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Some Results in the Theory of Arbitrage Pricing

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ABSTRACT

This paper derives a stronger version of Huberman's recent "preference free" pricing theorem. This pricing result relates the expected return on an asset to its factor responses *and* the covariance structure of the residuals from a linear factor model. It must characterize any infinite asset economy in which no arbitrage opportunities are present whether or not the factor model has uncorrelated residuals. This result provides the intuition for the role of residual risk in the pricing model and eliminates some classes of arbitrage opportunities still present under Huberman's bound. Some applications to empirical tests and performance measurement are also discussed.

THE ARBITRAGE PRICING THEORY is one of several financial pricing models which attempts to explain the cross-sectional variation in expected returns on assets. In certain respects it is similar to the Capital Asset Pricing Model, but there are both substantive and subtle differences. One of the key advantages of the APT is that it derives a simple linear pricing relation approximating that in the CAPM without some of the latter's objectionable assumptions. Perhaps the key disadvantage of the APT is that it provides no clues as to what might be important factors nor as to how to interpret the factor premiums which appear in the pricing equation.

This paper serves several purposes. After reviewing the linear factor model formulation in Section I, it derives in Section II a stronger version of Huberman's [9] recent "preference free" pricing theorem. This pricing result relates the expected return on an asset to its factor responses and the covariance structure of the residuals. This relation must characterize any infinite asset economy in which no arbitrage opportunities are present whether or not a linear factor model with uncorrelated residuals is the appropriate returns generating mechanism.

Section III specializes this result by eliminating the effects of residual covariances, and compares the results here to those of previous APT models. When applied to the standard linear factor model with uncorrelated residuals, this new pricing bound eliminates arbitrage opportunities which remain under Huberman's bound. This result also provides for the first time the intuition for the role of residual risk in the approximate pricing relation.

Section V provides an interpretation of the factor premiums and gives a valid asset pricing model interpretation to the APT which is consistent with Merton's

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[11] multi-beta asset pricing model. This interpretation overcomes the disadvantage of the APT discussed above.

Section VI discusses the problem of measurement error. There it is demonstrated that the new type of bound introduced here is, under certain circumstances, robust when the parameters are not known precisely.

I. Linear Factor Models

Throughout this paper it is assumed that the returns in excess of the riskless rate realized on all n assets under consideration are generated by the linear factor model

$$\tilde{x} - r1 = a + B\tilde{z} + \tilde{\epsilon} \tag{1a}$$

$$E(\tilde{\epsilon}) = 0 \tag{1b}$$

$$E(\tilde{z}) = 0 \tag{1c}$$

$$E(\tilde{z}\tilde{z}') = I \tag{1d}$$

$$E(\tilde{\epsilon}\tilde{z}') = 0 \tag{1e}$$

$$E(\tilde{\epsilon}\tilde{\epsilon}') = \Omega. \tag{1f}$$

The vectors $\tilde{x}$, a, and $\tilde{\epsilon}$ are each $n \times 1$ in dimension. They represent the realized returns, expected excess returns, and "residual" portions of the returns, respectively. It is assumed that the expected return on each asset is bounded, $| a_i | \leq a < \infty$. $\tilde{z}$ is a $K \times 1$ vector of the realizations on the common factors.[1]

B is the $n \times K$ matrix of factor loadings. It is assumed that each element is bounded in absolute value

$$| b_{ik} | \leq b < \infty. \tag{2}$$

This matrix is partitioned in two ways

$$B' = (b_1, b_2, \cdots, b_n) \tag{3a}$$

$$B = (\beta_1, \beta_2, \cdots, \beta_K). \tag{3b}$$

The components of the vector b_i are the factor loadings of the i^{th} asset for each of the K factors while the components of the vector β_k are the factor loadings on the k^{th} factor for each of the n assets.

Ω is the $n \times n$ positive definite variance-covariance matrix of the residuals. This paper differs from most other APT papers in considering explicitly the case of correlated residuals. We shall denote the diagonal matrix of residual variances

[1] The factor model describing a set of asset returns is not unique until the factors themselves are identified. In many cases the factors do not represent any economic quantities but are only random variables statistically extracted from the distribution of returns. In these cases any orthogonal transformation of the factors fits the data equally well. Thus if the factors z fit the returns with a factor loading matrix of B, the factors $z^* \equiv T'z$ with a factor loading matrix $B^* \equiv BT$ also satisfy (1) for any orthogonal matrix T. Unless otherwise noted we shall assume the factor vector to be the same in each step of any sequence considered.

constructed from the main diagonal of Ω by D_Ω and the correlation matrix of residuals by $R \equiv D_\Omega^{-1/2}\Omega D_\Omega^{-1/2}$. The covariance matrix of returns is $\Sigma = \Omega + BB'$.

The description thus far, apart from the boundedness assumptions, is completely general. For any given n, Conditions (1c) and (1d) can always be assured through a proper scaling and rotation of the factors. Conditions (1b) and (1e) can be satisfied for any factors with the correct choice of B and a.

The approach used here is not to consider a sequence of distinct economies as is sometimes done (e.g., in Huberman [9]). Rather a fixed infinite economy is initially assumed and a sequence of nested subsets of assets is examined. That is, for the n^{th} step, one new asset is added to the first $n - 1$ assets whose parameters do not change

$$a_n' = (a_{n-1}', a_n) \tag{4}$$

$$B_n' = (B_{n-1}', b_n). \tag{5}$$

The "size" of the economy will be denoted by the subscript n on the vectors and matrices; however, these will often be suppressed when the meaning is clear. We shall also work with subsequences[2] of assets in which not every integer is considered. As a reminder we index subsequences by ν. Unless otherwise noted all sequences and subsequences are infinite.

We assume that the matrix B_n has full column rank K for all sufficiently large n. Since the rank of B_n is nondecreasing on any subsequence, this assumption entails no loss of generality provided that the factor loading vectors are not collinear.[3] It is implicit in all our proofs that we consider sequences only after the point at which the rank of B_n is K.

