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Continuous Maturity Diversification of Default-Free Bond Portfolios and a Generalization of Efficient Diversification

W. JOHN HEANEY and PAO L. CHENG*

ABSTRACT

This paper presents a method for solving the mean-variance portfolio selection problem that is applicable to the case where the number of securities is nondenumerably infinite. Necessary conditions for the existence of an optimal portfolio density are obtained and an expression for the efficient frontier is derived. The conditions for the existence of an optimal portfolio of continuously maturing bonds when their covariance matrix is singular are used to derive an arbitrage-free bond pricing equation. A method for estimating the covariance matrix and the associated efficient frontier is presented.

THE CONSTRUCTION OF an optimal bond portfolio comprising bonds of all possible maturities requires the concept of a portfolio density defined on the continuous maturity of bonds. Obviously, the familiar paradigm of the discrete efficient set, e.g., Merton [5], needs to be extended to handle the continuous maturity problem. However, it turns out that we need, rather than an extension, a generalized approach to portfolio selections. We shall show that the generalized paradigm we are seeking will not only solve the problem of bond diversification with continuous maturities, but also the standard problem of diversification involving any finite set of risky assets. Moreover, the paradigm to be proposed in this paper also will solve the problem where the variance-covariance matrix of returns is singular. Therefore, in Section I, we generalize the traditional discrete paradigm, using the default-free discount bonds as a vehicle to illustrate the new approach. The generalization is equally valid for any finite set of risky assets, for instance, equity issues. In Section II, we expand the generalization for the discrete case to the case where the set of bonds have all possible maturities, i.e., the case of continuous maturity. We shall demonstrate the existence of optimal bond portfolio densities and the associated frontier. The reader will note in that section that the paradigm developed for the continuous case parallels the development of the discrete case in the section that precedes it. In Section III, we consider the case where the structure of the continuous maturity problem is "incomplete" or "degenerate," the analog of the discrete case when the variance-covariance matrix is singular. Given the tendency of bond yields of proximate maturities to move

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together, we anticipate that a degenerate variance-covariance matrix is the normal situation to be considered in a bond portfolio selection problem, although ultimately this is an empirical issue. In Section IV, we show how our results may be applied in a world with continuous trading opportunities and suggest a method for solving the bond portfolio selection problem in practice.

I. A Generalization of the Discrete Mean-Variance Efficient Diversification

We begin by using a finite set of default-free discount bonds to generalize the traditional paradigm of portfolio diversification, although the results are equally valid for any finite set of risky assets.

A dollar invested today in a default-free discount bond maturing in j periods, $j = 1, \cdots, N$, with a yield to maturity r_{0j} will generate a cash flow of $e^{jr_{0j}}$ in j periods. The value of this cash flow at the end of the first period is

$$e^{jr_{0j}-(j-1)\tilde{r}_{1j}}, \qquad j \geq 2 \tag{1}$$

where $\tilde{r}_{1j}$, $j \geq 2$, is the random default-free bond yield, with a maturity of $j - 1$ periods as of the end of period one. The expression in (1) may be viewed either as the random final wealth at the end of period one by investigating one dollar today in a j-period bond, or as the random rate of return plus one. For $j = 1$, (1) reduces to $e^{r_{01}}$, which is just the one period risk-free rate plus one.

Let $\tau = j - 1$ and define (1) as

$$\tilde{r}_\tau \equiv e^{(\tau+1)r_{0,\tau+1}-\tau\tilde{r}_{1,\tau+1}}, \qquad \tau = 1, \cdots, T \tag{2}$$

where τ is the maturity as of the end of period one. Denote by $\tilde{r}$ the random vector with elements $\tilde{r}_\tau$, $\tau = 1, \cdots, T$, with a mean vector μ and a variance-covariance matrix $\Sigma = \| \sigma_{\tau\tau'} \|$, $\tau, \tau' = 1, \cdots, T$. Note, for example, $\tilde{r}_\tau$'s will have a joint lognormal distribution if the yields $\tilde{r}_{1,\tau+1}$, $\tau = 1, \cdots, T$, have a joint normal distribution. Thus, let $\tilde{z}_\tau \sim N(\bar{z}_\tau, \sigma^2(\tilde{z}_\tau))$, where

$$\tilde{z} \equiv (\tau + 1)r_{0,\tau+1} - \tau\tilde{r}_{1,\tau+1}, \qquad \tau = 1, \cdots, T$$

$$\bar{z} \equiv (\tau + 1)r_{0,\tau+1} - \tau\bar{r}_{1,\tau+1}$$

$$\sigma^2(\tilde{z}_\tau) \equiv \tau^2\sigma^2(\tilde{r}_{1,\tau+1})$$

It follows that

$$\mu_\tau = e^{\bar{z}_\tau+(1/2)\sigma^2(\tilde{z}_\tau)}$$

$$\sigma_{\tau\tau} = e^{2\bar{z}+\sigma^2(\tilde{z}_\tau)}(e^{\sigma^2(\tilde{z}_\tau)} - 1)$$

$$\sigma_{\tau\tau'} = e^{\bar{z}_\tau+(1/2)\sigma^2(\tilde{z}_\tau)+\bar{z}_{\tau'}+(1/2)\sigma^2(\tilde{z}_{\tau'})}(e^{\mathrm{Cov}(\tilde{z}_\tau,\tilde{z}_{\tau'})} - 1)$$

where

$$\mathrm{Cov}(\tilde{z}_\tau, \tilde{z}_{\tau'}) = \tau\tau' \, \mathrm{Cov}(\tilde{r}_{1,\tau+1}, \tilde{r}_{1,\tau'+1}), \qquad \tau, \tau' = 1, \cdots, T$$

Given μ and Σ, the customary paradigm is to minimize[1] $x'\Sigma x$, subject to $x'\mu =$

[1] A prime to a vector or a matrix denotes transpose. Vectors without primes are taken to be column vectors.

