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Author(s): John E. Gilster, Jr. and William Lee
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The Effects of Transaction Costs and Different Borrowing and Lending Rates on the Option Pricing Model: A Note

JOHN E. GILSTER, JR. and WILLIAM LEE*

ABSTRACT

This paper modifies the Black-Scholes option pricing model to include the effects of transaction costs and different borrowing and lending rates. The paper demonstrates that these market imperfections tend to offset each other yielding a bounded range of prices for each option. The paper also shows that under some conditions the option pricing hedge may be society's lowest cost financial intermediary.

IN THEIR ANALYSIS of options, Black and Scholes [1] created a risk-free hedge by taking opposite positions in options and common stock. If the hedge consists of a long position in common stock and a short position in call options, the hedge will require a positive net investment which will *earn* the risk-free rate (an "investment hedge"). If the hedge consists of a long position in call options and a short position in stock, the hedge supplies funds which will *cost* the borrowing rate (a "borrowing hedge").[1]

If transaction costs are ignored, the option price appropriate to an investment hedge is the traditional Black-Scholes price (and therefore earns the risk-free rate) and the option price appropriate to a borrowing hedge is the Black-Scholes price with the investor's borrowing rate substituted for the Treasury bill rate

* This research was conducted while the authors were, respectively, an Assistant Professor of Finance and a Doctoral Candidate in Physics at the University of Illinois. They are now employed as a Visiting Assistant Professor of Finance at the University of New Mexico and a researcher for California Laser, respectively. The authors would like to thank the University of Illinois Investors in Business Education and the University of Illinois Research Board for financial support. Special thanks to George Constantinides who was smart enough (and kind enough) to point out a serious error in an earlier version of the paper. Remaining errors are, of course, the authors'.

[1] There are a variety of possible applications of the Black-Scholes option hedge concept. In its most elegant incarnation, a zero risk, zero return hedge is formed with either a long position in stock, a short position in call options, and borrowing; or short stock, long options, and lending. Unfortunately, in practice, market imperfections (i.e., discrimination against short sales, transaction costs, and different borrowing and lending rates) make this classical hedge position unprofitable.

Other uses for the option pricing hedge result from the fact that any two of the three elements of the hedge can be used to duplicate the third, thus providing the investor with a choice of mechanisms for achieving the same results. For example, the introduction of this paper makes use of the fact that a long position in stock and a short position in call options (rebalanced) can be used to duplicate the behavior of a Treasury bill, thus providing investors with a choice of risk-free investments. On the other hand, footnote 7 makes use of the fact that a call option (with rebalancing) can be used to duplicate the behavior of a long position in stock that has been partially acquired on margin.

(the hedge therefore *costs* the borrowing rate). Since borrowing rates are higher than lending rates and call option prices increase as interest rates increase, the option price appropriate to a borrowing hedge will be greater than the option price appropriate to an investment hedge.[2]

When transaction costs are considered, option prices must be adjusted so as to earn the investor the risk-free rate on an investment hedge or cost him the borrowing rate on a borrowing hedge *net* of transaction costs. In essence, the profit from an investment hedge comes from the declining time premium of an option that has been short sold (or "written"). The additional revenue needed to pay transaction costs can be generated by *raising* the initial option price to provide for a greater decline in time premium and a greater profit for the hedge's short position in options. In essence, the cost of a borrowing hedge results from the deteriorating time premium of the hedge's long position in options. To provide for borrowing at the market rate after transaction costs, the initial option price must be *reduced* so as to provide for less deterioration in time premium and more funds available to pay transaction costs.

The reader will note that if transaction costs and the borrowing and lending rate spread are of precisely the right size, the transaction cost adjusted option price for an investment hedge can equal the transaction cost adjusted option price for a borrowing hedge. In this case, the market imperfection of different borrowing and lending rates and the market imperfection of positive transaction costs cancel to produce a unique option price at which an investment hedge will yield (net of transaction costs) exactly the Treasury bill rate and a borrowing hedge will cost (net of transaction costs) exactly the borrowing rate.

If transaction costs are large relative to the spread between borrowing and lending rates, the option price at which an investment hedge yields the risk-free rate (net of transaction costs) will be higher than the option price at which a borrowing hedge costs the borrowing rate (net of transaction costs). Between these two transaction cost adjusted option prices will exist a no man's land in which option prices are too low for an investment hedge to compete with Treasury bills and too high for a borrowing hedge to compete with other forms of borrowing.