II. Arbitrage Pricing Theory

The primary tools for investigation of the APT are large portfolios with little or no residual risk. In this paper, portfolios will be denoted generically by the n vector α. The i^{th} component measures the dollars invested in the i^{th} risky asset. The remainder $1 - 1'\alpha$ is invested in the riskless asset. It is assumed that α is not a null vector; that is, the portfolio includes one or more long or short positions. A riskless arbitrage opportunity exists with the portfolio α if

$$\mathrm{Var}(\alpha'\tilde{x}) = \alpha'(\Omega + BB')\alpha = 0 \tag{6a}$$

$$E(\alpha'\tilde{x} - r) = \alpha'a > 0. \tag{6b}$$

Since Ω is positive definite (and BB' is positive semidefinite), riskless arbitrage

[2] For an infinite set of numbers with a specific order, $x_1, x_2, \cdots, x_n, \cdots$, a sequence is obtained by deleting a leading portion but retaining the same ordering on the remainder, i.e., $x_m, x_{m+1}, \cdots, x_{m+n}, \cdots$. A subsequence is obtained after any deletion which leaves an infinite number of elements in the same order, i.e., $x_{\nu_1}, x_{\nu_2}, \cdots, x_{\nu_n}, \cdots$ where $\nu_1 < \nu_2 < \cdots < \nu_n < \cdots$. The following important property of subsequences is used in this paper. A sequence x_n does not have a limit L if and only if there exists some subsequence x_ν which is bound away from L. That is, for some $\delta > 0$, $|x_{\nu_i} - L| > \delta$ for all ν_i.

[3] The rank of B_n is well defined for any finite n. For large n, however, the rows of B_n may be "nearly" collinear if the factor loadings are not "sufficiently different" across the assets. This issue is addressed in Lemma 5.

opportunities cannot be available. In our infinite economy, however, asymptotic arbitrage opportunities might exist. Asymptotic arbitrage opportunities have been used previously by Huberman [9]. A sequence of portfolios is said to generate an asymptotic arbitrage opportunity if along some subsequence

$$\lim_{\nu\to\infty}\mathrm{Var}(\alpha_\nu'\tilde{x}) = 0 \tag{7a}$$

$$\alpha_\nu' a \geq \delta > 0 \quad \text{for all } \nu. \tag{7b}$$

In his definition, Huberman [9] required that an asymptotic arbitrage opportunity have an infinite expected profit in the limit. There is no distinction between these two definitions. If an arbitrage opportunity of the type described in (7) exists, then it will always be possible to earn an infinite arbitrage profit as well.[4,5]

We now present the basic pricing relation based on the premise of an absence of asymptotic arbitrage opportunities. This theorem is an extension of a similar theorem by Huberman [9, Theorem 1].

THEOREM 1. *Assume that the returns on an infinite set of assets are generated by a linear factor model* (1) *and* (2). *If no arbitrage opportunities of the type described in Equation* (7) *are available, then there is a sequence of length K vectors of factor premiums* λ_n *which price all assets approximately,*

$$a \approx B\lambda. \tag{8}$$

Specifically there exists a positive number V such that the weighted sum of the squared pricing errors is uniformly bounded,

$$(a_n - B_n\lambda_n)'\Omega_n^{-1}(a_n - B_n\lambda_n) \leq V < \infty \quad \text{for all } n. \tag{9}$$

If asset variances are uniformly bounded then Ω *may be replaced by the correlation matrix of residuals R in* (9).

Proof: Since Ω is positive definite, it can be factored as $\Omega = CC'$ where C is a nonsingular matrix of size n. Now consider the orthogonal projection[6] of the

[4] For an exact arbitrage opportunity with a finite number of assets, an infinite profit could be earned by scaling up any portfolio satisfying (6) so that the portfolio weights became unboundedly large. Some care must be taken when applying this method to (7) since the original portfolio's variance is only asymptotically zero. One way to construct an infinite profit portfolio from a finite profit portfolio is to set $\hat{\alpha}_n = \alpha_n/\sqrt{\sigma_n}$ where σ_n^2 is the variance on the portfolio α_n. Since $\sigma_\nu \to 0$ and $\alpha_\nu' a$ is bounded away from zero for some subsequence, $\hat{\alpha}_\nu' a = \alpha_\nu' a/\sqrt{\sigma_\nu} > \delta/\sqrt{\sigma_\nu}$ becomes unbounded and $\lim \mathrm{Var}(\hat{\alpha}_\nu'\tilde{x}) = \lim \sigma_\nu^2/\sigma_\nu = 0$.

[5] Huberman [9] provides a brief discussion of his definition of arbitrage which also applies here. Briefly the existence of such an arbitrage provides an opportunity for an investor to achieve unboundedly large wealth with probability one. It does not, however, guarantee that the certainty equivalent wealth is also unboundedly large. Sufficient conditions for this to be true, as given by Ross [12], are that utility be bounded below or that the utility function be uniformly integrable. Necessary conditions remain unknown. Thus, Huberman's results and those here are not completely free of assumptions on preferences.

[6] Note that this projection is precisely the OLS estimation of Equation (10). It is not the GLS estimation of the equation $a = B\lambda + v$ since Ω is not the cross-sectional covariance structure of the residuals $E[vv']$. The "improper" covariance matrix is used because it is not our purpose, at least directly, to minimize unexplained expected returns in a statistical sense. Rather we wish to determine what characteristics of unanticipated returns can lead to deviations in expected returns.

vector $C^{-1}a$ into the column space of $C^{-1}B$

$$C^{-1}a = C^{-1}B\lambda + u$$

$$\lambda \equiv (B'\Omega^{-1}B)^{-1}B'\Omega^{-1}a. \tag{10}$$

Define the pricing errors as

$$v \equiv a - B\lambda = Cu. \tag{11}$$

By the orthogonality of the projection,

$$0 = B'C'^{-1}u = B'\Omega^{-1}v. \tag{12}$$

Thus, $\alpha = \Omega^{-1}v(v'\Omega^{-1}v)^{-1}$ is a portfolio with no factor risk whose expected excess return and variance are

$$\alpha'a = (v'\Omega^{-1}v)^{-1}v'\Omega^{-1}a = (v'\Omega^{-1}v)^{-1}v'\Omega^{-1}(B\lambda + v) = 1 \tag{13a}$$

$$\mathrm{Var}(\alpha'\tilde{x}) = (v'\Omega^{-1}v)^{-2}v'\Omega^{-1}(BB' + \Omega)\Omega^{-1}v = (v'\Omega^{-1}v)^{-1}. \tag{13b}$$

Therefore if (9) is violated, the variance vanishes along some subsequence, and an arbitrage opportunity exists.

If asset variances are bounded then $D_\Omega \leq dI$ and

$$v'R^{-1}v \leq dv'D_\Omega^{-1/2}R^{-1}D_\Omega^{-1/2}v = dv'\Omega^{-1}v. \tag{14}$$

The quadratic form with Ω is bounded as just demonstrated so that the one with R must be as well. Q.E.D.