E and $x'\iota = 1$, where E is any desired expected portfolio return, x is the vector of discrete portfolio weights x_τ, $\tau = 1, \cdots, T$, and ι is the vector of one's. The well-known first-order condition for the problem is

$$\Sigma x = ½(w_1\mu + w_2\iota) \tag{3}$$

where w_1 and w_2 are the Lagrangian multipliers corresponding to the two constraints. Assuming Σ is not singular, it is usual to premultiply (3) by Σ^{-1} to obtain x, and then impose the two constraints to determine w_1 and w_2. An expression analogous to (3) also occurs as the first-order condition to the continuous portfolio selection problem (see Section II), but in this latter case the meaning of Σ^{-1} is far from obvious. We can avoid explicit consideration of Σ^{-1} and solve the portfolio selection problem by introducing the eigenvectors and eigenvalues of Σ, namely, those vectors y_τ and numbers λ_τ such that

$$\Sigma y_\tau = \lambda_\tau y_\tau, \qquad \tau = 1, \cdots, T \tag{4}$$

Assuming nonsingular Σ (the case of singular Σ is treated below in Section III), the eigenvectors of Σ are independent and complete, i.e., any T by one vector can be expressed as a linear combination of the y_τ. Thus, we can write

$$x = \textstyle\sum_{\tau=1}^{T} x_\tau y_\tau \tag{5}$$

and

$$f \equiv ½(w_1\mu + w_2\iota) = \textstyle\sum_{\tau=1}^{T} f_\tau y_\tau \tag{6}$$

where x_τ and f_τ are appropriate scalars. In addition, the eigenvectors of Σ are orthogonal,[2] i.e.,

$$y_\tau' y_{\tau'} = 0, \qquad \tau \neq \tau', \qquad \tau, \tau' = 1, \cdots, T,$$

and normalized, i.e.,

$$y_\tau' y_\tau = 1, \qquad \tau = 1, \cdots, T$$

Thus

$$x_\tau = y_\tau' x$$

and

$$f_\tau = ½ w_1 y_\tau' \mu + ½ w_2 y_\tau' \iota \tag{7}$$

Substituting (5) and (6) into (3) and using (4), we obtain

$$\textstyle\sum_{\tau=1}^{T} (x_\tau \lambda_\tau - f_\tau) y_\tau = 0 \tag{8}$$

Since the y_τ's are independent, we have

$$x_\tau = \lambda_\tau^{-1} f_\tau \tag{9}$$

[2] Since Σ is a real symmetric matrix of full rank, its eigenvectors may be chosen to form an orthonormal set (see Mirsky [7]).

and

$$x = \sum_{\tau=1}^{T} \lambda_\tau^{-1} f_\tau y_\tau = \tfrac{1}{2} w_1 \sum_{\tau=1}^{T} \lambda_\tau^{-1} (y_\tau' \mu) y_\tau + \tfrac{1}{2} w_2 \sum_{\tau=1}^{T} \lambda_\tau^{-1} (y_\tau' \iota) y_\tau \tag{10}$$

Using the following definitions, which are equivalent to those of Merton's [5], x may be expressed in familiar form:[3]

$$a \equiv \iota'(\textstyle\sum_\tau \lambda_\tau^{-1} y_\tau' \mu y_\tau) = \sum_\tau \lambda_\tau^{-1} (y_\tau' \mu)(y_\tau' \iota)$$

$$b \equiv \mu'(\textstyle\sum_\tau \lambda_\tau^{-1} y_\tau' \mu y_\tau) = \sum_\tau \lambda_\tau^{-1} (y_\tau' \mu)^2$$

$$c \equiv \iota'(\textstyle\sum_\tau \lambda_\tau^{-1} y_\tau' \iota y_\tau) = \sum_\tau \lambda_\tau^{-1} (y_\tau' \iota)^2$$

$$x_p \equiv (1/a) \textstyle\sum_\tau \lambda_\tau^{-1} (y_\tau' \mu) y_\tau$$

$$x_q \equiv (1/c) \textstyle\sum_\tau \lambda_\tau^{-1} (y_\tau' \iota) y_\tau$$

Thus, we can express (10) as

$$x = \tfrac{1}{2} w_1 a x_p + \tfrac{1}{2} w_2 c x_q \tag{11}$$

We have followed a circuitous route to obtain (11) from (3), but the payoff is that the method may be generalized in a straightforward fashion to handle the continuous portfolio selection problem. In addition, the same approach is applicable when Σ is singular.

