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Non-Standard C.A.P.M.'s and the Market Portfolio

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RICHARD ROLL (8) CHANGED OUR WAY OF thinking about the CAPM model, both in terms of theoretical content and empirical testing. One of Roll's key contributions was to show that if the market portfolio were efficient, the Sharpe-Lintner-Mossin or zero beta form of the CAPM *must* follow.

The purpose of this article is to extend Roll's work to non-standard forms of the CAPM. In recent years, the literature has contained over a dozen extensions of the simple CAPM. Of particular interest have been models which incorporate uncertain inflation, differential taxes on income and capital gains, heterogenous expectations, non-marketable assets, missing assets and international securities. In this article we will show that models that have been proposed in each of these areas follow directly from the assumption that investors act as if some particular market portfolio is efficient. We believe that doing so is important for several reasons. First, it is a generalization of Roll's work which shows that his analysis and many of his conclusions can be extended well beyond his initial contribution. Second, it allows us to derive alternative equilibrium models using only algebra in a very simple and straightforward way. Furthermore, since the derivations are almost exactly parallel, having mastered one, the reader can move easily through the others. Third, we show that the differences between most non-standard forms of the CAPM come about from a difference in which market portfolio is assumed efficient and an assumption about how to change from the definition of returns used to define the market portfolio to the usual meaning of returns (dividends plus capital gains). This analysis highlights exactly what are the differences between models and shows that many of the asserted differences are nonexistent.

The final reason this paper is important is that it stresses that each model arises directly from the efficiency of a *particular* version of the market portfolio. Therefore, on a theoretical level, one can think about which version of a portfolio is likely to be efficient and on an empirical level the correctness of any model is likely to be decided by which "market portfolio" is efficient.

The following sections of the paper are divided by type of non-standard CAPM. The types discussed incorporate non-marketable assets, international assets, inflation, taxes, and heterogenous expectations.

I. The Effect of Missing and Non-marketable Assets

Mayers (7) derived the impact of both missing and nonmarketable assets on general equilibrium returns. Actually Mayers' treatment of the missing asset case

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was the first demonstration of the type of derivation we shall use throughout the remainder of this paper.

In the case of missing assets the goal is to express equilibrium in terms of the characteristics of the set of assets which has improperly been excluded from the market portfolio as well as the characteristics of the set of assets which has been included:

Let

r'_M be the return on all assets—the true market return

r_M be the return on those assets which have been included in the incomplete definition of the market portfolio

r_H be the return on those assets which have been inappropriately excluded from the definition of the market portfolio

P_M be the market value (at the beginning of the period) of all assets included in the incomplete definition of the market portfolio

P_H be the market value (at the beginning of the period) of all assets inappropriately excluded from the market portfolio

r_i be the return in asset i

r_F be the riskless lending and borrowing rate

β'_i be the Beta coefficient between the return on asset i and the true market portfolio

Following Roll or Mayer the true market portfolio should be efficient and in the presence of unlimited riskless lending and borrowing a version of the standard CAPM holds

$$\bar{r}_i = r_F + \frac{\operatorname{cov}(r_i r'_M)}{\operatorname{var}(r'_M)} (\bar{r}'_M - r_F) \tag{1}$$

The return on the true market portfolio is simply a weighted average of the return on the misestimated market portfolio and the missing assets as follows:

$$r'_M = \frac{P_M}{P_H + P_M} r_M + \frac{P_H}{P_H + P_M} r_H \tag{2}$$

substituting equation (2) into equation (1) and simplifying

$$\bar{r}_i = r_F + \left[(\bar{r}_M - r_F) + \left(\frac{P_H}{P_M}\right)(\bar{r}_H - r_F)\right] \frac{\operatorname{cov}(r_i r_M) + \left(\frac{P_H}{P_M}\right)\operatorname{cov}(r_i r_H)}{\operatorname{var}(r_M) + 2\left(\frac{P_H}{P_M}\right)\operatorname{cov}(r_H r_M) + \left(\frac{P_H}{P_M}\right)^2 \operatorname{var}(r_H)} \tag{3}$$