COROLLARY. *Under the conditions of Theorem* 1, *there exists a fixed vector* λ^* *which prices with bounded total error.*

Proof: This proof is omitted since it is similar to Huberman's [9] proof of his second theorem and corollary. The arbitrage portfolio's weights are multiplied by Ω^{-1} and normalized as in Theorem 1 above. Q.E.D.

Theorem 1 and its corollary must hold regardless of the number of factors which are extracted from the assets' returns. This feature permits the following characterization of the accuracy condition (9). The maximum Sharpe measure, the ratio of expected excess return to standard deviation, achievable with a set of assets is $[a'\Sigma^{-1}a]^{1/2}$. When no factors are extracted in the linear model, the value which is bounded in (9) is the square of the Sharpe measure. One requirement of the accuracy condition is, therefore, that there be a finite tradeoff between expected return and standard deviation. Similarly when one or more factors are extracted, the accuracy condition requires that there be a finite tradeoff between the remaining expected return unexplained by the existing factors and residual standard deviation.

III. A Comparison with Previous Models

The usual assumption in linear factor models is that the residuals are uncorrelated, i.e., $\Omega = D$, a diagonal matrix. If this assumption is adopted, then the accuracy bound in (9) is $v'D^{-1}v \leq V < \infty$. With the additional assumption of an upper limit d on residual variances, the accuracy bound becomes $v'v \leq dv'D^{-1}v \leq dV < \infty$. This last condition is that derived by Huberman [9] under precisely these two additional assumptions.

In certain respects these bounds are more pleasing than those in (9). In particular, each asset's contribution to the error depends only on its own return structure. On a more practical basis, inversion of a large matrix is not required. To achieve uncorrelated residuals, however, typically requires using many more factors than are actually needed for an unweighted bounded total pricing error. Furthermore, in many cases it is simply impossible to describe the covariance structure of returns by a linear model with uncorrelated residuals and a fixed number of factors for all n even when a covariance-free pricing bound is valid. These simplified bounds are valid in a diagonal factor economy and a factor economy with bounded variation, respectively.

Definition: A factor economy representation is *diagonal* if the norm of the correlation matrix of residuals is uniformly bounded; $\| R_n \| \leq r < \infty$ for all n.[7] A factor economy has bounded residual variation if the norm of the covariance matrix of residuals is uniformly bounded; $\| \Omega_n \| \leq w < \infty$ for all n.

Since every element of a correlation matrix is less than one in absolute value, the norm of R can be large only if there is "substantial" correlation among the residuals. For example, at the extreme, if all the residuals are perfectly correlated then $\| R_n \| = n$ and the economy is not diagonal. An economy can fail to have bounded variation for only one of two reasons. Either it is not diagonal or its residual variances are not bounded. A diagonal economy with bounded asset variances must have bounded variation since $\| \Omega \| \leq \| D_\Omega \| \; \| R \|$. Note that the converse is not true. An economy with bounded variation need not be diagonal. Bounded variation does imply diagonality, however, if residual variances are uniformly bounded away from zero since $\| R \| \leq \| D_\Omega^{-1} \| \; \| \Omega \|$. Also note that a bound on asset variances implies that residual variances are bounded but does not guarantee "bounded residual variation" in the absence of diagonality.

THEOREM 2. *Assume that the returns on an infinite set of assets are generated by a linear factor model with arbitrarily correlated residuals and that there are no asymptotic arbitrage opportunities. A sufficient condition for the covariance-free pricing bound*

$$(a - B\lambda)' D_\Omega^{-1} (a - B\lambda) \leq V < \infty \quad \text{for all } n \tag{15}$$

to obtain is that the economy be diagonal. A sufficient condition for the equally

[7] A matrix norm is a nonnegative scalar number assigned to a matrix as a measure of its "magnitude." Throughout this paper we use the matrix norm induced by the Euclidean norm (or length) of a vector. $\| A \| \equiv \sup(x'A'Ax/x'x)$; $x \neq 0$. This norm is also known as the spectral norm of a matrix. For symmetric matrices it is equal to the largest eigenvalue. Except in Theorem 2, the matrices whose norms are required are of fixed dimension for all n and we are only concerned if they vanish in the limit. Since any valid matrix norm is a continuous function of the matrix's elements and is zero if and only if all elements of the matrix are zero, any matrix norm could be used in these cases. One such useful norm of a $k \times k$ matrix is $M(A) \equiv k \max | a_{ij} |$.

weighted pricing bound

$$(a - B\lambda)'(a - B\lambda) \le V < \infty \quad \text{for all } n \tag{16}$$

to obtain is that the economy have bounded variation.

Proof: Using Lemma 0 (in the Appendix)

$$v'D_\Omega^{-1}v \le v'D_\Omega^{-1/2}R^{-1}D_\Omega^{-1/2}v \,\| R \| \equiv v'\Omega^{-1}v \,\| R \|. \tag{17}$$

The second factor is bounded by assumption, and, in the absence of arbitrage opportunities, the first factor is also bounded as proved in Theorem 1. Therefore, the product is bounded.

Similarly using Lemma 0 for the bound in (16)

$$v'v \le v'\Omega^{-1}v \,\| \Omega \|. \tag{18}$$

Again the two factors on the right-hand side of (18) are bounded so their product must be as well. Q.E.D.

The accuracy bound in (16) is the same as that derived by Huberman [9] for uncorrelated residuals. Chamberlain and Rothschild [4] and Stambaugh [16] have independently proved this same stronger result (under seemingly different conditions[8]) for correlated residuals.

At first it might appear that the bound in (15) is only a minor modification of that in (16) useful for little more than application in the case of large residual variances (i.e., large $\| D_\Omega \|$). While Equation (15) certainly does provide this service, it is also a tighter bound on pricing in the opposite extreme when some variances are small, as the following example demonstrates.

Consider an economy with no factors, uncorrelated residuals, and parameters: $a_i = i^{-1}$, $d_{ii} = i^{-2}$, for i even; and $a_i = -(i+1)^{-1}$, $d_{ii} = (i+1)^{-2}$ for i odd. Clearly for all even n, $\lambda_n = 0$ is the best fit, and

$$v_n'v_n = \tfrac{1}{2} \textstyle\sum_1^{n/2} i^{-2} \tag{19}$$

converges to $\pi^2/6$ so (16) is satisfied. Equation (15) is violated, however, since

$$v_n'D_n^{-1}v_n = \textstyle\sum_1^n 1 = n. \tag{20}$$

Thus, the portfolio $\alpha_i = -\alpha_{i-1} = i/n$, for i even, represents an asymptotic arbitrage opportunity not precluded by (16). Since the economy in this example has no correlation among residuals, the improvement of (15) over (16) obviously applies to Huberman's bound as well.