To complete the analysis of this section, we use the two constraints $E = \mu' x$ and $1 = \iota' x$ to obtain from (11) two linear equations

$$E = \tfrac{1}{2} w_1 b + \tfrac{1}{2} w_2 a$$

$$1 = \tfrac{1}{2} w_1 a + \tfrac{1}{2} w_2 c$$

with solutions

$$w_1 = 2\left(\frac{cE - a}{bc - a^2}\right)$$

$$w_2 = 2\left(\frac{b - aE}{bc - a^2}\right) \tag{12}$$

Thus, the optimal portfolio weights for the T maturities are the vector

$$x = a\left(\frac{cE - a}{bc - a^2}\right) x_p + c\left(\frac{b - aE}{bc - a^2}\right) x_q \tag{13}$$

The expression for the efficient frontier follows by using (3), (12), and (13):

$$x' \Sigma x \equiv S^2 = \frac{1}{bc - a^2} (cE^2 - 2aE + b) \tag{14}$$

Note that (12), (13), and (14) are familiar results (see Merton [5]).

[3] x_p and x_q below are, respectively, the familiar "tangent portfolio" and the absolute minimum variance portfolio (see Merton [5]).

II. Continuous Maturity and the Optimal Bond Portfolio Density

The reformulation of the bond portfolio problem with discrete maturities in the preceding section essentially carries over to the case of continuous maturities. However, a slight modification of the preliminaries is necessary.

A dollar invested today in a default free-discount bond maturing beyond the current period with a yield to maturity $r_0(t)$ will generate a cash flow of $e^{tr_0(t)}$ at time t. The value of this cash flow at the end of the current period with length Δt as a unit of time is

$$\tilde{r}(t) \equiv e^{tr_0(t)-(t-1)\tilde{r}_1(t)}, \qquad t > 1$$

where $\tilde{r}_1(t)$ is the random bond yield with a maturity of $t - \Delta t = t - 1$ as of the end of the current period. Let $\mu(t)$ be the mean of $\tilde{r}(t)$, and $\sigma(t, t')$ be the variance-covariance function of $\tilde{r}(t)$ and $\tilde{r}(t')$, $t, t' > 1$. If $t = t'$, $\sigma(t, t')$ is the variance of $\tilde{r}(t)$, otherwise it is the covariance of $\tilde{r}(t)$ and $\tilde{r}(t')$. Of course, the so-called current period can be made as short as desired. Given $\mu(t)$ and $\sigma(t, t')$, the bond portfolio problem may be stated as

$$\begin{aligned} \underset{x(t)}{\text{Min}}\ L = {} & \int_1^T dt \int_1^T dt' x(t)\sigma(t, t')x(t') \\ & + w_1\left(E - \int_1^T x(t')\mu(t')\, dt'\right) \\ & + w_2\left(1 - \int_1^T x(t')\, dt'\right) \end{aligned} \tag{15}$$

where T is the maximum maturity entertained and $x(t)$ is the bond portfolio density, a function we seek to minimize L.

The existence of a solution to the problem calls for several restrictions on the functions entering (15):[4]

(a) $x(t)$ and $\mu(t)$ must be square integrable,[5] i.e.,

$$\int_1^T x^2(t)\, dt < \infty \quad \text{and} \quad \int_1^T \mu^2(t)\, dt < \infty$$

(b) $\int_1^T dt' \int_1^T dt\, \sigma(t, t') < \infty$.

For the ensuing exposition, it will be convenient to adopt the following simple notations when confusion does not arise.

[4] See Mikhlin [6]. We use the word function to mean square integrable functions.

[5] The fact that $x(t)$ must be square integrable rules out the possibility of choosing a finite set of bonds from the available continuum of maturities. In that case, the density would be located at a finite number of points and would have to be infinite at each point so that $\int_1^T x(t)\, dt = 1$. For example, if only one bond of maturity t^* were chosen, $x(t) = \delta(t^* - t)$ where $\delta(t^* - t)$ is the Dirac Delta function $\int_1^T \delta(t^* - t)\, dt = 1$, but $\int_1^T \delta^2(t^* - t)\, dt = \infty$.

(a) If any two functions $h(t)$ and $g(t)$ are square integrable, their "inner product" is denoted by

$$(h, g) = (g, h) \equiv \int_1^T dt\, h(t)g(t)$$

This is analogous to the inner product of two vectors.

(b) The multiplication of a "matrix," e.g., $\sigma(t, t')$ with any "vector" $h(t)$ is denoted by

$$\sigma h \equiv \int_1^T dt'\, \sigma(t, t')h(t')$$

which may in turn be viewed as a "vector" as in the discrete case. It follows that the "inner product" of σh and any function g can be written as

$$(g, \sigma h) \equiv \int_1^T dt \int_1^T dt'\, g(t)\sigma(t, t')h(t')$$

If $g = h$, for example, it is analogous to the "quadratic form" in the discrete case.

With this convention, we restate the portfolio problem in (15):

$$\underset{x(t)}{\text{Min}}\ L = [(x, \sigma x) - 2(x, f) + w_1 E + w_2] \tag{16}$$

where

$$f(t) \equiv \tfrac{1}{2}[w_1 \mu(t) + w_2] \tag{17}$$

$(x, x) < \infty$, $(\mu, \mu) < \infty$, and $(f, f) < \infty$, and $\sigma(t, t')$ is such that:

(a) it is symmetric, i.e., $\sigma(t, t') = \sigma(t', t)$,
(b) it is positive definite, i.e., $(x, \sigma x) > 0$ for any $x(t)$, and
(c) $(g, \sigma h)^2/(g, \sigma g)(h, \sigma h) \leq 1$ for any g and h.[6]

To show that $x(t)$ exists, we invoke three theorems.