This equation must hold for all risky assets, including the market portfolio of risky assets. Setting $\bar{r}_i = \bar{r}_M$ and rearranging

$$\frac{\frac{P_H}{P_M}(\bar{r}_H - r_F) + (\bar{r}_M - r_F)}{\mathrm{var}(r_M) + 2\frac{P_H}{P_M}\mathrm{cov}(r_H r_M) + \left(\frac{P_H}{P_M}\right)^2 \mathrm{var}(r_H)} = \frac{\bar{r}_M - r_F}{\mathrm{var}(r_M) + \frac{P_H}{P_M}\mathrm{cov}(r_M r_H)} \tag{4}$$

Substituting equation (4) into equation (3) yields

$$\bar{r}_i = r_F + \frac{\bar{r}_M - r_F}{\mathrm{var}(r_M) + \frac{P_H}{P_M}\mathrm{cov}(r_H r_M)}\left[\mathrm{cov}(r_i r_M) + \frac{P_H}{P_M}\mathrm{cov}(r_i r_H)\right] \tag{5}$$

This is simply the return form of Mayers (7) equation (27) which is expressed in terms of prices. This analysis is very similar to the analysis presented in Mayer (7). The major reason for including it is that it exactly parallels several of the proofs that follow in latter sections of this article but is less complex.

Now let us derive the relationship for non-marketable assets. Let r_M' now stand for the return on the aggregate of all risky assets including non-marketable assets, the subscript $_H$ stand for non-marketable risky assets and r_M stand for the portfolio of risky marketable assets.[1] Let r_{pi} be the return on investor i's optimum portfolio which includes marketable and non-marketable assets. Then following Roll for any marketable asset we can write[2]

$\bar{r}_k = r_F + \lambda_i \mathrm{cov}(r_{pi} r_k)$

λ_i is a coefficient representing the ratio of excess return to variance on the optimum portfolio held by investor i (6)

Define w_i as the dollars of wealth which investor i will place in risky assets (including non-marketable risky assets). Dividing both sides of equation (6) by λ_i and multiplying both sides of w_i

$$\frac{w_i}{\lambda_i}(\bar{r}_k - r_F) = \mathrm{cov}(r_k, w_i r_{pi})$$

Dividing both sides of this equation by $\sum_i w_i$ summing over all i and recognizing that $\frac{\sum_i w_i r_{pi}}{\sum_i w_i} = r_M'$ (since the sum of all risky assets is the market portfolio of risky assets) we obtain after rearranging

$$\bar{r}_k = r_F + \frac{\sum_i w_i}{\sum_i \frac{w_i}{\lambda_i}}\mathrm{cov}(r_k r_M') \tag{7}$$

[1] We follow Mayer in treating r_H and P_H as observable variables for expository purposes. This proof differs from that presented in Mayers.

[2] Note this expression does not hold for any non-marketable asset. An investor can not purchase and sell the non-marketable assets. Thus if derivatives were taken to arrive at an equation like this for non-marketable assets a Lagrangian multiplier associated with each asset would be included in the expression.

This equation must hold for all marketable risky assets or portfolios of marketable risky assets. It does not have to hold for non-marketable risky assets because equation (6) did not have to hold for non-marketable assets. One portfolio it must hold for is the market portfolio of all marketable risky assets. Using equation (7) to define $\bar{r}_M$, solving for $\dfrac{\sum_i w_i}{\sum_i \dfrac{w_i}{\lambda_i}}$ and substituting into (7) we obtain

$$\bar{r}_k = r_F + \frac{\operatorname{cov}(r_k r'_M)}{\operatorname{cov}(r_M r'_M)} (\bar{r}_M - r_F)$$

Recall that

$$\operatorname{cov}(r_k r'_M) = \frac{P_M}{P_M + P_H} \operatorname{cov}(r_k r_M) + \frac{P_H}{P_M + P_H} \operatorname{cov}(r_k r_H)$$

and

$$\operatorname{cov}(r_M r'_M) = \frac{P_M}{P_M + P_H} \operatorname{var}(r_M) + \frac{P_H}{P_M + P_H} \operatorname{cov}(r_M r_H).$$

Substituting these two expressions into the previous equation and simplifying

$$\bar{r}_k = r_F + \frac{\bar{r}_M - r_F}{\operatorname{var}(r_M) + \dfrac{P_H}{P_M} \operatorname{cov}(r_M r_H)} \left[\operatorname{cov}(r_k r_M) + \frac{P_H}{P_M} \operatorname{cov}(r_k r_H)\right]$$

which is essentially equation (5) with the definition of the terms modified as explained above. This is the general equilibrium equation presented in Mayer (7) equation 20 page 230. The other versions of the general equilibrium relationships presented in Mayer are just algebraic manipulations of this equation.

