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Ordering Uncertain Options under Inflation: A Note

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ABSTRACT

Stochastic dominance rules (SD) have been extended to the case where investors are allowed to borrow and lend at the riskless interest rate. Stochastic dominance rules with a riskless asset (SDR) are much more effective than SD rules. However, it seems that this benefit is eliminated by an uncertain inflation, since riskless assets become risky once uncertain inflation is considered.

We prove in this paper that SDR criteria are valid also in the face of uncertain (and independent) inflation. Moreover, while the mean-variance (MV) efficient set increases with uncertain inflation, the stochastic dominance efficient sets decrease.

STOCHASTIC DOMINANCE CRITERIA (SD) for ordering risky assets were developed mainly in the late sixties, following the development of the mean-variance rule (MV) and the capital asset pricing model (CAPM). By analogy to the extension of Sharpe and Lintner of the MV criterion, Levy and Kroll [10] have extended the SD rules by allowing borrowing and lending at the riskless interest rate. This extension sharply decreases the size of the efficient set, and in some cases an "empirical separation" is obtained (or almost a separation) where the efficient set of risky assets includes only one or two risky portfolios (see Levy and Kroll [11]). Note, however, that these important economic results rely on the assumption that a riskless asset indeed exists, an assumption which cannot be defended in a period of uncertain inflation.

Chen and Boness [2], Roll [17], and Friend et al. [5] investigate the impact of inflation on the CAPM. Chen and Boness assume a quadratic utility function and also make the assumption that the real rate of return is given by the nominal rate of return less the inflation, hence ignoring the cross-product of inflation and rates of return, which greatly simplifies the mathematical analysis. Roll also assumes a quadratic utility function and further makes the assumption that a riskless asset in real terms exists. Friend et al. assume that returns are generated by a Wiener process, the investment horizon is infinitesimal, and that investors are characterized by constant proportional risk aversion. Under these assumptions they derive a formula which is analogous to the CAPM, where inflation (a random variable) is incorporated in the formula. It should be noted that by an infinitesimal investment horizon, they do away with the cross-product of inflation and the rates of return, which is equal to zero in this specific case.

The relationship between the inflation rate and stock prices has been a subject

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of several studies. Lintner [13] observes that over a long period of time there is no significant correlation between the annual percentage change in stock prices and the annual change in the wholesale price index (Lintner [13], p. 32). Fama and Schwert [3] find that common stock returns are negatively related to the *expected* inflation rate. However, although common stocks are also found to be negatively related to the *unexpected* inflation rate, this result is not very conclusive since the regression coefficient of the unexpected inflation is not always significantly different from zero (see Fama and Schwert [3], Table 8, p. 140). Bodie [1] also finds the return on common stocks to be negatively correlated with the inflation rate.

In this paper, we assume that inflation is a random variable which is independent of the return on a portfolio of risky assets. In some cases, however, this assumption can be relaxed to account for possible dependency between inflation and the return on various assets. We neither assume a quadratic utility function, nor an infinitesimal investment horizon. Namely, the planned investment horizon can be a month, a year, etc. Hence, the cross-product term does not vanish and we do not utilize any approximations that do away with this term.

We investigate conditions under which the various decision rules are valid in the face of uncertain inflation. Section I provides a brief review of the rules discussed in this paper. Section II analyses the SD criteria, while Section III is devoted to the MV criterion. Concluding remarks are given in Section IV.

I. Review of Various Decision Rules

We deal in this paper with different classes of utility functions U_k, $k = 1, 2, \cdots$. We denote by $u(x)$ a utility function and by $u^{(i)}(x)$ the i^{th} order derivative of $u(x)$, $u^{(i)}(x) = \dfrac{\partial^i u(x)}{\partial x^i}$. $u(x) \in U_k$ if $u^{(i)}(x) \geq 0$ for odd numbered i and $u^{(i)}(x) \leq 0$ for even numbered i, $i = 1, 2, \cdots k$.

The decision rules appropriate for the different classes of utility functions U_k are known as stochastic dominance rules (SD). The decision rules pertaining to the classes U_1, U_2, and U_3 are known as first degree, second degree, and third degree stochastic dominance, respectively (denoted as FSD, SSD, and TSD). (See Quirk and Saposnik [16], Fishburn [4], Hadar and Russell [6], Hanoch and Levy [7], Rothschild and Stiglitz [19], and Whitmore [22].)

