $\psi_1, \psi_2, \cdots, \psi_p$. The solution is

$$\underline{\psi} = (\underline{I} + \underline{P}\underline{E})^{-1}\underline{P}\underline{F} \tag{A7}$$

where $\underline{I}$ is a $p \times p$ identity matrix,

$$\underline{P} = \begin{bmatrix} s_{11} & \cdots & s_{1p} \\ \vdots & & \vdots \\ s_{p1} & \cdots & s_{pp} \end{bmatrix},$$

$$\underline{E} = \text{diag}[(\underline{C}_1'\underline{D}_1^{-1}\underline{C}_1) \cdots (\underline{C}_p'\underline{D}_p^{-1}\underline{C}_p)],$$

and

$$\underline{F} = [\underline{C}_1'\underline{D}_1^{-1}(\underline{\bar{R}}_1 - R_f\underline{1}_1) \cdots \underline{C}_p'\underline{D}_p^{-1}(\underline{\bar{R}}_p - R_f\underline{1}_p)]'.$$

The first Kuhn-Tucker condition for the same portfolio optimization problem disallowing short sales is Equation (A3) with a column vector $\underline{U} = [\nu_1 \cdots \nu_n]'$ added to its right-hand side. Thus, by replacing $\underline{\bar{R}}_h - R_f\underline{1}_h + \underline{U}_h$ (where $\underline{U}_h = [\nu_{n_{h+1}+1} \cdots \nu_{n_h}]'$) for $\underline{\bar{R}}_h - R_f\underline{1}_h$ in the expression for $\underline{Z}_h$ of the short sales case, the corresponding expression for $\underline{Z}_h$ of the no short sales case is obtained. Explicitly,

$$\underline{Z}_h = \underline{D}_h^{-1}(\underline{\bar{R}}_h - R_f\underline{1}_h + \underline{U}_h - \underline{\psi}_h^*\underline{C}_h), \qquad \text{for} \quad h = 1, 2, \cdots, p. \tag{A8}$$

Here, ψ_h^* is obtained from $\underline{\psi}^* = [\psi_1^* \cdots \psi_p^*]' = (\underline{I} + \underline{P}\underline{E})^{-1}\underline{P}\underline{F}^*$, where $\underline{F}^* = [\underline{C}_1'\underline{D}_1^{-1}(\underline{\bar{R}}_1 - R_f\underline{1}_1 + \underline{U}_1) \cdots \underline{C}_p'\underline{D}_p^{-1}(\underline{\bar{R}}_p - R_f\underline{1}_p + \underline{U}_p)]'$. By imposing the complementarity conditions that $\nu_i = 0$ for $z_i > 0$ and $\nu_i \geq 0$ for $z_i = 0$, ψ_h^* becomes ψ_h, for $h = 1, 2, \cdots, p$, which is computed via Equation (A7) using only input data for securities included in the optimal portfolio. Thus, the first Kuhn-Tucker condition given by Equation (A8) is equivalent to Equation (2).

Appendix B

The variance-covariance matrix of MIM-G is

$$\underline{V} = \textstyle\sum_{h=1}^{p} \underline{B}_h\underline{B}_h' + \underline{D}, \tag{B1}$$

where $\underline{B}_h = \sqrt{\text{var}(I_h)}\ [b_{ih} \cdots b_{nh}]'$ and $\underline{D} = \text{diag}[\delta_1^2 \cdots \delta_n^2]$. To use Equation (A3) to solve for $\underline{Z}$ for the short sales case requires $\underline{V}^{-1}$. To invert $\underline{V}$, Equation (B1) can be expressed recursively as

$$\underline{Q}_{h-1} = \underline{B}_h\underline{B}_h' + \underline{Q}_h, \qquad \text{for} \quad h = 1, 2, \cdots, p, \tag{B2}$$

with $\underline{Q}_0 = \underline{V}$ and $\underline{Q}_p = \underline{D}$. Since $\underline{Q}_{h-1}$ can be decomposed into an outer product of two vectors $\underline{B}_h\underline{B}_h'$ and a square matrix $\underline{Q}_h$, its inverse is

$$\underline{Q}_{h-1}^{-1} = \underline{Q}_h^{-1}(\underline{I} - \underline{A}_h), \tag{B3}$$

where $\underline{I}$ is an $n \times n$ identity matrix and $\underline{A}_h = \alpha_h\underline{B}_h\underline{B}_h'\underline{Q}_h^{-1}$ with the scalar $\alpha_h = (1 + \underline{B}_h'\underline{Q}_h^{-1}\underline{B}_h)^{-1}$.