II. International C.A.P.M.

Three of the best known and most often quoted C.A.P.M.s are the models presented by Grauer, Litzenberger and Stehle (4). Solnick (14), and Senbet (11). These models are concerned with exchange risk and the effect of exchange risk on both the price of goods and the return on assets.

Solnick assumes that investors consume a domestic good but can invest in their own assets and assets of other countries. Exchange rate risk adds to the uncertainty of the return of foreign assets. However, Solnick argues that this risk can be eliminated through the use of lending in some countries and borrowing in others. This differentiates Solnick's model from other models discussed in this section.

Grauer, Litzenberger, and Stehle (4) assume that all assets and all commodities are freely traded. Exchange rate fluctuations are simply changes in the relative prices of the goods offered by the various countries. This results in the value of a security not being affected by the country of domicile of the investor.

Senbet does not make an explicit assumption about the prices of consumption goods and exchange rate movements. Rather, he assumes the existence of a supercurrency which can be used to price all goods and local currencies.

A. Solnick Model

Solnick's C.A.P.M. follows from two assumptions. The first is that investors can eliminate exchange risk through hedging. The second is that investors consume only in their own country. Since investors only consume in their home currency, the appropriate way in which to examine portfolio returns is in units of the investor's home currency. Thus the relevant efficient frontier for any investor will be in units of his or her home currency. Domestic assets are already in the appropriate units. Foreign assets are converted to the appropriate units by hedging activity that eliminates exchange risk.

All investors are assumed to have homogeneous expectations regarding both an asset's return in its home currency and exchange rate fluctuations. These two assumptions together imply that investors in any one country have homogeneous expectations about returns on foreign securities as well as domestic securities, both before and after hedging to eliminate exchange risk. Thus all investors in any one country face the same frontier of risky securities. Furthermore, a riskless lending and borrowing rate is assumed to exist in each country although the rate may differ between countries. Homogeneous expectations and riskless lending and borrowing insure that a single portfolio is optimal for all investors in any one country (though it may differ between countries).

Under the conditions just discussed, from Roll's analysis the Sharpe-Lintner model must hold with respect to the optimal portfolio held by the investors in any country (e.g. country k). If we let r_k^* be the return on the optimal portfolio for an investor living in country k given that the appropriate hedged situation is taken, then

$$\bar{r}_{ik}^* - r_{Fk} = \frac{\operatorname{cov}(r_{ik}^* r_k^*)}{\operatorname{var}(r_k^*)} (\bar{r}_k^* - r_{Fk})$$

where

1. $\bar{r}_{ik}^*$ is the return on asset i to an investor in country k given that he has hedged away exchange risk.
2. r_{FK} is the riskless lending and borrowing rate in country k

If r_i is the return on an asset of country k in units of its own currency, then $r_i = r_{ik}^*$ and for assets of country k and we have

$$\bar{r}_i - r_{Fk} = \frac{\operatorname{cov}(r_i r_k^*)}{\operatorname{var}(r_k^*)} (\bar{r}_k^* - r_{Fk}) \tag{8}$$

To get from this equation to Solnick's final equation it is necessary to define the hedging strategy. Solnick's hedge consists of borrowing a sum of money in the foreign market equal to the amount of assets purchased in that country and investing the proceeds from the loan in the domestic riskless asset. Thus any loss on the foreign investment due to exchange rates should be approximately offset by gains on the foreign debt while gains on investment from changes in exchange rates are offset by losses on the foreign debt.