The mean-variance criterion (MV) is better known; option F dominates option G if $E_F(X) \geq E_G(X)$ and $\sigma_F^2 \leq \sigma_G^2$ (with at least one strict inequality). If $u(x) \in U_2$ ($u'(x) \geq 0$, $u''(x) \leq 0$) and $F(x)$ and $G(x)$ are normally distributed, then F dominates G by the MV criterion if and only if there is dominance by the SSD rule (see Hanoch and Levy [7], p. 343).

Levy and Kroll [10] showed that even if F does not dominate G by one of the above SD rules, dominance may exist if one creates new random variables $\{F_\alpha\} = \{\alpha X + (1 - \alpha)r\}$ and $\{G_\beta\} = \{\beta Y + (1 - \beta)r\}$, where r stands for the riskless interest rate and α, β are nonnegative investment proportions representing diversification strategies between the riskless and risky assets, respectively. A necessary and sufficient condition for dominance of the (infinite) set $\{F_\alpha\}$ over the (infinite) set $\{G_\beta\}$ is that one can find at least one diversification strategy $F_\alpha = \alpha X + (1 - \alpha)r$ which dominates the unlevered risky option G.

We turn now to the analysis of stochastic dominance rules in the face of uncertain inflation.

II. Stochastic Dominance with Uncertain Inflation

For convenience only and without loss of generality, assume that X and Y stand for the return (terminal wealth) on investments F and G, respectively. We denote the general price index (a random variable) by P, where, for simplicity, in the base period $P = 1$ and where Π stands for the reciprocal of P, namely $\Pi = 1/P$. Hence, the real returns on F and G are given by $X_R = X \cdot \Pi$ and $Y_R = Y \cdot \Pi$, respectively, where X_R and Y_R stand for the real returns.

We prove below that under certain assumptions if X dominates Y by one of the SD rules then also X_R dominates Y_R by the same rule.

THEOREM 1. *If the random variable P is independent of X and Y and $\Pi > 0$, then dominance of X over Y in nominal terms implies dominance in real terms. In short, $XD_kY \rightarrow X_RD_kY_R$, where D_k denotes stochastic dominance of the k^{th} order. Using the notation D_k we are able to prove FSD, SSD, and TSD and, in fact, any k^{th} order stochastic dominance rule in one theorem.*

Proof: The joint density functions of X and Π and Y and Π are given by $f(X, \Pi)$ and $g(Y, \Pi)$, respectively, where $f(X)$ and $g(Y)$ are the marginal densities of X and Y. Furthermore, $m(\Pi)$ denotes the marginal density function of Π. By the theorem's assumption it is given that FD_kG, namely $E_Fu(X) \geq E_Gu(Y)$ for all $u \in U_k$ ($k = 1, 2, \cdots$). By definition, the expected utility of the real return on option F is given by

$$E_Fu(X\Pi) = \int_0^\infty \int_0^\infty u(X\Pi) f(X, \Pi)\, dX\, d\Pi \tag{1}$$

Using the independence assumption of X and Π we have, $f(X, \Pi) = f(X)m(\Pi)$. Hence

$$E_Fu(X\Pi) = \int_0^\infty \int_0^\infty u(X\Pi) f(X) m(\Pi)\, dX\, d\Pi$$

which can be written as

$$E_Fu(X\Pi) = \int_0^\infty m(\Pi) \left[\int_0^\infty u(X\Pi) f(X)\, dX \right] d\Pi \tag{2}$$

Similarly, for option G we have

$$E_Gu(Y\Pi) = \int_0^\infty m(\Pi) \left[\int_0^\infty u(Y\Pi) g(Y)\, dY \right] d\Pi \tag{3}$$

We have to prove that, for all $u \in U_k$ ($k = 1, 2, \cdots$), $Eu_F(X\Pi) \geq E_Gu(Y\Pi)$. To show this claim, recall that by the theorem's assumption FD_kG (for $k = 1, 2, \cdots$). Thus, for all $u \in U_k$ ($k = 1, 2, \cdots$), $E_Fu(X) \geq E_Gu(Y)$.

Define $V(X) = u(X\Pi)$. Thus, $V^{(i)}(X) = \partial V^i(X)/\partial X^i = \Pi^i u^{(i)}(X\Pi)$, namely the sign of $V^{(i)}(X)$, the i^{th} derivative of $V(X)$, is identical to the sign of $u^{(i)}(X)$, since $\Pi \geq 0$. Hence, if $u \in U_k$ $(k = 1, 2, \cdots)$ also $V(X) \in U_k$ and $V(Y) \in U_k$ $(k = 1, 2, \cdots)$. Since by the theorem's assumption XD_kY (or FD_kG), $(k = 1, 2, \cdots)$, we have (for every $\Pi > 0$)

$$\int_0^\infty u(X\Pi)f(X)\,dX = \int_0^\infty V(X)f(X)\,dX \geq \int_0^\infty V(Y)g(Y)\,dY$$
$$= \int_0^\infty u(Y\Pi)g(Y)\,dY$$

The density $m(\Pi)$ is common to Equations (2) and (3) which implies by the last inequality that

$$E_F u(X\Pi) \geq E_G u(Y\Pi) \quad \text{for all} \quad u \in U_k \ (k = 1, 2, \cdots)$$

which completes the proof.