[8] Chamberlain and Rothschild [4] assume that only K eigenvalues of the returns' covariance matrix become unbounded. Stambaugh [16] assumes that it is possible to decompose the returns' covariance matrix as $\Sigma = BB' + D - A$ where D is a diagonal matrix with bounded elements and A is nonnegative definite. Stambaugh [16, footnote 11] has demonstrated that these two conditions are equivalent. The second assumption given in Theorem 2 is likewise equivalent. This is most easily seen when compared to Stambaugh's representation. The residual covariance matrix is identified as $\Omega = D - A$. With bounded elements in D and A nonnegative definite, $\| \Omega \|$ is clearly bounded by $\max(d_{ii})$. To show the converse, choose any bounded D with $\min(d_{ii}) \ge \| \Omega \|$ and take A as the residual $A \equiv D - \Omega$, then by Lemma 0, quadratic forms in A are bounded by $x'Ax = x'(D - \Omega)x \ge x'x[\min(d_{ii}) - \| \Omega \|] \ge 0$ so A is nonnegative definite.

The intuition as to why the weighted squared error bound in (15) dominates the unweighted bound in (16) is that an asset's usefulness in arbitrages is limited by its residual variance. Roughly speaking, the smaller an asset's residual variance the more extreme its weighting can be in a portfolio, while maintaining diversification, and the more extreme is its weighting, the bigger is its effect on the portfolio's expected return. Thus, an asset with a small residual variance can have a major impact on a diversified portfolio's expected return, and to prevent arbitrage its own expected return must be more nearly in line with the APT's prediction. Assets with larger residual variances can only have small effects on a diversified portfolio's expected return, so their own expected returns need not be as closely in line with the prediction.

One application of Theorem 2 is for determining the "factor size" of an economy. The economy should have small residual covariances to assure that all relevant factors have been removed and priced. On the other hand, we also want the economy to have no extraneous factors which make negligible contributions to pricing. These issues are discussed in detail in the next section.

IV. Complete Factor Representations

In Theorem 2 it was proved that under certain conditions not all of the covariances among assets' returns needed to be considered when developing the no-arbitrage linear pricing rule. The basic requirement was that the correlation matrix of the residuals remained "small." No discussion was given at that point as to how one might select factors so that this would be true. This section discusses the difference between important (priced) factors and unimportant contributions to residual covariances.

Before proceeding with the analysis, some definitions and lemmas are collected at this point. Proofs of the lemmas are in the Appendix.

LEMMA 1. *For any fixed set of n assets and given order of factor extraction, the norm of the residual covariance matrix is nonincreasing in the number of factors.*

This lemma guarantees that if a particular factor economy has bounded residual variation, then the economy resulting from the extraction of additional factors also has bounded residual variation. The characteristics of the factor loadings are also important. Following Connor[9] we define:

Definition: A factor economy is *regular* if

$$\lim_{n\to\infty} \| Q_n^{-1} \| = 0 \tag{21}$$

where $Q_n \equiv B_n'B_n$. If the limit in (21) is positive then the factor representation is *irregular.*

LEMMA 2. *The regularity measure* $\| Q^{-1} \|$ *is nonincreasing in the number of assets n for any fixed representation (i.e., fixed columns).*

Lemma 2 together with the nonnegativity of norms guarantees that the limit in (21) always exists. Therefore, every representation is either regular or irregular, and for the latter $\| Q^{-1} \|$ is bounded away from zero.

[9] In an earlier version of [6].

LEMMA 3. *The regularity measure is nondecreasing in the number of factors for any fixed set of assets.*

Like Lemma 1 this lemma guarantees that if a particular factor economy is regular, then the economy resulting from replacing a factor back into the residual covariance matrix is also regular.

Definition: A factor loading matrix B with ρ columns is a *complete factor representation* if it is regular and has bounded residual variation.

LEMMA 4. *Let B_ρ be a complete factor representation, then any representation with factors removed has unbounded variation and any representation with added factors is irregular.*

Lemma 4 guarantees that, for a given economy, complete factor representations can only come in a particular (possibly infinite) size ρ. In this sense, it is meaningful to talk of the number of factors in an economy even if residual covariances remain. In a complete economy, no additional factors can be extracted from the covariance matrix of residuals which are pervasive in influence and distinct in effect. This notion can be made precise as follows.

LEMMA 5. *Let B be a complete factor representation and let β be a vector of loadings on a new factor which could be extracted from the residuals. Then β is close to collinear with the columns of B; i.e., there exist vectors w and c such that $\beta = Bc + w$ with $w'w$ uniformly bounded for all n.*

With these lemmas the main result of this section can now be proved. In a complete factor representation there is a unique fixed vector of factor premiums which prices assets with bounded squared error.

THEOREM 3. *If asset returns are generated by a complete linear factor economy,*[10] *then the sequence of factor premiums $\lambda_n \equiv (B_n'B_n)^{-1}B_n'a_n$ has a limit λ_∞ which is the unique fixed vector of premiums satisfying*

$$v_n'v_n \equiv (a_n - B_n\lambda_n)'(a_n - B_n\lambda_n) \le V < \infty \quad \text{for all } n. \tag{22}$$

Proof: From Theorem 2 the sequence λ_n prices with bounded total squared error. By Huberman's [9] second theorem or by analogy with the corollary to Theorem 1, some fixed vector λ^* with bounded total squared error also exists. We wish to prove that λ^* is unique and $\lambda_n \to \lambda^*$.

Consider a fixed premium vector μ with errors u_n at step n. Then

$$\begin{aligned} u_n'u_n &\equiv (a_n - B_n\mu)'(a_n - B_n\mu) \\ &= [v_n - B_n(\mu - \lambda_n)]'[v_n - B_n(\mu - \lambda_n)] \\ &= v_n'v_n + (\mu - \lambda_n)'B_n'B_n(\mu - \lambda_n) \\ &\ge (\mu - \lambda_n)'(\mu - \lambda_n)/\|(B_n'B_n)^{-1}\|. \end{aligned} \tag{23}$$

[10] Throughout the rest of this paper, we will assume that for some ρ sufficiently large $\|\Omega_\rho\|$ is uniformly bounded so that some finite complete factor representation exists. If this is not the case then regularity can be defined as $\|B'\Omega^{-1}B\| \to 0$ and a complete representation is the regular representation with the fewest number of factors. This and the remaining theorems have obvious analogs for the pricing bound $v'\Omega^{-1}v$ which was introduced in Theorem 1.