Let r_k be the return on the optimal portfolio for an investor in country k when the return on each security is defined in its own (home) currency. Then, the after hedged return on the portfolio selected by investors in k is related to the before hedged return as shown below.

$$\bar{r}_k^* = \bar{r}_k - \sum_{\substack{j=1 \\ j\neq k}}^{N} W_j^k r_{Fj} + \sum_{\substack{j=1 \\ j\neq k}}^{N} W_j^k r_{Fk} = \bar{r}_k - \textstyle\sum_{j=1}^{N} W_j^k r_{Fj} + r_{Fk} \tag{9}$$

(1) W_j is the proportion of total wealth (across all countries) invested in the assets of country j

(2) W_j^k is the proportion of their wealth which investors from country k invest in country j

(3) $N =$ the total number of countries

substituting equation (9) into equation (8) yields:

$$\bar{r}_i - r_{Fk} = \frac{\operatorname{cov}(r_i r_k)}{\operatorname{var}(r_k)} \left(\bar{r}_k - \textstyle\sum_{j=1}^{N} W_j^k r_{Fj}\right) \tag{10}$$

Note that we have dropped the asterisk in variance and covariance terms. This is permissible since the substitution into these expressions for r_k^* involves r_k and constant (non-stochastic) terms and the constant terms drop out in taking expected values. Defining λ_k as

$$\frac{(\bar{r}_k - \sum_{j=1}^{N} W_j^k r_{Fj})}{\operatorname{var}(r_k)}$$

Multiplying both sides by $\dfrac{W_k}{\lambda_k}$, summing, recalling $\sum_{k=1}^{N} W_k r_k = r_M$ and rearranging yields

$$\bar{r}_i - r_{Fk} = \frac{1}{\sum_k^N \dfrac{W_k}{\lambda_k}} \operatorname{cov}(r_i r_M)$$

Solving this for the market portfolio and using this to estimate the summation containing (W_k/λ_k)

$$\bar{r}_i - r_{Fk} = \frac{\operatorname{cov}(r_i r_M)}{\operatorname{var}(r_M)} (\bar{r}_M - r_{FM}) \tag{11}$$

This is Solnick's C.A.P.M. and his equation (16).

B. Senbet Model

In Senbet's (11) model, each investor can consume a basket of goods including both domestic and foreign commodities. Similarly, each investor is free to invest in both domestic and international assets. The unique aspect of Senbet's model is that exchange rates are defined in terms of a "token supercurrency" or numeraire.[3] Wealth expressed in the numeraire represents real purchasing power

[3] To understand the concept of a numeraire currency, think in terms of a market basket of goods and services. The numeraire currency is a unit of the market basket and the value of all currencies is expressed in terms of how many units of the market basket a unit of that currency will purchase.

and the investor should be concerned with the terminal value of his wealth in terms of real purchasing power. Investment risk arises from changes in the number of units of the numeriare currency that a unit of a particular local currency will command and from the risk associated with the returns from any asset denominated in its local currency.

Senbet assumes homogenous expectations concerning both asset values one period hence expressed in units of the asset's own currency and exchange rates between each currency and the numeraire currency. These assumptions imply homogeneous expectations with respect to asset returns expressed in units of the numeraire currency. Thus, all investors face the same efficient frontier.

Senbet assumes a lending and borrowing rate that is riskless in the currency in which it is denominated, but is, of course, risky in the numeraire currency. Thus each investor will hold a different portfolio on the efficient frontier constructed in the numeraire currency. However, since 1) each investor will select an efficient portfolio from the same efficient frontier, 2) the aggregate of all investors' portfolios must be the market portfolio, and 3) combinations of efficient portfolios are efficient, the market portfolio (expressed in the numeraire) must be efficient.[4]

As Roll (8) has shown, if the market portfolio is efficient and there is no riskless lending and borrowing, the zero Beta form of the C.A.P.M. holds, or

$$\bar{R}_i - \bar{R}_Z = \frac{\operatorname{cov}(R_i R_M)}{\operatorname{var}(R_M)} (\bar{R}_M - \bar{R}_Z) \tag{12}$$

where capital letters are used to denote that the relevant variables are defined in terms of the numeraire currency.[5]

Define R_{Fi} as the return in the numeraire currency of the riskless asset of the country to which asset i belongs. Then equation (12) becomes

$$\bar{R}_{Fi} - \bar{R}_Z = \beta_{Fi}(\bar{R}_M - \bar{R}_Z)$$

Solving for $\bar{R}_Z$

$$\bar{R}_Z = \frac{\bar{R}_{Fi} - \beta_{Fi}\bar{R}_M}{1 - \beta_{Fi}}$$

Eliminating $\bar{R}_Z$ by substituting the expression for $\bar{R}_Z$ into equation (12) and simplifying yields

$$\bar{R}_i = \bar{R}_{Fi} + \frac{\beta_i - \beta_{Fi}}{1 - \beta_{Fi}} (\bar{R}_M - \bar{R}_{Fi}) \tag{13}$$

There is a nominally riskless asset for each currency. Define R_{FM} as the return on a portfolio of nominally riskless assets where the weights represent the proportion that each country's risky assets represents of the world market portfolio. Since this has the same weights as the market portfolio, its change in

[4] See Roll (9) for a proof. None of the weights can be negative and the weights must sum to one. But this is, of course, true for the case in the text.