If transaction costs are low relative to the spread between borrowing and lending rates, the option price at which an investment hedge yields the risk-free rate (net of transaction costs) will be lower than the option price at which a borrowing hedge costs the borrowing rate (net of transaction cost). Any option price between these two transaction cost adjusted prices can be used to form a risk-free investment hedge yielding more than Treasury bills, while *simultaneously* being used to form a borrowing hedge which costs less than the borrowing

[2] In this model, borrowing is assumed to be risk-free (i.e., the person doing the borrowing knows he will have to repay in full and on time). If an individual feels he might be able to escape repayment he would tend to prefer buying stock with borrowed money over an equivalent option position.

Some may wonder how riskless borrowing and lending rates can differ. Here are two possible reasons: (1) In real life, intermediation is not costless; therefore, borrowing and lending rates can differ even if both are risk-free. (2) The model derived in this paper requires borrowing to be risk-free from the perspective of the borrower (i.e., he knows he will have to repay in full), but the borrowing need not be risk-free from the perspective of the lender (possibly due to an information asymmetry).

rate. This is, in effect, society's lowest cost financial intermediary—a financial black hole. A further discussion of this phenomenon is presented in Section II.

I. Transaction Costs of Hedge Rebalancing

The Black-Scholes model can be modified to yield rebalancing transaction costs (in addition to yielding the usual risk-free return) by adding a rebalancing transaction cost term, $\alpha E\,|g|$, to the right-hand side of the original Black-Scholes equation:

$$r^i w - r^i x w_1 - \tfrac{1}{2}\sigma^2 x^2 w_{11} - w_2 = \alpha E\,|g| \tag{1}$$

where r^i, x, and σ^2 are the risk-free rate, the stock price, and the stock variance, respectively, and α is the transaction cost rate applied to the change in capital required for rebalancing, g. Subscripts refer to the partial derivatives of option price, $w(x, t)$, with respect to its first or second argument.

The change in capital required for rebalancing, g, consists of the change in the number of options, Δn, at the changed price, $-w(x + \Delta x, t + \Delta t)$, i.e.,

$$\begin{aligned} g &= -w(x + \Delta x, t + \Delta t)\Delta n \\ &= -[w + w_1\Delta x + \tfrac{1}{2}w_{11}(\Delta x)^2 + w_2\Delta t] \\ &\quad \cdot[n + n_1\Delta x + \tfrac{1}{2}n_{11}(\Delta x)^2 + n_2\Delta t - n] \\ &= (-\Delta x w n_1/\Delta t - w n_2 - \tfrac{1}{2}\sigma^2 x^2 w n_{11} - \sigma^2 x^2 w_1 n_1)\Delta t \end{aligned} \tag{2}$$

Replacing n, n_1, n_{11}, and n_2 by $1/w_1$, $-w_{11}/w_1^2$, $(2w_{11}^2 - w_1 w_{111})/w_1^3$, and $-w_{12}/w_1^2$, respectively, yields:[3]

$$g = \Delta x w w_{11}/(w_1\Delta t) + w w_{12}/w_1 + \sigma^2 x^2(-w w_{11}^2/w_1^2 + \tfrac{1}{2}w w_{111}/w_1 + w_{11}) \tag{3}$$

Equation (3) makes it clear that, as $\Delta t \to 0$, g approaches infinity, yielding the horrifying (although not entirely unexpected) result that, with continuous rebalancing, transaction costs will be infinite for any positive transaction cost rate. Fortunately, this result is more of a mathematical artifact than a practical problem. Tables I and II show that daily rebalancing results in very reasonable transaction costs.[4]

[3] This paper assumes that all rebalancing is conducted by adjusting the option portion of the hedge. A generally similar analysis would apply to common stock rebalancing but in this case:

$$g = w_{12}x + \tfrac{1}{2}\sigma^2 x^3 w_{111} + x^2\sigma^2 w_{11} + w_{11}x\Delta x/\Delta t$$

This paper assumes option rebalancing because it generally involves smaller dollar amounts.

[4] Discrete rebalancing interval applications of the continuous time option pricing model seem to pose no great problem. Footnote 7 of the original Black and Scholes paper argued that the risk associated with short interval rebalancing is uncorrelated with the market, a position later supported in more detail by Boyle and Emanuel [2]. Moreover, Rubinstein [5] and Brennan [3] have demonstrated that discrete interval applications of the Black-Scholes model will be valid under assumptions of constant proportional risk aversion and a bivariate lognormal distribution between market and underlying asset returns.

However, the reader should be warned that, unlike most option papers which derive their model in a purely continuous time framework and then worry about discrete time applications, this paper has a discrete time approximation within the continuous derivation itself.