The cross-product term is zero from the orthogonality of the projection. The inequality follows from Lemma 0 proved in the Appendix.

If we set $\mu = \lambda^*$, then the right-hand side of (23) must remain bounded for all n. The matrix norm in the denominator vanishes by assumption so this can be true only if $\lambda_n \rightarrow \lambda^*$, element by element. Thus the limit λ_∞ exists.

Obviously there cannot be two distinct fixed premium vectors with bounded total weighted squared error. If there were, the sequence λ_n would have to have two different limits. Q.E.D.

V. Interpreting the APT Factor Premiums

Much of the popularity of the arbitrage pricing model is due to its derivation of the "multifactor CAPM-like equation" without the latter's objectionable assumption of a joint normal (or another elliptical[11]) distribution for all asset returns and without the objectionable conclusion that all investors hold the market portfolio. The Arbitrage Pricing Theory has its own drawbacks. As mentioned previously the pricing relation is only approximate. In addition it need not be unique, nor are the factor premiums identified as they are in the multifactor asset pricing model. The second problem was addressed in the previous section. Here it is demonstrated that under the same conditions the factor premiums can be identified.

In the APT, portfolios which have been diversified to remove "all" of the idiosyncractic risk take the place of the mutual funds of Ross' [14] separating distributions or the market and zero-beta portfolios of the CAPM. These portfolios also are important in pricing.

Definition: A *well-diversified portfolio sequence* is any sequence of portfolios in which the fractional investments α_n satisfy

$$\lim_{n\rightarrow\infty}\alpha_n'\Omega\alpha_n = 0. \tag{24}$$

That is, a well-diversified portfolio sequence has vanishingly small residual risk in the limit.

Exactly which portfolio sequences are well diversified depends on the factor representation chosen. In an economy with bounded residual variation, portfolio sequences satisfying $\alpha_n = o(1/\sqrt{n})1$ are well diversified. So, for example, an equally weighted portfolio sequence is well diversified. If residual variances are unbounded, however, this is not sufficient, since a few assets can contribute enormously to the residual variance of a portfolio. In a diagonal economy portfolio, sequences satisfying $\alpha_n = o(1/\sqrt{n})D_\Omega^{-1/2}1$ are well diversified.

[11] Recently Chamberlain [3] has completely characterized all multivariate distributions for which any portfolio's mean return and variance are sufficient statistics for its entire distribution. When a riskless asset is available, these distributions are the elliptical ones (so called because the loci of equal probability density are ellipsoidal). Any elliptical distribution, for which the means, μ, covariance matrix, Σ, and density function, $f(x)$, are defined, satisfies $f(x) \propto |\Sigma|^{-1/2}g[(x-\mu)'\,\Sigma^{-1}(x-\mu)]$ for some strictly positive univariate function $g(\cdot)$. If returns and state variables are joint elliptically distributed, then all portfolios are completely characterized by their mean and variance of return and their covariance with each state variable. In this case a multi-beta CAPM equation describes expected returns.

It is appropriate to contrast this notion of diversification under the APT with that in the CAPM. In the latter model a portfolio is *fully* diversified only if it is perfectly correlated with the market portfolio (or some combination of it and the zero-beta portfolio in the two-factor version of the CAPM). In the APT essentially all large and roughly balanced portfolios are well diversified. The corresponding positive prediction is that investors hold large portfolios and not necessarily just the levered market. This prediction is certainly more in keeping with the usual intuitive explanations of the benefits of diversification and, as suggested before, accounts for the APT's popularity.

THEOREM 4. *The expected excess returns on all well-diversified portfolio sequences are correctly given with zero error in the limit by any linear pricing model satisfying Theorem* 1. *Precisely, if* $\bar{x}_n^\alpha - r \equiv a_n^\alpha = \alpha_n' a_n$ *is the expected excess return on the* n^{th} *portfolio in the well-diversified portfolio sequence* α_n, *then*

$$\lim_{n\to\infty}(a_n^\alpha - \alpha_n' B_n \lambda_n) = 0 \tag{25}$$

for any premium vector sequence satisfying (9). *If* $\| \Omega_n^{-1} \|$ *is uniformly bounded then well-diversified portfolios are correctly priced by any factor premium vector sequence satisfying* (16) *in Theorem* 2.

Proof: The expected excess return on the portfolio in the n^{th} step of the sequence is $a_n^\alpha = \alpha_n' B_n \lambda_n + \alpha_n' v_n$. Denoting the (nonsingular) "square root" of Ω by C as in Theorem 1, then by the Cauchy-Schwarz inequality the square of the second term is bounded by

$$(\alpha_n' v_n)^2 = (\alpha_n' C_n C_n^{-1} v_n)^2 \leq (\alpha_n' \Omega_n \alpha_n)(v_n' \Omega_n^{-1} v_n). \tag{26}$$

Thus

$$0 \leq (a_n^\alpha - \alpha_n' B_n \lambda_n)^2 \leq (\alpha_n' \Omega_n \alpha_n)(v_n' \Omega_n^{-1} v_n). \tag{27}$$

The rightmost expression in (27) is the product of a term which converges to zero as n approaches infinity (by assumption) and a bounded term (by Theorem 1). Therefore (25) must hold. If $\| \Omega_n^{-1} \| < \infty$, then the last factor in (27) is bounded whenever the pricing result in (16) is valid. Q.E.D.

A portfolio sequence can be well diversified in two distinctly different fashions. If in the limit the investment is entirely in the riskless asset, $1'\alpha_n \to 0$, then Theorem 4 is trivially satisfied since $a_n^\alpha \to 0$ and $B_n'\alpha_n \to 0$. The theorem has force when the investment in the risky assets remains substantial even in the limit. In this case the residual risk has actually been diversified away and not simply avoided.

We now turn our attention to giving an asset pricing model interpretation to the vector of factor premiums. As with the CAPM there are two versions of the APT. One corresponds in form to the Black or zero-beta version of the asset model; the other is in form similar to the Sharpe-Lintner version of the model. While the forms are similar, the interpretations are not identical. The zero-beta CAPM arises when there is no riskless asset. The "zero-beta" APT obtains even though a separate riskless asset is available in the economy.

Which of the two forms is valid depends upon the factor structure. If it is possible to construct a well-diversified portfolio that is also free of factor risk, the Sharpe-Lintner version is appropriate. If this construction is not possible, then the zero-beta version is correct. Arbitrage pricing theory is not based on the equilibrium between the risky and riskless assets as is the CAPM. It is based upon the absence of arbitrage opportunities. If there is no way to create a portfolio of risky assets which is free of risk, no arbitrage comparisons can be made between the risky and riskless assets.