[5] Senbet assumes that lending equals borrowing so that nominally riskless assets are not part of the market portfolio defined in the numeraire currency.

purchasing power is the same as the change in the purchasing power of the market portfolio. This asset is, of course, risky in terms of the numeraire currency and equation (13) must hold with respect to this asset. Utilizing equation (13) to get a relationship between R_{FM} and R_{Fi}, solving for R_{Fi} and substituting for the R_{Fi} in the term involving $\bar{R}_M$ in equation (13) and simplifying this equation

$$\bar{R}_i = \bar{R}_{Fi} + \frac{\beta_i - \beta_{Fi}}{1 - \beta_{FM}} (\bar{R}_M - \bar{R}_{FM}) \tag{14}$$

To get from this equation to Senbet's final model, we must first express returns defined in the numeraire currency in terms of returns expressed in each asset's own local currency.

If we let π_i be the change in the purchasing power, in terms of the numeraire currency, of the currency for the country to which asset i belongs, then[6]

$$R_i = r_i - \pi_i \; (15) \quad \text{and} \quad R_{Fi} = r_{Fi} - \pi_i \; (16)$$

If π_M is the change in the purchasing power of the market portfolio in terms of the numeraire currency, then

$$R_M = r_M - \pi_M \; (17) \quad \text{and} \quad R_{FM} = r_{FM} - \pi_M \; (18)$$

Note that the π's have different subscripts since each of the assets or portfolios involve different amounts of exchange risk. Utilizing the above expressions we have

1. $\text{Cov}(R_i R_M) = \text{cov}(r_i r_M) - \text{cov}(r_i \pi_M) - \text{cov}(\pi_i r_M) + \text{cov}(\pi_i \pi_M)$ (19)
2. $\text{var}(R_M) = \text{var}(r_M) - 2\,\text{cov}(r_M \pi_M) + \text{var}(\pi_M)$ (20)
3. $\text{cov}(R_{FM} R_M) = -\text{cov}(\pi_M r_M) + \text{var}(\pi_M)$ (21)
4. $\text{cov}(R_{Fi} R_M) = -\text{cov}(\pi_i r_M) + \text{cov}(\pi_i \pi_M)$ (22)

Substituting equations (15), (16), (17), (18), (19), (20), (21), and (22) into equation (14) yields:

$$\bar{r}_i - \bar{\pi}_i = r_{Fi} - \bar{\pi}_i + \frac{\text{cov}(r_i r_M) - \text{cov}(r_i \pi_M)}{\text{var}(r_M) - \text{cov}(r_M \pi_M)} [(\bar{r}_M - \bar{\pi}_M) - (r_{FM} - \bar{\pi}_M)]$$

Simplifying yields:

$$\bar{r}_i - r_{Fi} = \frac{\text{cov}(r_i r_M) - \text{cov}(r_i \pi_M)}{\text{var}(r_M) - \text{cov}(r_M \pi_M)} (\bar{r}_M - r_{FM})$$

This is Senbet's (11) and Senbet and Chen's (12) C.A.P.M. and is Senbet's equation (4), page 459.

C. Grauer, Litzenberger, and Stehle Model

Grauer, Litzenberger, and Stehle (4) have developed an international capital asset pricing model based on free international transferability of goods and

[6] These expressions are approximations. The correct expression is $1 + R_i = \frac{1 + r_i}{1 + \pi_i}$. Expanding the expression in a Taylor series and ignoring all terms except the first two yields the expression given in the text.

assets. All investors have homogeneous expectations about return on the assets expressed in the asset's own currency. Furthermore, there is one international market basket that serves as a price index.