THEOREM 1. *If σ is positive definite, the function $x(t)$ which minimizes the functional*

$$F(x) = (x, \sigma x) - 2(x, f)$$

satisfies

$$\sigma x = f$$

and conversely, if $\sigma x = f$ has a solution, $F(x)$ assumes its minimum value for the solution.

For a proof of the theorem, see Mikhlin [6]. It follows that the first-order

[6] This is analogous to $\rho^2 \leq 1$, where ρ denotes the correlation coefficient.

condition for the portfolio problem in (16) is

$$\int_1^T dt' \ \sigma(t, t')x(t') = f(t) \tag{18}$$

Equation (18) is a Fredholm integral equation of the first kind and has been exhaustively studied.[7] $\sigma(t, t')$ is known as the kernel of the integral equation, which is said to be *closed* if there does not exist a function $h(t)$ such that

$$\sigma h \equiv \int_1^T \sigma(t, t')h(t') \, dt' = 0$$

THEOREM 2. *If σ is a closed symmetric kernel, there exists a denumerably infinite set of eigenfunctions $\phi_n(t)$ and eigenvalues λ_n satisfying*

$$\sigma\phi_n \equiv \int_1^T dt' \ \sigma(t, t')\phi_n(t') = \lambda_n\phi_n, \qquad n = 1, 2, \cdots$$

Moreover,

(*a*) *the eigenfunctions are orthogonal and normalized, i.e.,*

$$(\phi_n, \phi_{n'}) \equiv \int_1^T dt \ \phi_n(t)\phi_{n'}(t) = 0, \qquad n \neq n'$$

$$(\phi_n, \phi_n) \equiv \int_1^T dt \ \phi_n^2(t) = 1$$

(*b*) *the eigenfunctions are complete, i.e., given a function $g(t)$ there exists a set of numbers g_n such that*

$$g(t) = \textstyle\sum_{n=1}^{\infty} g_n\phi_n(t)$$

where $g_n \equiv (g, \phi_n) \equiv \int_1^T g(t)\phi_n(t) \, dt.$

It follows that

$$\int_1^T g^2(t) \, dt = \textstyle\sum_{n=1}^{\infty} g_n^2 < \infty$$

since $g(t)$ is square integrable, $(\phi_n, \phi_n) = 1$ and $(\phi_n, \phi_{n'}) = 0$. The proof of the theorem can be found in [8].

THEOREM 3. *For a closed symmetric kernel, the Fredholm equation $\sigma x = f$ has a solution if and only if*

$$\textstyle\sum_{n=1}^{\infty} f_n^2/\lambda_n^2 < \infty$$

where

$$f_n \equiv (f, \phi_n)$$

[7] See, for example, Pogorzelski [8].

The solution is

$$x(t) = \sum_{n=1}^{\infty} f_n \lambda_n^{-1} \phi_n(t) \tag{19}$$

We give the proof as follows, invoking Theorem 2 where necessary.

Since the eigenfunctions of σ are complete, we can write

$$x(t) = \sum_{n=1}^{\infty} x_n \phi_n(t) \tag{20}$$

where

$$x_n \equiv (x, \phi_n)$$

We can write the first-order condition in (18) as

$$\int_1^T \sigma(t, t') \sum_n x_n \phi_n(t')\, dt' = \sum_n f_n \phi_n(t)$$

which yields

$$\sum_n x_n \lambda_n \phi_n(t) = \sum_n f_n \phi_n(t) \tag{21}$$

Now (21) implies[8] that $x_n \lambda_n = f_n$ and we obtain

$$x_n = \lambda_n^{-1} f_n, \qquad n = 1, 2, \cdots \tag{22}$$

Since by construction $x(t)$ is to be square integrable, then by Theorem 2,

$$\int_1^T x^2(t)\, dt = \sum_n x_n^2 < \infty$$

Hence,

$$\sum_n x_n^2 = \sum_n f_n^2 \lambda_n^{-2} < \infty$$

Conversely, if $\sum_n x_n^2 < \infty$, then $x(t)$ is square integrable, with $x_n = f_n \lambda_n^{-1}$. Q.E.D.

Given these three theorems, the solution to the portfolio problem in (12) follows immediately. First, define

$$\mu_n \equiv \int_1^T \mu(t) \phi_n(t)\, dt \equiv (\mu, \phi_n)$$

$$e_n \equiv \int_1^T \phi_n(t)\, dt \equiv (1, \phi_n), \qquad n = 1, 2, \cdots$$

Since the eigenfunctions are complete, we have that

$$\mu(t) = \sum_{n=1}^{\infty} \mu_n \phi_n(t) \quad \text{and} \quad 1 = \sum_{n=1}^{\infty} e_n \phi_n(t)$$

From the definition of $f(t)$ in (17) we have

$$f_n = \tfrac{1}{2}(w_1 \mu_n + w_2 e_n)$$

[8] This is again analogous to the discrete case where y_τ, $\tau = 1, \ldots, T$ are the eigenvectors of Σ. When two linear combinations of y_τ's in T space are equal, then they must be identical linear combinations, where the y_τ are independent.