Definition: A factor representation has no *essential* (*factor*) *risk* if it is possible to construct a well-diversified portfolio sequence, fully invested in the risky assets, with a vanishingly small loading on each factor. That is, there exists a sequence of portfolios α_n satisfying

$$1'\alpha_n = 1 \tag{28a}$$

$$B_n'\alpha_n = 0 \tag{28b}$$

$$\lim_{n\to\infty}\alpha_n'\Omega_n\alpha_n = 0. \tag{28c}$$

If there is no essential factor risk then the portfolio sequence α_n defines a (limiting) riskless asset since it has no factor risk and vanishingly small residual risk. If a factor economy has essential factor risk then it is impossible to construct a portfolio fully invested in the risky assets which is completely free from risk even in the limit.[12]

LEMMA 6. *Essential factor risk is absent in complete factor economy if the augmented factor loading matrix* $\hat{B} \equiv (1, B)$ *is regular*; $\lim \| (\hat{B}'\hat{B})^{-1} \| = 0$. *If* $\| \Omega^{-1} \|$ *is uniformly bounded away from zero, then this condition is necessary as well.*

The intuition for this result can be explained by analogy with a multivariate regression model. The norm of $B'B$ vanishes since the factor representation is complete. Therefore, if the norm of $\hat{B}'\hat{B}$ does not vanish some columns of the factor loading matrix have "near-extreme multicollinearity" with a vector of ones. In this case any fully invested well-diversified portfolio will have a nonzero factor loading on a factor associated with one of these columns.

Whenever this "near-extreme multicollinearity" occurs, it is possible to construct a different factor representation in which the collinearity is captured by a single pervasive factor with a common impact. That is, the factor representation can be expressed as

$$\tilde{x} - r1 = a + \tilde{\zeta}1 + B\tilde{z} + \tilde{\epsilon} \tag{29}$$

where B and z now represent the effects of the $\rho - 1$ common factors and $\tilde{\zeta}$ is the common or pervasive risk. This representation is complete and $\hat{B}$ is regular.[13]

[12] The condition in Lemma 6 below is equivalent to the condition in Proposition 2 of Chamberlain and Rothschild [4].

[13] This representation is constructed by first rotating the original factors so that the "collinearity" among the columns of B and a vector of ones is confined to a single column of the new factor loading matrix. This column of the new factor loading matrix will be approximately constant. Precisely it will be $b_0 1 + w$ with $w'w$ uniformly bounded (by Lemma 5). This extra common risk can now be

From now on we will assume that any complete factor representation is expressed as in (29) where $\tilde{\zeta}$ is possibly a degenerate random variable identically zero when essential factor risk was originally absent.

It is clear from the representation in (29) that all portfolios of risky assets have the same ζ-risk. Well-diversified portfolios will contain no residual risk and the full component of ζ-risk. In addition it will be possible to construct well-diversified portfolios with any pattern of factor loadings (on the new factors in (29)). Since well-diversified portfolios are priced exactly, the following asset pricing model interpretation can be given to the APT.

THEOREM 5. *If B is a complete factor loading as expressed in* (29), *then the unique bounded squared error pricing relation may be expressed as*

$$a \approx a^0 1 + B(a^* - a^0 1) \tag{30}$$

where a^0 is the excess expected return on a well-diversified portfolio with only ζ risk, ($a^0 = 0$ if there is no essential factor risk) and a_k^* is the excess expected return on a well-diversified portfolio with a loading of one on the k^{th} factor and a zero loading on all the rest.

Proof: The proofs for the two cases are nearly the same. We show that for the case of no essential factor risk and discuss how it is modified.

We first prove by construction that the desired diversified portfolio sequences exist. Consider the sequences of portfolios α_m which are the solutions to the problems $m = 1, \cdots, \rho$

$$\begin{aligned} &\text{Min } \alpha_m'\alpha_m \\ &\text{subject to } \quad 1'\alpha_m = 1 \\ &\qquad\qquad\quad B'\alpha_m = \iota_m \end{aligned} \tag{31}$$

where ι_m is the m^{th} column of a $\rho \times \rho$ identity matrix. $\hat{B}'\hat{B}$ is nonsingular so the solutions to (31) are

$$\alpha_m = \hat{B}(\hat{B}'\hat{B})^{-1}\begin{pmatrix} \iota_m \\ 1 \end{pmatrix}. \tag{32}$$

These sequences are well diversified since

$$\alpha_m'\alpha_m = (\iota_m', 1)(\hat{B}'\hat{B})^{-1}\begin{pmatrix} \iota_m \\ 1 \end{pmatrix} \le 2 \; \| (\hat{B}'\hat{B})^{-1} \|. \tag{33}$$

The final equality follows from Lemma 0, and this norm vanishes because $\hat{B}$ is complete.

By Theorem 4 $a_{mn}^* \equiv \alpha_{mn}' a_n$ satisfies

$$\lim_{n\to\infty}(a_{mn}^* - \alpha_{mn}'\hat{B}_n\lambda_n) = \lim_{n\to\infty}[a_{mn}^* - (\iota_m', 1)\lambda_n] = 0. \tag{34}$$

But the limit of λ_n exists by Theorem 3 and ι_m is fixed so the limit a^* exists.

subsumed into the residuals giving a new residual covariance matrix of $\Omega + ww'$. Since the original economy had bounded variation and $\| ww' \| = w'w$, which is bounded, the new representation also has bounded variation.

Finally by Theorem 4 such portfolios are priced exactly

$$a^* = B(\alpha_1, \cdots, \alpha_p)\lambda = I\lambda = \lambda \tag{35}$$

so the proposed pricing is valid with bounded squared error.

When there is essential factor risk, the last portfolio is constructed to satisfy $B'\alpha_p = 0$. In place of (35) we will then have

$$\begin{aligned} a^0 &= (0', 1)\lambda = \lambda_p \\ a^* &= [B(\alpha_1, \cdots, \alpha_{p-1}), 1]\lambda \\ a_k^* &= \lambda_k + \lambda_p. \end{aligned} \tag{35'}$$

Solving (35′) for λ and substituting into the pricing result gives (30). Q.E.D.