Grauer, Litzenberger, and Stehle assume that investors look at real returns; that is, returns after adjusting for the effect of inflation. Since the same price index is applicable to all investors and since they are assumed to have homogeneous expectations with respect to the index and asset returns, all investors have homogeneous expectations with respect to real returns. Additionally, Grauer, Litzenberger, and Stehle assume a common riskless lending and borrowing rate in real terms. With homogeneous expectations concerning real returns and a real riskless and borrowing rate, the market portfolio is the tangency portfolio and as Roll has shown, this implies the Sharpe-Lintner C.A.P.M. This is the equation that Grauer, Litzenberger, and Stehle derived on page 242. Note that the GLS equation is expressed in real returns rather than in nominal returns. Thus it is simpler than the previous international asset pricing models.

III. Inflation and Equilibrium Returns

Two of the best known and most often quoted articles on uncertain inflation and equilibrium prices are Friend, Landskroner, and Losq (3) (hereafter FL&L) and Chen and Boness (2) (hereafter C&B). The results of both sets of authors follow directly from the assumption that a particular definition of the market portfolio is efficient. Differences in their results stem entirely from differences in the way they have incorporated inflation in their analysis and their treatment of the statistical process generating returns and inflation.

FL&L assume that nominal returns and inflation are both generated by a continuous Gaussian (Wiener) process, while C&B assume discrete returns. The equation used by FL&L to relate real returns to nominal returns is simply the continuous analog of the equation used by C&B. However, each set of authors approximates this equation differently, and uses different approximations for the cross-product terms involving inflation and security returns.

Both sets of authors assume investors use real returns to choose between portfolios. Investors are interested in maximizing their utility of end of period wealth. In a world of inflation, they should be interested in maximizing the utility of the purchasing power of end of period wealth or utility of wealth defined in real terms. Both sets of authors, by assuming homogeneous expectations with respect to both nominal returns and inflation rates, imply that investors have homogeneous expectations concerning real returns. Thus, all investors face the same efficient frontier. Without riskless lending and borrowing in real terms, investors will in general select different efficient portfolios. However, combinations of efficient portfolios are efficient. Since the market portfolio is a combination of all investors' efficient portfolios, the market portfolio is efficient.

If we denote the market portfolio by M', we can think of it as the portfolio of all assets which are risky in real terms. Note that this portfolio includes assets which are riskless in nominal terms. As Roll has shown, the Zero Beta C.A.P.M. must hold with respect to M'. Thus,

$$\bar{R}_i = \bar{R}_Z + \beta_i'(\bar{R}_M' - \bar{R}_Z)$$

$\bar{R}_i$ is the expected real return on stock i. (23)

$\bar{R}'_M$ is the expected real return on the market portfolio of all securities which are risky in real terms.
$\bar{R}_Z$ is the expected return on a portfolio which has returns uncorrelated with M' in real terms.
β'_i is $\text{cov}(R_i R'_M)$ divided by $\text{var}(R'_M)$

Note that the nominally riskless asset must be risky in real terms. Thus equation (23) must describe the real return on the riskless asset. Using this relationship to get an expression for $\bar{R}_Z$ and substituting into equation (23) yields

$$\bar{R}_i = R_F + \frac{\text{cov}(R_i, R'_M) - \text{cov}(R_F, R'_M)}{\text{var}(R'_M) - \text{cov}(R_F, R'_M)} (\bar{R}'_M - R_F). \tag{24}$$

C&B related real and nominal return by

$$R_i = r_i - \pi. \tag{25}$$

Using this approximation and $R'_M = \alpha R_M + (1 - \alpha) R_F$ we have

$$\text{cov}(R_i, R'_M) = \alpha \text{ cov}(r_i, r_M) - \alpha \text{ cov}(r_M, \pi) - \text{cov}(r_i, \pi) + \text{var}(\pi) \tag{26}$$

$$\text{cov}(R_F, R'_M) = -\alpha \text{ cov}(r_M, \pi) + \text{var}(\pi), \tag{27}$$

and

$$\text{var}(R'_M) = \alpha^2\text{var}(r_M) - 2\alpha \text{ cov}(r_M, \pi) + \text{var}(\pi) \tag{28}$$

Substituting these expressions and (25) into (24) and rearranging yields

$$\bar{r}_i = r_F + \frac{\text{cov}(r_i, r_M) - \dfrac{\text{cov}(r_i, \pi)}{\alpha}}{\text{var}(r_M) - \dfrac{\text{cov}(r_M, \pi)}{\alpha}} (\bar{r}_M - r_F) \tag{29}$$

This is exactly the C.A.P.M. presented by C&B [2] in their equation (7a) which is equivalent to their equation (7).