Table I

The Effects of Transaction Costs and Different Borrowing and Lending Rates on Option Prices

Annual Standard Deviation of Common Stock Returns = 0.15*

		Investing at 12%					Borrowing at 15%			
Stock Price	Time To Exp (Months)	B&S Price (1)	Rebal Adj (2)	Init -End Adj (3)	Net Adj Price (4)	Net Price Spread (5)	Net Adj Price (6)	Init -End Adj (7)	Rebal Adj (8)	B&S Price (9)
34	1	0.000	0.000	0.000	0.000	−0.000	0.000	−0.000	−0.000	0.000
	5	0.211	0.048	0.007	0.266	−0.062	0.204	−0.009	−0.058	0.270
	9	0.825	0.173	0.020	1.019	−0.174	0.845	−0.025	−0.195	1.066
40	1	0.905	0.030	0.014	0.949	−0.031	0.918	−0.015	−0.030	0.963
	5	2.679	0.150	0.035	2.863	−0.042	2.821	−0.040	−0.150	3.011
	9	4.160	0.260	0.049	4.470	0.016	4.485	−0.058	−0.237	4.780
46	1	6.398	0.000	0.048	6.446	0.000	6.447	−0.050	−0.000	6.497
	5	7.989	0.045	0.063	8.097	0.246	8.343	−0.071	−0.036	8.450
	9	9.525	0.106	0.078	9.709	0.429	10.138	−0.091	−0.080	10.308

* Exercise Price = $40.

Table II

The Effects of Transaction Costs and Different Borrowing and Lending Rates on Option Prices

Annual Standard Deviation of Common Stock Returns = 0.45*

		Investing at 12%					Borrowing at 15%			
Stock Price	Time To Exp (Months)	B&S Price (1)	Rebal Adj (2)	Init -End Adj (3)	Net Adj Price (4)	Net Price Spread (5)	Net Adj Price (6)	Init -End Adj (7)	Rebal Adj (8)	B&S Price (9)
34	1	0.282	0.027	0.009	0.318	−0.062	0.255	−0.009	−0.028	0.292
	5	2.441	0.289	0.052	2.782	−0.547	2.235	−0.055	−0.296	2.585
	9	4.265	0.554	0.078	4.896	−0.981	3.916	−0.083	−0.561	4.559
40	1	2.266	0.085	0.038	2.390	−0.198	2.192	−0.039	−0.086	2.317
	5	5.548	0.396	0.083	6.027	−0.715	5.312	−0.087	−0.395	5.793
	9	7.790	0.673	0.110	8.572	−1.136	7.437	−0.116	−0.666	8.218
46	1	6.740	0.067	0.064	6.870	−0.176	6.694	−0.065	−0.066	6.825
	5	9.818	0.403	0.112	10.333	−0.694	9.639	−0.116	−0.394	10.149
	9	12.157	0.690	0.139	12.986	−1.108	11.878	−0.147	−0.672	12.697

* Exercise Price = $40.

Equation (3) also presents a problem because, in principle, the transaction cost adjusted option price should be used to calculate the transaction cost adjustment. Fortunately, the adjustment is small enough so that g can be approximated using unadjusted Black-Scholes values. $E|g|$ is then evaluated by splitting g into positive and negative components and using Sprenkle's formula (see [6, p. 17]) to evaluate each truncated distribution yielding:

$$E|g| = |K + J| + |K - J|$$

where

$$K = x(ww_{11}/(w_1\Delta t))(e^{\rho\Delta t}N((\rho + \sigma^2/2)\Delta t^{1/2}/\sigma) - N((\rho - \sigma^2/2)\Delta t^{1/2}/\sigma))$$

where $N(\ )$ is the cumulative standard normal distribution function and ρ is the expected average rate of growth of the stock price. J consists of the last two terms on the right-hand side of Equation (3).

Borrowing from Cox and Ross [4], we observe that Equations (1) and (3) do not involve risk preferences and can be solved assuming risk neutrality. Therefore, the transaction cost adjustment, c, to the Black-Scholes price will be the present value of the expected value of future transaction costs:[5]

$$c = \alpha \int_0^\infty \int_t^{t^*} e^{-r(t^*-u)} E|g| L'(x)\, du\, dx \tag{4}$$

where $L'(x)$ is the lognormal density function of stock price.[6]

II. Combined Effects of Transaction Costs and Different Borrowing and Lending Rates

Tables I and II illustrate the interest rate/transaction cost effect under plausible assumptions. Column (1) of the tables is the Black-Scholes option price. Column (9) is the Black-Scholes model value with a borrowing rate substituted for the risk-free rate. The common stock standard deviations listed in the tables roughly correspond to the 10th and 90th deciles of the standard deviations observed by Whaley [7].