Given input data $\underline{Q}_p$ $(=\underline{D})$ and $\underline{B}_h$, for $h = 1, 2, \cdots, p$, Equation (B3) when used recursively will provide all matrix elements in $\underline{V}^{-1}$. The sequence is as follows. The inverse of $\underline{Q}_p$ is computed first. This is straightforward because $\underline{Q}_p$ is a diagonal matrix. Given $\underline{Q}_p^{-1}$ and $\underline{B}_p$, all elements in $\underline{Q}_{p-1}^{-1}$ can be obtained. Then, $\underline{Q}_{p-2}^{-1}, \underline{Q}_{p-3}^{-1}, \cdots, \underline{Q}_0^{-1}$ $(=\underline{V}^{-1})$ are computed sequentially.

To obtain an explicit expression of $\underline{Z}$ in terms of the input data for the analysis, Equation (B3) is written as

$$\begin{aligned}\underline{Q}_0^{-1} &= \underline{Q}_1^{-1}(\underline{I} - \underline{A}_1)\\ &= \underline{Q}_2^{-1}(\underline{I} - \underline{A}_2)(\underline{I} - \underline{A}_1)\\ &\vdots\\ &= \underline{Q}_p^{-1}(\underline{I} - \underline{A}_p)(\underline{I} - \underline{A}_{p-1}) \cdots (\underline{I} - \underline{A}_1),\end{aligned}$$

which is equivalent to

$$\underline{V}^{-1} = \underline{D}^{-1}\{I - \underline{A}_1 - \underline{A}_2(\underline{I} - \underline{A}_1) - \underline{A}_3(\underline{I} - \underline{A}_2)(\underline{I} - \underline{A}_1) - \cdots - [\underline{A}_p(\underline{I} - \underline{A}_{p-1}) \cdots (\underline{I} - \underline{A}_1)]\}. \quad \text{(B4)}$$

Therefore,

$$\begin{aligned}\underline{Z} &= \underline{V}^{-1}(\underline{\bar{R}} - R_f\underline{1})\\ &= \underline{D}^{-1}(\underline{\bar{R}} - R_f\underline{1} - \textstyle\sum_{h=1}^{p} \phi_h\underline{B}_h).\end{aligned} \quad \text{(B5)}$$

Here, each

$$\phi_h = \alpha_h\underline{B}_h'\underline{Q}_h^{-1}\underline{G}_h, \quad \text{(B6)}$$

for $h = 1, 2, \cdots, p$, is a scalar, where

$$\underline{G}_1 = \underline{\bar{R}} - R_f\underline{1},$$

and (B7)

$$\underline{G}_h = \underline{G}_{h-1} - \phi_{h-1}\underline{B}_{h-1}, \qquad \text{for} \quad h = 2, 3, \cdots, p.$$

Equation (B5) is equivalent to Equation (6) where $\psi_h = \sqrt{\text{var}(I_h)}\,\phi_h$.

In the no short sales case, it can be shown that the first Kuhn-Tucker condition is

$$\underline{Z} = \underline{D}^{-1}(\underline{\bar{R}} - R_f\underline{1} - \textstyle\sum_{h=1}^{p} \phi_h^*\underline{B}_h + \underline{U}) \quad \text{(B8)}$$

where

$$\begin{aligned}\underline{U} &= [\nu_1 \cdots \nu_n]'\\ \phi_h^* &= \alpha_h\underline{B}_h'\underline{Q}_h^{-1}\underline{G}_h^*, \qquad \text{for} \quad h = 1, 2, \cdots, p,\\ \underline{G}_1^* &= \underline{\bar{R}} - R_f\underline{1} + \underline{U},\end{aligned}$$

and

$$\underline{G}_h^* = \underline{G}_{h-1}^* - \phi_{h-1}^*\underline{B}_{h-1}, \qquad \text{for} \quad h = 2, 3, \cdots, p.$$

Also, by imposing the complementarity conditions, ϕ_h^* becomes ϕ_h, for $h = 1, 2, \cdots, p$, which is computed via Equations (B6) and (B7) using only input data for securities included in the optimal portfolio. What it amounts to is that the first Kuhn-Tucker condition given by Equation (B8) is equivalent to Equation (6) with a ν_i/δ_i^2 term added to its right-hand side.

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