The FL&L approximation is

1. $\bar{R}_i = \bar{r}_i - \bar{\pi} - \text{cov}(r_i, \pi)$,
2. $R_i\, dt = \bar{r}_i\, dt + \sqrt{\text{var}(r_i)}\, Y_i \sqrt{dt} - \bar{\pi}\, dt - \sqrt{\text{var}(\pi)}\, Y_\pi \sqrt{dt}$ $- \sqrt{\text{var}(r_i)\text{var}(\pi)}\, Y_i Y_\pi\, dt + \text{var}(\pi) Y_\pi^2\, dt$,

and

3. all terms involving dt to a power greater than $3/2$ can be ignored.

The FL&L expressions are all in continuous time, $\text{cov}(r_i, \pi)$ is the continuous covariance between r_i and π, $\text{var}(\pi)$ and $\text{var}(r_i)$ are variances, Y's are standard normal variates and dt is the derivative of time. With these definitions one obtains the same variance-covariance expressions which are given in (26), (27) and (28) with the qualification that they are in continuous time. Using $R'_M = \text{R}_M + (1 - \alpha)\text{R}_F$ and condition 1 and substituting (26), (27) and (28) in equation (24) and simplifying yields

$$\bar{r}_i - \text{cov}(r_i, \pi) = r_F + \frac{\text{cov}(r_i, r_M) - \dfrac{\text{cov}(r_i, \pi)}{\alpha}}{\text{var}(r_M) - \dfrac{\text{cov}(r_M, \pi)}{\alpha}} (\bar{r}_M - r_F - \text{cov}(r_M, \pi)) \tag{30}$$

This is exactly the C.A.P.M. presented by FL&L in their equation (12). The proofs above make it clear that the difference between equation (29) and (30) derive primarily because of the difference in the relationship between expected real returns and nominal returns assumed by FL&L and C&B.

IV. Differential Taxes

Brennan (1) derived the first general equilibrium model that incorporated differential taxes on income and capital gains. Brennan derived his results from the maximization of investor utilities and market clearing conditions. We should and do get Brennan's result utilizing Roll's (8) insight and a little algebra. Brennan assumes homogeneous expectations with respect to pre-tax returns and a common riskless lending and borrowing rate.

Furthermore he assumes investors make their portfolio choices in terms of after-tax returns. The rate of taxation is independent of return, and taxes on losses and borrowing are treated as a subsidy. Furthermore the rate of taxation may differ between investors. Investors in different tax brackets will face a different efficient frontier and a different riskless lending and borrowing rate in the relevant after tax terms. Consequently Roll's (9) insight must be utilized in a slightly different manner than in prior sections. With a constant lending and borrowing rate the tangency portfolio is optimal for each investor and Roll's insight is that the Sharpe Lintner equation holds between each assets expected after tax return and each investor's tangency portfolio thus for investor i,

$$\bar{R}_{ik} = R_{iF} + \beta_{k,ip}(\bar{R}_{ip} - R_{iF}) \tag{31}$$

where capital letters denote after tax values, the i subscript indicates investor i; and the ip subscript indicates the optimum portfolio of investor i.

Let lower case r's be before-tax returns, d_k be a securities dividend yield, t_{oi} the ordinary income tax rate of investor i, and t_{ci} the capital gains rate then

$$R_{ik} = (r_k - d_k)(1 - t_{ci}) + d_k(1 - t_{oi}) = r_k(1 - t_{ci}) - d_k(t_{oi} - t_{ci})$$

$$R_{iF} = r_F(1 - t_{oi})$$

Note that since there are homogeneous assumptions about pre-tax returns there is no i subscript on the before-tax terms. Making Brennan's assumption that dividends are riskless, we can substitute the expression for after-tax returns into equation (31) to obtain

$$(\bar{r}_k - r_F)(1 - t_{ci}) - (d_k - r_F)(t_{oi} - t_{ci}) = [(\bar{r}_{ip} - r_F)(1 - t_{ci}) - (d_{ip} - r_F)(t_{oi} - t_{ci})]$$
$$\frac{\text{cov}(r_k(1 - t_{ci}), r_{ip}(1 - t_{ci}))}{\text{var}(r_{ip}(1 - t_{ci}))}$$