Note that by Theorem 1, XD_kY is a sufficient but not a necessary condition for X_RDY_R. To prove this claim, we present below an example for which $X_RD_kY_R$ in spite of the fact that $X\not{D}_kY$. We shall demonstrate this for FSD, namely $k = 1$. Let X and Y have the following distributions: $X = 2$ and $X = 10$, both with probability ½; $Y = 1$ with probability ¼ and $Y = 4$ with probability ¾. Assume that Π has the following distribution: 1 and 2 with equal probabilities of ½. It can easily be verified that neither X dominates Y nor Y dominates X by the FSD rule (i.e., the distributions intersect). However, assuming independence of the returns and inflation, it can be verified that the real return $X\Pi$ dominates the real return $Y\Pi$ by FSD.

Similar examples can be constructed for the SSD and TSD criteria. Thus, with uncertain and independent inflation, the size of the efficient set may decrease, to the benefit of the individual investor.

In our opinion this result is counterintuitive. It asserts that, while in the absence of inflation one may not be able to establish dominance between two risky assets, the introduction of uncertain inflation which is independent of the returns on the two risky options may yield additional information, and dominance between the two risky assets can be established. Thus, the efficient set in real terms is no larger (and may be smaller) than the efficient set in nominal terms.

While in general the conditions of Theorem 1 are sufficient but not necessary, they are necessary and sufficient in the following two cases:

(a) X, Y, and Π are lognormally distributed and $u \in U_2$.
(b) The utility function is of the power form ($u(x) = x^\alpha$, with the logarithmic function as a special case); namely, the function exhibits constant absolute risk aversion.

Case (b) is trivial. Let us turn to case (a). If X and Y are lognormally distributed, Levy [8, 9] proved that X dominates Y for all risk averters if and only if $E(X) \geq E(Y)$ and $\sigma_X/E(X) \leq \sigma_Y/E(Y)$ (with at least one inequality). Assuming the above two conditions hold for $X\Pi$ and $Y\Pi$ (real returns), it can easily be shown that they hold for X and Y as well (nominal returns).

Theorem 1 can be applied to several economic issues. In particular, the stochastic dominance criteria with a riskless asset (which we denote by SDR to distinguish from SD) developed by Levy and Kroll [10] are valid even though a riskless asset in real terms does not exist. Thus, the drastic reduction in the stochastic dominance efficient set, due to the inclusion of a *nominal* riskless asset, holds in real terms as well. This conclusion is summarized in the following corollary.

COROLLARY. *Let F and G be as in Theorem* 1. *Suppose that* $F \not{D} G$ *and* $G \not{D} F$, *by the stochastic dominance criterion of order k* ($k = 1, 2, \cdots$). *However, there is some* $\alpha > 0$ *such that* $F_\alpha D_k G$ ($k = 1, 2, \cdots$) *where* F_α *stands for the cumulative distribution of* $\alpha X + (1 - \alpha)r$ *and r is the riskless interest rate. Then* $\{F_\alpha\}_R D_k \{G_\beta\}_R$ ($k = 1, 2, \cdots$) *where the subscript R denotes the return expressed in real terms, when the inflation (a random variable) is independent of the nominal returns. This dominance exists even though r in real terms is no longer riskless.*

The proof of this corollary is by a straight application of Theorem 1.

So far we have assumed that inflation is independent of the nominal returns. However, Theorem 1 (and the corollary) can be extended to the case of dependency between the return on stock prices and inflation (as found in many of the empirical studies). In this case, the sufficiency conditions of Theorem 1 are stated in terms of the conditional distributions of X and Y, given every possible value of P (denoted by X/P and Y/P). The proof of the theorem is completely analogous to the proof of Theorem 1. The theorem obtained in the case of dependency can be applied to the following situation. Suppose that the stock market changes in some systematic way in relationship to the general price index (e.g., stock prices are a bad hedge against inflation and are negatively correlated with inflation, see Fama and Schwert [3]) but there is no change in the relative location of the distribution of stock prices with a change in the inflation rate (e.g., with low inflation all distributions of the rates of return are shifted to the right by some constant and the opposite occurs with high inflation). In this case, dominance of X over Y, implies also dominance of X/Π over Y/Π (for all Π) which by the theorem implies that $X\Pi$ dominates $Y\Pi$.