Columns (2) and (8) ("Rebal Adj") are the difference between the traditional Black-Scholes option price and the daily (i.e., $\Delta t = 1/260$) rebalancing transaction cost price derived in Section I. In addition to the assumptions specified in the tables, one-way transaction costs for options are assumed to be 2% and the underlying stock's expected return is 17% per year.

Columns (3) and (7) ("Init-End Adj") are estimates of the adjustment to the option price required to cover the cost of acquiring the hedge and finally liquidating it. Common stock transaction costs, α_x, are assumed to be 1.25%.[7]

[5] The authors would like to thank an anonymous referee for suggesting the discounted expected value approach to solving Equation (1). Our original solution was considerably more complicated.

[6] Equation (4) was evaluated by numerical integration.

[7] Initiation and termination costs will be lowest for investors for whom buying (or selling) the common stock portion of the hedge is a by-product of other activities. For example, an investor who wishes to sell common stock has the alternative of keeping the stock and hedging against its movements by writing (and rebalancing) $1/w_1$ call options against it and, in effect, selling the stock. Later, when the option expires and the stock is actually sold, the investor saves the present value of the transaction costs on the amount by which the stock price exceeds the option's exercise price. Therefore, the incremental costs of initiating and terminating the hedge are:

$$\text{Cost} = \alpha w/w_1 - \alpha_x w$$

where α_x is the transaction cost rate for common stock.

Columns (4) and (6) ("Net Adj Price") are Black-Scholes option prices with both types of transaction costs added (for the investment hedge) or subtracted (for the borrowing hedge). Column (4) therefore shows the option price which can be used to form an investment hedge yielding (net of transaction costs) exactly the Treasury bill rate; and Column (6) shows the option price which can be used to form a borrowing hedge costing (net of transaction costs) exactly the borrowing rate.

Column (5) ("Net Price Spread") is the result of subtracting Column (4) from Column (6). It should be interpreted as follows.

1) Negative values indicate that the option price is the bounded range between the prices specified in Columns (4) and (6). For option prices within this range, neither investment hedges nor borrowing hedges are particularly attractive. The reader will note that the highest option price which produces an attractive borrowing hedge (Column (6), Net Adj Price – Borrowing) is frequently higher than the Black-Scholes price (Column (1)). When this occurs, an option which sells for the Black-Scholes price can be used to create excess profits by forming a borrowing hedge and (in effect) borrowing at below the market rate.

2) A zero value indicates a unique option price (i.e., the value shown in both Column (6) and Column (4)). This occurs when transaction cost effects and borrowing and lending effects precisely cancel. This unique option price is *never* the Black-Scholes price except in the trivial case where all prices are zero.

3) A positive value indicates that the option hedge can be viewed as a financial intermediary with a lower spread than traditional intermediaries. If the option sells at a price between the prices specified in Columns (4) and (6), the option hedge is *simultaneously* a higher return risk-free investment than Treasury bills and a lower cost source of funds than traditional borrowing. In this case, there is *no* price to which the option can adjust which will eliminate excess profits. For example, if the option price were to drop low enough for the investment hedge to no longer be attractive, this would only make a borrowing hedge even *better*. It may be that the only thing that prevents all short-term borrowing and lending from being sucked into these financial "black holes" is the limited number of investors in the special transaction cost situations described in footnote 7.

This permanent disequilibrium is bounded between the Column (6) and Column (4) prices. If the option price is above the Column (6) price, the option is *not* attractive as a part of a borrowing hedge but a short position in the option will be *very* desirable as part of an investment hedge. This unbalanced selling pressure should drive the option price below the Column (6) price at which time the option is desirable as *both* an investment hedge *and* a borrowing hedge. This presumably results in a better balance between supply and demand for the option.

Similarly, if the option were to sell below the Column (4) price it would be very attractive as a part of a borrowing hedge but there would be no interest in forming investment hedges. The resulting net buying pressure should push the option price back above the Column (4) value.

Therefore, when the Column (5) value is positive, it indicates a bizarre form of bounded disequilibrium.[8]

[8] The existence of these permanent excess profit situations is not particularly sensitive to the assumptions used to create Tables I and II. For example, the phenomenon does not disappear if the transaction cost rate is doubled.

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