Thus, from (19), we obtain the solution $x(t)$ in terms of w_1 and w_2:

$$x(t) = \textstyle\sum_{n=1}^{\infty} \tfrac{1}{2}\lambda_n^{-1}(w_1\mu_n + w_2 e_n)\phi_n(t)$$
$$= \tfrac{1}{2}w_1 \textstyle\sum_n \lambda_n^{-1}\mu_n\phi_n(t) + \tfrac{1}{2}w_2 \textstyle\sum_n \lambda_n^{-1}e_n\phi_n(t) \tag{23}$$

where $\sum_n \lambda_n^{-1}\mu_n\phi_n(t)$ and $\sum_n \lambda_n^{-1}e_n\phi_n(t)$ may be viewed as two "linear combinations" in infinite space, i.e., as two "vectors."

We now define, using the two "vectors,"

$$a \equiv (\textstyle\sum_n \lambda_n^{-1}\mu_n\phi_n(t),\ 1)$$
$$\equiv \int_1^T \textstyle\sum_n \lambda_n^{-1}\mu_n\phi_n(t)\,dt = \textstyle\sum_n \lambda_n^{-1}\mu_n e_n$$
$$b \equiv (\textstyle\sum_n \lambda_n^{-1}\mu_n\phi_n(t),\ \mu(t))$$
$$\equiv \int_1^T \textstyle\sum_n \lambda_n^{-1}\mu_n\phi_n(t)\mu(t)\,dt = \textstyle\sum_n \lambda_n^{-1}\mu_n^2$$
$$c \equiv (\textstyle\sum_n \lambda_n^{-1}e_n\phi_n(t),\ 1)$$
$$\equiv \int_1^T \textstyle\sum_n \lambda_n^{-1}e_n\phi_n(t)\,dt = \textstyle\sum_n \lambda_n^{-1}e_n^2$$
$$x_p(t) \equiv (1/a)\textstyle\sum_n \lambda_n^{-1}\mu_n\phi_n(t)$$
$$x_q(t) \equiv (1/c)\textstyle\sum_n \lambda_n^{-1}e_n\phi_n(t)$$

and write (23) as

$$x(t) = \tfrac{1}{2}w_1 a x_p(t) + \tfrac{1}{2}w_2 c x_q(t) \tag{24}$$

which is analogous to (11) in Section II. In addition, we have that

$$E = \tfrac{1}{2}w_1 b + \tfrac{1}{2}w_2 a$$

and

$$1 = \tfrac{1}{2}w_1 a + \tfrac{1}{2}w_2 c$$

by using the constraints $(x, \mu) = E$ and $(x, 1) = 1$. Analogously, we obtain the solutions

$$w_1 = 2\left(\frac{cE - a}{bc - a^2}\right) \quad \text{and} \quad w_2 = 2\left(\frac{b - aE}{bc - a^2}\right) \tag{25}$$

and

$$x(t) = a\left(\frac{cE - a}{bc - a^2}\right) x_p(t) + c\left(\frac{b - aE}{bc - a^2}\right) x_q(t) \tag{26}$$

The associated efficient frontier follows, since

$$(x, \sigma x) = (x, f) = (x, \tfrac{1}{2}(w_1\mu + w_2)) \equiv S^2$$

and by substituting for x, w_1, and w_2 from (25) and (26), we obtain

$$S^2 = \frac{1}{bc - a^2}(cE^2 - 2aE + b) \tag{27}$$

In concluding this section, it is interesting to consider the role played by the eigenfunctions $\phi_n(t)$. Let a portfolio n have the portfolio density

$$z_n(t) \equiv \phi_n(t)/\int_1^T \phi_n(t)\, dt \equiv \phi_n(t)/e_n$$

so that

$$S_n^2 = (z_n, \sigma z_n) = \lambda_n/e_n^2$$

and

$$S_{nn'} = (z_n, \sigma z_{n'}) = 0$$

where $z_{n'}(t) = \phi_{n'}(t)/e_{n'}$, $n \neq n'$. Thus we see that portfolios z represent independent and complete sources of uncertainty. The analog for the discrete case is also easily identified as follows. Define

$$z_\tau = y_\tau/\iota' y_\tau, \qquad \tau = 1, \cdots, T$$

where y_τ is the τ^{th} eigenvector of Σ. Then it follows that

$$\textstyle\sum_{\tau=1}^T z_\tau' \sum z_\tau = \iota' R \iota = \sum_\tau \lambda_\tau = \text{trace of } \Sigma$$

where R is the diagonal matrix of eigenvalues. Note that the trace of Σ is the sum of all the variances of the T risky securities.

III. Optimal Bond Portfolio Density with Degenerate Kernel

We now consider the case when the kernel $\sigma(t, t')$ is not closed, often referred to as the degenerate kernel. The degeneracy is defined as follows. There exists a function $h(t) \neq 0$ such that $\int_1^T \sigma(t, t')h(t')\, dt' = 0$, where $h(t)$ is square integrable. The consequence of degeneracy is that $\sigma(t, t')$ only has a finite set of eigenfunctions and eigenvalues. That is to say, the eigenfunctions are incomplete and can only span a subspace of finite number of dimensions.