As a final aspect of our comparison of the two models note that the limit portfolios of the sequences α_m are perfectly correlated with their corresponding factors

$$\begin{aligned} \mathrm{Corr}(\alpha'_{mn}\tilde{x}, \tilde{z}_m) &= \alpha'_{mn}\beta_m[\alpha'_{mn}(BB' + \Omega)\alpha_{mn}]^{-1/2} \\ &= [1 + \alpha'_{mn}\Omega\alpha_{mn}]^{-1/2} \rightarrow 1. \end{aligned} \tag{36}$$

Thus, these portfolios are equivalent to the portfolios of "best hedges" described by Merton.[14]

VI. Linear Models with Measurement Error

In most potential applications of the APT, we will not know how well a linear generating model fits the true distribution of returns. If we do not insist on uncorrelated residuals, then the existence of many linear generating models satisfying all the assumptions of (1) and (2) except for bounds on the parameters is guaranteed. But even in this case the parameters may remain unknown. In this section we derive a pricing restriction when the factor loadings are only known subject to measurement error.

This result has potential applications in at least two areas. If the measurement error is statistical in nature, then Theorem 6 can be used to relate average realized returns to estimated factor loadings. If the measurement error is the difference between the true parameters and the parameters of an investor's own predictive density for the returns, then Theorem 6 places restrictions on the beliefs any investor must have so that no arbitrage opportunities are perceived. These restrictions could then be useful in performance measurement.

THEOREM 6. *Assume that the returns on an infinite set of assets are generated by the linear factor model in Equations* (1) *and* (2) *but that the parameters a and B*

[14] For any finite n the portfolio α_m is not generally the best hedge. The portfolio with investments in proportion to $(BB' + \Omega)^{-1}\iota_m$ provides the highest correlation with the m^{th} factor. Note that it depends upon Ω which does not affect α_m. Furthermore if the residual variances are strictly positive then a perfect hedge is not obtainable except in the limit.

are observed with error

$$a^o = a + \tilde{u}$$
$$B^o = B + \tilde{U}. \tag{37}$$

If these errors are independent of the realizations on the factors and the residuals and if no arbitrage opportunities are perceived, then there exists a sequence of vectors of factor premiums λ_n^o which prices all assets approximately. Specifically there exists a positive number V such that

$$(a_n^o - B_n^o\lambda_n^o)'T_n^{-1}(a_n^o - B_n^o\lambda_n^o) \le V < \infty, \quad \text{for all } n \tag{38}$$

where $T \equiv E[\tilde{u}\tilde{u}' + \tilde{U}\tilde{U}'] + \Omega$. (Note: It need not be assumed that the observations B^o are unbiased.[15])

Proof: Rewrite the arbitrage model as

$$\tilde{x} - r1 = a^o + B^o\tilde{z} + \tilde{\epsilon}^o$$
$$\tilde{\epsilon}^o \equiv \tilde{\epsilon} - \tilde{u} - \tilde{U}\tilde{z}. \tag{39}$$

Then the apparent covariance matrix of the residuals is

$$T \equiv E(\tilde{\epsilon}^o\tilde{\epsilon}^{o\prime}) = E[\tilde{u}\tilde{u}' + \tilde{\epsilon}\tilde{\epsilon}' + \tilde{U}\tilde{z}\tilde{z}'\tilde{U}'] = E[\tilde{u}\tilde{u}' + \tilde{U}\tilde{U}'] + \Omega \tag{40}$$

where the simplification follows from the assumed independence. The remainder of the proof is similar to that of Theorem 1. If (38) is violated, then, by holding the portfolio $\alpha = T^{-1}v(v'T^{-1}v)^{-1}$, an investor can achieve a profit with an expectation of one and a vanishingly small variance. Thus, to insure that no asymptotic arbitrage opportunity is available, the bound in (38) must hold. Q.E.D.

The most obvious application of this theorem is in empirical work where the parameters must be estimated. Suppose we adopt the standard assumption that the returns vector is independent and identically distributed over time and estimate the parameters a and B using a given time series of observations. Any errors in the estimates a^o and B^o depend only upon past realizations of $\tilde{z}$ and $\tilde{\epsilon}$. By assumption these are independent of future realizations so Theorem 6 applies and the point estimates cannot display apparent arbitrage opportunities since these could be realized in the future. The indicated portfolio is a true (asymptotic) arbitrage since the portfolio weights chosen eliminate any estimation risk as well as the residual risk. If Theorem 6 is refuted by the evidence, then either arbitrage is possible or the returns vector is not i.i.d.

The derived factor premiums will be measured subject to error since their estimators depend upon B^o and a^o. Equation (38) automatically corrects for this measurement error, however, as well as that in the original parameters by including the estimation covariances in the bound.

A second potential application is in the context of a heterogeneous beliefs model. Under the appropriate assumptions, Theorem 6 restricts each investors'

[15] The observations a^o are assumed to be unbiased under the observer's expectations by definition.

beliefs about expected returns to be approximately linear to his beliefs about B. Note that investors may have different beliefs about the factor premiums.

VII. Conclusion

This paper has derived a number of propositions about arbitrage pricing as it relates to linear returns generating models. Theorems 1 and 2 provide stronger no-arbitrage conditions in the spirit of Huberman's "simple approach" to the APT. Also developed are sufficient conditions for the APT pricing relation to be unique and for it to have a multi-beta asset pricing model interpretation. A final result deals with arbitrage pricing under imperfect information.

Appendix

This appendix provides the proofs of the Lemmas.

LEMMA 0. *All quadratic forms of a symmetric positive definite matrix A are bounded by*

$$x'x\,\|A^{-1}\|^{-1} \leq x'Ax \leq x'x\,\|A\|. \tag{A1}$$

Proof: Let $c_1(A) \geq \cdots \geq c_k(A) \geq 0$ denote the eigenvalues of A in descending order. Then

$$\|A\|^2 \equiv \sup_{x\neq 0}\left(\frac{x'A^2x}{x'x}\right) = c_1(A^2) = c_1^2(A)$$

$$\|A\| = c_1(A) = \sup_{x\neq 0}\left(\frac{x'Ax}{x'x}\right) \tag{A2}$$

from which the upper bound follows. Similarly

$$\|A^{-1}\|^2 \equiv \sup\left(\frac{x'A^{-2}x}{x'x}\right) = c_1(A^{-2}) = c_k^{-2}(A)$$

$$\|A^{-1}\|^{-1} = c_k(A) = \inf\left(\frac{x'Ax}{x'x}\right) \tag{A3}$$

from which the lower bound follows. Q.E.D.

We now give the proofs of the six lemmas on regularity in the body of the paper. We will use a subscript n or ρ on the matrices Q and B to denote the number of assets or factors, respectively.

Proof of Lemma 1: Let β be the next factor to be extracted, then $\Omega_\rho = \Omega_{\rho+1} - \beta\beta'$ so $\|\Omega_{\rho+1}\| \leq \|\Omega_\rho\|$. Q.E.D.