III. The Mean-Variance Rule and Inflation

Suppose that X dominates Y by the mean-variance rule, when returns are expressed in nominal terms. Namely, in the absence of inflation, we have $EX \geq EY$ and $\sigma_X^2 \leq \sigma_Y^2$. We shall use the notation $1/P = \Pi$ as in the previous section and assume that returns are stated in terms of terminal wealth (i.e., one plus the rate of return). Assuming inflation is independent of the nominal returns we obtain $EX_R \equiv E(X\Pi) = EXE\Pi$ and $EY_R \equiv E(Y\Pi) = EYE\Pi$ where the subscript R denotes real returns. It is obvious that $EX \geq EY \to EX_R \geq EY_R$ as long as $E\Pi \geq 0$. Since $\Pi = 1/P$ where P is the general price index, it is safe to assume that $E\Pi > 0$. With regards to the variance, we have

$$\mathrm{Var}(X_R) = \mathrm{Var}(X\Pi) = E(X\Pi)^2 - (E(X\Pi))^2$$

Using the independence assumption, we get

$$\mathrm{Var}(X_R) = E(X^2)E(\Pi^2) - (EX)^2(E\Pi)^2$$

Substituting $\sigma_X^2 + (EX)^2$ for $E(X^2)$ and $\sigma_\Pi^2 + (E\Pi)^2$ for $E(\Pi^2)$ yields

$$\mathrm{Var}(X_R) = [\sigma_X^2 + (EX)^2][\sigma_\Pi^2 + (E\Pi)^2] - (EX)^2(E\Pi)^2$$

which can be reduced to

$$\mathrm{Var}(X_R) = \sigma_X^2\sigma_\Pi^2 + (EX)^2\sigma_\Pi^2 + (E\Pi)^2\sigma_X^2$$

Similarly, for the other risky option we obtain

$$\mathrm{Var}(Y_R) = \sigma_Y^2\sigma_\Pi^2 + (EY)^2\sigma_\Pi^2 + (E\Pi)^2\sigma_Y^2$$

Since the real variance is an increasing function of both the nominal expected return and the nominal variance, $\sigma_X^2 < \sigma_Y^2$ does not necessarily imply that $\sigma_{X_R}^2 \leq \sigma_{Y_R}^2$. However, if both X and Y appear in the nominal MV efficient set, they also both appear in the real MV efficient set. Thus, the MV efficient set in real terms is larger than the efficient set in nominal terms. This result seems to contradict the results of the previous section, whereby the SD efficient set decreases with the introduction of uncertain and independent inflation. However, this paradox is solved once we bear in mind that SSD and MV yield the same ordering if the distributions are assumed to be normal and one further assumes risk aversion. Assuming that X and Y are normally distributed, we indeed obtain the same ordering of options by the MV and SSD criteria. However, if X and Y are normally distributed, $X\Pi$ and $Y\Pi$ are *not* normally distributed even if X and Π (and Y and Π) are independent and even if they are both normally distributed. Thus, while normality holds (by assumption) when returns are stated in nominal terms, this assumption does not hold in real terms. Hence, the fact that the MV rule is not able to distinguish between $X\Pi$ and $Y\Pi$ simply indicates that the MV rule is not valid (due to the violation of the normality assumption) and not that such dominance does not exist.

IV. Concluding Remarks

Stochastic dominance criteria as well as the MV rule become significantly more effective once riskless borrowing and lending are allowed.

Facing uncertain inflation induces the riskless asset to be a risky one, and it seems that the dominance criteria with borrowing and lending, at least in their classic form, are invalid.

We prove in this paper the following.

1) Dominance in nominal terms implies dominance in real terms as long as the uncertain inflation is independent of returns. This assumption is supported by some of the empirical evidence.

2) If, however, inflation is dependent on returns, then conditional dominance in nominal terms implies dominance in real terms.

3) Point 1 above implies that the stochastic dominance efficient set is reduced by the introduction of uncertain inflation, a result which is counterintuitive.

4) The stochastic dominance rules with riskless borrowing and lending developed by Levy and Kroll [10] are valid even though riskless borrowing and lending in real terms do not actually exist.

5) Unlike the stochastic dominance rules, the mean-variance efficient frontier increases rather than decreases by the introduction of uncertain inflation. This expansion is a technical one, and no risk averter would select a portfolio taken from the segment added to the efficient frontier.

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