Suppose the degenerate $\sigma(t, t')$ has N eigenfunctions and eigenvalues, i.e.,

$$\int_1^T \sigma(t, t')\phi_n(t')\, dt = \lambda_n \phi_n(t), \qquad n = 1, \cdots, N$$

Let $\phi_n(t)$ span a subspace S of dimension N. Given an arbitrary function $g(t)$, it is possible to decompose it into two parts: $g_s(t)$ that lies in S and $g_o(t)$ that is orthogonal to S such that

$$g(t) = g_s(t) + g_o(t)$$

It follows that

$$(g_o, \phi_n) \equiv \int_1^T g_o(t)\phi_n(t)\,dt = 0, \qquad n = 1, \cdots, N \tag{28}$$

and the result

$$(g, \phi_n) = (g_s + g_o, \phi_n) = (g_s, \phi_n) \equiv g_{sn}, \qquad n = 1, \cdots, N$$

Moreover, we state without proof[9] the theorem that a function which is orthogonal to the eigenfunctions of $\sigma(t, t')$ is also orthogonal to $\sigma(t, t')$ itself, i.e.,

$$\sigma g_o \equiv \int_1^T \sigma(t, t')\, g_o(t')\, dt' = 0 \tag{28'}$$

Given the above preliminaries, we consider the portfolio density problem with the first-order condition as before, i.e.,

$$\sigma x = f \tag{29}$$

but with σ having a finite number of N eigenfunctions.

THEOREM 4. *A necessary condition for a solution to the portfolio problem to exist when $\sigma(t, t')$ is degenerate is that there exists a number k such that $\mu(t) - k \in S$.*

Proof: Let

$$x(t) = x_s(t) + x_o(t)$$
$$f(t) = f_s(t) + f_o(t)$$

Then, since

$$\begin{aligned} \sigma x &= \sigma(x_s + x_o) \\ &= \sigma x_s \qquad \text{using (28)} \\ &= \sigma \textstyle\sum_n x_n \phi_n \\ &= \textstyle\sum_n x_n \lambda_n \sigma_n \in S, \end{aligned}$$

a necessary condition for a solution to (29) is that $f \in S$. Thus, we require

$$f_o(t) = f(t) - f_s(t) = 0$$

or

$$\tfrac{1}{2}[w_1\mu(t) + w_2] - \tfrac{1}{2}[w_1 \textstyle\sum_n^N \mu_n\phi_n(t) + w_2 \sum_n^N e_n\phi_n(t)] = 0$$

Rearranging, we obtain

$$\mu(t) - k = \textstyle\sum_n^N (\mu_n - ke_n)\phi_n(t) \in S$$

where

$$k \equiv -w_2/w_1 \qquad \text{Q.E.D.}$$

[9] For a proof, see [8].

Define

$$\nu_s(t) \equiv \textstyle\sum_{n=1}^{N} \ (\mu_n - ke_n)\phi_n(t) = \sum_n^N \nu_{sn}\phi_n(t)$$

so that $\mu(t) = k + \nu_s(t)$. Decomposing $x(t) = x_s(t) + x_o(t)$ and using (28), we now reformulate for this "degenerate" case the portfolio problem as expressed in (15):

$$\begin{aligned}\operatorname*{Min}_{x(t)} &\int_1^T dt \int_1^T dt' \ x_s(t)\sigma(t, t')x_s(t') \\ &+ w_1\left[E - k - \int_1^T x_s(t')\nu_s(t') \ dt'\right] \\ &+ w_2\left[1 - \int_1^T x_s(t') \ dt' - \int_1^T x_o(t') \ dt'\right]\end{aligned} \tag{30}$$

Noting that $\int_1^T x_s(t') \ dt'$ is not restricted, we can reformulate (30) as follows:

$$\begin{aligned}\operatorname*{Min}_{x(t)} &\int_1^T dt \int_1^T dt' \ x_s(t)\sigma(t, t')x_s(t') \\ &+ w_1\left[E - k - \int_1^T x_s(t')\nu_s(t') \ dt'\right]\end{aligned} \tag{31}$$

Theorem 5. *The solution to the reformulated portfolio problem when σ is degenerate is*

$$x_s(t) = (E - k)\frac{a_s}{b_s} x_{ps}(t)$$

where

$$\begin{aligned} x_{ps}(t) &\equiv (1/a_s)\textstyle\sum_n \lambda_n^{-1}\nu_{sn}\phi_n(t) \\ a_s &\equiv \textstyle\sum_n \lambda_n^{-1}\nu_{sn}e_n \\ b_s &\equiv \textstyle\sum_n \lambda_n^{-1}\nu_{sn}^2 \end{aligned}$$

with

$$\begin{aligned}\int_1^T x_{ps}(t) \ d(t) &= (1/a_s)\int_1^T \textstyle\sum_n \lambda_n^{-1}\nu_{sn}\phi_n(t) \ dt \\ &= (1/a_s)\textstyle\sum_n \lambda_n^{-1}\nu_{sn}e_n = 1\end{aligned}$$

Proof: The theorem follows from using the first-order condition

$$\int_1^T \sigma(t, t')x_s(t') \ dt' = w_1 \int_1^T \nu_s(t') \ dt'$$

expanding x_s and ν_s in terms of the eigenfunction ϕ_n as in Theorem 3 and using the constraint in (31). Q.E.D.

If we solve (31) with the constraint $\int_1^T x_s(t) \ dt = 1$, we see, by inspecting (31), that the efficient frontier associated with $x_s(t)$ is a hyperbola in the $E - k$

standard deviation plane. It follows that the unconstrained choice $x_s(t)$ corresponds to a fraction of wealth $(E - k)\frac{a_s}{b_s}$ invested in the portfolio located on the constrained frontier where a tangent drawn to the frontier would intersect the origin in the $E - k$ standard deviation plane.