Proof of Lemma 2: Partitioning the factor loading matrix by assets

$$Q_n^{-1} = (Q_{n-1} + b_n b_n')^{-1} = Q_{n-1}^{-1} - \left(\frac{1}{1 + b_n' Q_{n-1}^{-1} b_n}\right) Q_{n-1}^{-1} b_n b_n' Q_{n-1}^{-1}. \tag{A4}$$

Thus

$$\| Q_n^{-1} \| \leq \| Q_{n-1}^{-1} \| \tag{A5}$$

where the inequality follows because the final matrix in (A4) is positive semidefinite while both Q_n and Q_{n-1} are positive definite. Q.E.D.

Proof of Lemma 3: Let A and C be symmetric positive definite matrices related by

$$C = \begin{pmatrix} A & a \\ a' & \alpha \end{pmatrix}. \tag{A6}$$

Clearly $\inf(x'Ax/x'x) \geq \inf(x'Cx/x'x)$. Therefore, by (A3) $\| A^{-1} \| \leq \| C^{-1} \|$. Since

$$Q_\rho = \begin{pmatrix} Q_{\rho-1} & B'_{\rho-1}\beta_\rho \\ \beta'_\rho B_{\rho-1} & \beta'_\rho \beta_\rho \end{pmatrix} \tag{A7}$$

this result applies, and $\| Q_{\rho-1}^{-1} \| \leq \| Q_\rho^{-1} \|$. Q.E.D.

Proof of Lemma 4: Let β denote a component vector of B, and $B_{\rho-1}$ be B with β removed. Then

$$\| Q_\rho^{-1} \| = \left\| \begin{pmatrix} B'_{\rho-1}B_{\rho-1} & B'_{\rho-1}\beta \\ \beta' B_{\rho-1} & \beta'\beta \end{pmatrix}^{-1} \right\| \geq 1/\beta'\beta \tag{A8}$$

where the inequality follows from Lemma 3. Since this norm vanishes, $\beta'\beta \to \infty$. Now $\Omega_{\rho-1} = \Omega_\rho + \beta\beta'$ so

$$\| \Omega_{\rho-1} \| \geq \| \beta\beta' \| = \beta'\beta \to \infty, \tag{A9}$$

proving that smaller representations have unbounded variation.

Now let β denote an additional factor loading vector to be appended to B forming $B_{\rho+1} = (B, \beta)$. Then as in (A8)

$$\| Q_{\rho+1}^{-1} \| \geq 1/\beta'\beta. \tag{A10}$$

But as in (A9) $\| \Omega_\rho \| \geq \beta'\beta$. Since Ω_ρ has bounded variation its norm is finite for all n so $\beta'\beta$ is also finite. Thus, $B_{\rho+1}$ is irregular.

By Lemma 3 the regularity measure is nondecreasing so both parts of this lemma remain valid if more than a single factor is added or removed. Q.E.D.

Proof of Lemma 5: Consider the orthogonal projection of the new vector of factor loadings β into the column space of B

$$\beta = Bc + w$$
$$c = (B'B)^{-1}B'\beta. \tag{A11}$$

By the orthogonality of the projection $B'w = 0$ for all n. Now define $\eta \equiv (w'w)^{-1}$. The regularity measure of (β, B) is

$$\| Q_{\rho+1}^{-1} \| = \| ((\beta, B)'(\beta, B))^{-1} \| =$$
$$\left\| \begin{pmatrix} c'Q_\rho c + \eta^{-1} & c'Q_\rho \\ Q_\rho c & Q_\rho \end{pmatrix}^{-1} \right\| = \left\| \begin{pmatrix} \eta & -\eta c' \\ -\eta c & Q_\rho^{-1} + \eta cc' \end{pmatrix} \right\|. \tag{A12}$$

Now if η vanishes in the limit, then the norm in (A12) is asymptotically equal to $\| Q_\rho^{-1} \|$. This latter norm vanishes since B is regular. But by assumption B is also complete so, by Lemma 4, $B_{\rho+1}$ cannot be regular. Therefore η does not vanish. Q.E.D.

Proof of Lemma 6: Assume that, for all n above some value sufficiently large, the rank of $\hat{B}_n$ is one more than the rank of B_n. If this is not the case, then $\hat{B}_n'\hat{B}_n$ is singular for all n so the norm of its inverse becomes unbounded. Also there is no solution to (28a) and (28b) for any n so the conditions can not be satisfied.

If this rank condition is met then $\hat{B}_n$ is of full column rank for large n since B_n has full column rank in a complete economy. Thus, the problem

$$\begin{aligned} &\text{Min } \alpha'\alpha \\ &\text{subject to} \quad 1'\alpha = 1 \\ &\qquad\qquad\quad B'\alpha = 0 \end{aligned} \tag{A13}$$

has the solution

$$\alpha_n = \hat{B}_n(\hat{B}_n'\hat{B}_n)^{-1}\begin{pmatrix}1\\0\end{pmatrix}. \tag{A14}$$

This sequence satisfies (28a,b) by construction and is well diversified since for all n

$$\begin{aligned} \alpha'\Omega\alpha &\le \| \Omega \| \ (1, 0')(\hat{B}'\hat{B})^{-1}\begin{pmatrix}1\\0\end{pmatrix} \\ &\le \| \Omega \| \ \| (\hat{B}'\hat{B})^{-1} \|. \end{aligned} \tag{A15}$$

Both inequalities follow from Lemma 0. Since the economy is complete, $\| \Omega \|$ is bounded. The other norm vanishes by assumption so (28c) is satisfied.

To prove necessity, consider the projection in (A11) for $\beta = 1$.

$$w = 1 - Bc. \tag{A16}$$

Now if $\| (\hat{B}_n'\hat{B}_n)^{-1} \|$ does not have a limit of zero, it must be bounded away from zero since, as in Lemma 2, this norm is nonincreasing. Thus by (A12) $w'w \le W$. For any portfolio sequence satisfying (28a,b), $\alpha'w = \alpha'1 - \alpha'Bc = 1$. By Lemma 0 residual variance is bounded by

$$\alpha'\Omega\alpha \ge \alpha'\alpha / \| \Omega^{-1} \|. \tag{A17}$$

The denominator is finite by assumption while the numerator is bounded away from zero by the Cauchy-Schwarz inequality.

$$\alpha'\alpha \ge (\alpha'w)^2/w'w = 1/w'w \ge 1/W > 0. \tag{A18}$$

Thus (28c) cannot be satisfied.

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