When the kernel $\sigma(t, t')$ is degenerate, riskless (i.e., zero variance) portfolios may be constructed from the assets. Unless all such portfolios yield the same return k, infinite profits (riskless) would result by shorting a riskless portfolio with low return and investing the proceeds in one with a higher return. That all riskless portfolios yield the same return is guaranteed by Theorem 4, i.e.,

$$(x_o, \mu) = k\left[1 - \frac{a_s}{b_s}(E - k)\right]$$

Thus, the optimal portfolio consists of an amount $1 - \frac{a_s}{b_s}(E - k)$ in a riskless portfolio yielding k, and the rest in the tangent portfolio yielding

$$(x_{ps}, \mu) = k + \frac{b_s}{a_s}$$

Results analogous to Theorems 4 and 5 may be proven in the case where the portfolio selection problem is discrete. Thus, given N assets with mean return vector μ and variance-covariance matrix Σ, we state without proof the following theorem.

THEOREM 4′. *If Σ is the rank $J < N$, the eigenvectors of Σ span a subspace S of dimension $J < N$, and a necessary condition for a solution to the mean-variance portfolio selection problem is that there exists a number k such that $\mu - k\iota \in S$.*

IV. Continuous Trading, Continuous Maturity Diversification, and Conclusion

Theorem 4 or 4′ can be used as the basis for formulating arbitrage arguments with continuous trading opportunities. In this case, at each calendar time τ, the trading horizon is at $\tau + d\tau$. By way of example, if there are two assets where the price of asset 1 follows the stochastic process

$$\frac{dp_1}{p_1} = \alpha\, d\tau + \sigma\, dz$$

where dz is the standard normal Wiener process and the price of asset 2 is a function of p_1, i.e., $p_2 = p_2(p_1, \tau)$, then the variance-covariance matrix per unit time, Σ, is singular. Using Itô's lemma, we have

$$\Sigma = \begin{pmatrix} \sigma^2 & \sigma^2 \dfrac{p_1}{p_2}\dfrac{\partial p_2}{\partial p_1} \\ \sigma^2 \dfrac{p_1}{p_2}\dfrac{\partial p_2}{\partial p_1} & \sigma^2\left(\dfrac{p_1}{p_2}\cdot\dfrac{\partial p_2}{\partial p_1}\right)^2 \end{pmatrix}$$

and there is only one eigenvector

$$y_1 = \begin{pmatrix} 1 \\ \dfrac{p_1}{p_2} \cdot \dfrac{\partial p_2}{\partial p_1} \end{pmatrix}$$

According to Theorem 4′, for there to be a solution to the portfolio problem there must exist a number k such that the instantaneous expected return on the two assets in excess of k is a vector parallel to y_1. Thus, we have two equations:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} - k\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ \dfrac{p_1}{p_2} \cdot \dfrac{\partial p_2}{\partial p_1} \end{pmatrix}$$

where c is a constant, $\mu_1 = \alpha$, and using Itô's lemma,

$$\mu_2 = \frac{1}{p_2}\left(\frac{\partial p_2}{\partial \tau} + \alpha p_1 \frac{\partial p_2}{\partial p_1} + \frac{1}{2}\sigma^2 p_1^2 \frac{\partial^2 p_2}{\partial p_1^2}\right)$$

Substituting μ_2 into the second equation above and eliminating c via the first equation, we obtain the Black-Scholes [1] equation for p_2, where k is interpreted as the riskless rate.

More generally, suppose that in the case of a portfolio of pure discount bonds with continuous maturity, the price of a bond may be written as $p(s, t)$ where s is a vector of N state variables following the joint Wiener process

$$ds_i = \alpha_i \, d\tau + \sigma_i \, dz_i$$

$$E(dz_i \, dz_j) = \rho_{ij} \, dt, \; \rho_{ii} = 1$$

and t is time to maturity of the bond. The instantaneous variance-covariance matrix per unit time for the portfolio is, using Itô's lemma,

$$\sigma(t, t') = \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \rho_{ij} \dfrac{1}{p(t)} \dfrac{1}{p(t')} \dfrac{\partial p(t)}{\partial s_i} \dfrac{\partial p(t')}{\partial s_j} \qquad (32)$$

Note that (32) is degenerate since there are only N eigenfunctions which lie in the subspace spanned by the vectors[10]

$$\frac{1}{p} \frac{\partial p(t)}{\partial s_i}, \qquad i = 1, \cdots, N$$

Thus, using Theorem 4 and Itô's Lemma, we have

$$-\frac{\partial p}{\partial t} + \textstyle\sum_i \alpha_i \dfrac{\partial p}{\partial s_i} + \tfrac{1}{2} \sum_i \sum_j \sigma_i \sigma_j \rho_{ij} \dfrac{\partial^2 p}{\partial s_i \partial s_j} - kp = \sum_{i=1}^{N} c_i \dfrac{\partial p}{\partial s_i} \qquad (33)$$

where c_i are independent of time to maturity t.

Results such as (33) are well known[11] in the literature but their relationship to

[10] See Theorem 6 below.

[11] See, e.g., Brennan and Schwartz [2].

the structure of the variance-covariance matrix and the associated bond portfolio problem have not been stressed.

In practice, the solution to the bond portfolio problem requires that we estimate $\sigma(t, t')$ and solve for its eigenfunctions and eigenvalues. We will assume that σ is degenerate and of rank N and write[12]

$$\sigma(t, t') = \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} f_i(t) f_j(t') \tag{34}$$

where $f_i(t)$ are known independent functions of t. For example, if $N = 1$ and $f(t) = t$, comparison of (32) and (34) may suggest the interpretation of a single state variable s, which is the yield to maturity and the same for all bonds, in which case,

$$b_{11} = \sigma_1^2 \quad \text{and} \quad f_1(t) = -\frac{1}{p}\frac{\partial p}{\partial s} = t = \text{duration}$$

In this case the optimal portfolio density for the risky bond portfolio $x_s(t)$ is linear in t. Analogously, in the one-factor square-root interest rate model of Cox et al. [4], where $x_s(t)$ is monotonic and increasing with t,[13] the bonds of long maturities predominate the efficient risky bond portfolio. The density asymptotically approaches an upper limit so that, given the values of the parameters of their model, bonds of more than 3 years to maturity would be held in roughly equal amounts in the efficient risky bond portfolio.

In general, b_{ij} and N are to be estimated by fitting to the historical data and no illuminating interpretation may be possible. Define the matrix F^* with elements

$$F_{ij}^* = \int_1^T dt f_i(t) f_j(t)$$

and solve the eigenvalue problem

$$BF^* A_j = \lambda_j A_j \tag{35}$$

where B is the matrix $\| b_{ij} \|$. The solution to the portfolio problem can then be constructed with the help of Theorem 6.

THEOREM 6. *The eigenvalues of $\sigma(t, t')$ are the eigenvalues of BF^*, i.e., $\lambda_1, \cdots, \lambda_N$ in (35), while the eigenfunctions of $\sigma(t, t')$ are*

$$\phi_j(t) = \textstyle\sum_{i=1}^{N} a_{ij} f_j(t), \qquad j = 1, \cdots, N$$

where a_{ij} are the elements of the eigenfunctions A_j of BF^*.[14]

It is interesting at this juncture to compare our work with the research by

[12] A degenerate symmetric kernel can always be expressed in the form (34). See Pogorzelski [8].

[13] In their model, the state variable is the instantaneous riskless rate and

$$f_1(t) = 2(e^{\gamma t - 1})/[(\gamma + k + \lambda)(e^{\gamma t - 1}) + 2\gamma]$$

where γ, k, and λ are constants defined in [4].

[14] We omit the proof, which is available on request, in the interest of saving space.

Chamberlain and Rothschild [3].[15] Both papers are formulated in a Hilbert space, both treat an infinite set of assets, and both employ the ideas of zero arbitrage and factor structure. However, the Hilbert spaces employed are quite different. In Chamberlain and Rothschild's (CR) paper, the elements of the space are random variables with a distribution function P defined. The space is $L_2(P)$. In our work, the elements of the space are functions. The portfolio weights themselves are elements of the space. The space is L_2. Regarding the infinite set of assets, CR consider a countable, in the limit countably infinite, set of assets. We, in the main, are concerned with a noncountably infinite set of assets. The connection between these two approaches in this regard is as subtle as that between the continuum and the integers.[16] With respect to zero arbitrage, CR define it in terms of the behavior of a cost function and mean return function. Their definition ensures that the cost and mean return are continuous linear functionals on $L_2(P)$. They exploit this fact to show that the mean-variance efficient set is generated by two portfolios. We, however, do not define a cost function, and while the mean return on a portfolio is a linear functional in L_2, no use is made of this fact. We endeavor to construct mean-variance efficient portfolios as linear combinations of the eigenfunctions of the kernel $\sigma(t, t')$ and show this is possible if the kernel is closed. If the kernel is degenerate, we show that this is possible only if the excess expected return vector is a linear combination of the eigenfunction of σ corresponding to nonzero eigenvalues. When σ is degenerate there are riskless portfolios. By adopting the condition on the excess expected return vector, we ensure the result that the excess return on all riskless portfolios is zero, thereby ruling out arbitrage possibilities. Finally, CR are concerned with approximate factor structures. We, however, view the eigenfunctions of the kernel as "factor loadings," in which case the "factor structure" would be exact.

The purpose of this paper has been to generalize the mean-variance approach to handle the bond portfolio selection problem with continuous maturities. We have demonstrated that the problem is already implicit in continuous trading paradigm when bonds are assumed to mature at each point in time and have shown that when the instantaneous variance-covariance matrix is singular, the instantaneous excess return on bonds over the riskless rate must be a linear combination of the eigenfunctions corresponding to the nonzero eigenvalues of the covariance matrix to prevent arbitrage. Using Itô's lemma, we observed that this latter result leads to pricing equations for bonds. Finally, we have outlined a practical approach for estimating the covariance matrix for bonds and shown how it may be used to solve the normative problem of choosing the optimal risky bond portfolio.

[15] The authors are grateful to an anonymous referee for bringing to our attention the complementary work by Chamberlain and Rothschild [3].

[16] The results of Section I and Theorem 4′ can be extended to the case where the assets are countably infinite by imposing restrictions on the covariance matrix and vectors x and μ, which are analogous to the two restrictions (a) and (b) of Section II. Namely, vectors should be "square summable."

$$\sum_{t=1}^{\infty} x_t^2 < \infty \quad \text{and} \quad \sum_{t=1}^{\infty} \sum_{t'=1}^{\infty} \sigma_{t,t'} < \infty$$

This space is l_2.

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