Define t_i as $\dfrac{t_{oi} - t_{ci}}{1 - t_{ci}}$. Dividing both sides of the above equation by $(1 - t_{ci})$ and divide both the covariance and variance term by $(1 - t_{ci})^2$ yields

$$\bar{r}_k = r_F + [(\bar{r}_{ip} - r_F) - (d_{ip} - r_F)t_i] \frac{\operatorname{cov}(r_k, r_{ip})}{\operatorname{var}(r_{ip})} + (d_k - r_F)t_i$$

defining λ_i as $\dfrac{(\bar{r}_{ip} - r_F) - (d_{ip} - r_F)t_i}{\operatorname{var}(r_{ip})}$ and rearranging yields

$$\frac{1}{\lambda_i}(\bar{r}_k - r_F) - \frac{t_i}{\lambda_i}(d_k - r_F) = \operatorname{cov}(r_k, r_{ip}) \tag{32}$$

Define w_i as the fraction of the total investment in risky assets which is made by investor i. Multiplying both sides of equation (32) by w_i suming across all investors and recognizing that $\sum_i w_i r_{ip} = r_M$ where r_M is the return on the market portfolio yields

$$\sum_i \frac{w_i}{\lambda_i}(\bar{r}_k - r_F) - \sum_i \frac{w_i t_i}{\lambda_i}(d_k - r_F) = \operatorname{cov}(r_k r_M)$$

Define

$$1.\ \lambda = \sum_i \frac{\lambda_i}{w_i} \qquad \text{and} \qquad 2.\ T_M = \sum_i \frac{w_i t_i}{\lambda_i} \bigg/ \sum_i \frac{w_i}{\lambda_i}$$

then equation (33) can be written as

$$(\bar{r}_k - r_F) - (d_k - r_F)T_M = \lambda \operatorname{cov}(r_k r_M)$$

This equation holds for each security and thus must hold for the market, therefore,

$$\lambda = \frac{(\bar{r}_M - r_F) - (d_M - r_F)T_M}{\sigma_M^2} \tag{35}$$

Equation (34) and (35) are the versions of the general equilibrium equation presented by Brennan (1).

V. Heterogeneous Expectations

The best known research specifying equilibrium relationships when investors hold heterogeneous expectations about expected payoffs, variances and covariances are Lintner (5) and Rabinovitch and Owen (9) and Sharpe (13). In previous sections we derived equilibrium relationships by showing that a relevant measure of the market portfolio was efficient. In dealing with heterogeneous expectations, Roll's insight has to be employed differently for the meaning of the market portfolio is not clear. Although later we will see that under certain restrictive assumptions about utility it can be employed in a manner parallel to other sections of this paper.

To start this section we will assume that each investor i holds some portfolio which he or she believes will be ex-post efficient. If the investor believes this portfolio is efficient, then the expected return on any security must be related to the return on this portfolio by a linear equation like equation (6).

Expressing this equation in terms of prices using matrix notation we have

$$\frac{1}{R_F}\,{}_0P = {}_1\bar{P}_i - \frac{1}{t_i}\textstyle\sum_i X_i$$

${}_0P$ = the N component vector of present prices.
${}_1\bar{P}_i$ = the N component vector of the payoff expected by investor i from each stock.
X_i = the N component vector of number of shares held by investor i.
$\sum_i$ = the NXN variance covariance matrix of the payoffs from securities as perceived by the i^{th} vector.

Rearranging,

$$\frac{1}{R_F}\,t_i \textstyle\sum_i^{-1}\,{}_0P = t_i \textstyle\sum_i^{-1}\,{}_1\bar{P}_i - X_i$$

Summing over all investors, recognizing that $\sum_i X_i = n$ where n is the N component vector of the number of shares outstanding for each stock, and solving for $\frac{1}{R_F}\,{}_0P$, we obtain

$$\frac{1}{R_F}\,{}_0P = [\textstyle\sum_i t_i \sum_i^{-1}]^{-1}[\sum_i t_i \sum_i^{-1}\,{}_1\bar{P}_i - n]$$

This is exactly the result obtained by Rabinovitch and Owen, equation (6), page 578.

Litner has two principal results. The first can be shown to follow trivially from Rabinovitch and Owen.[7] Lintner has derived his results under a special form of the investor utility function, namely, $u^i(w_i) = -e^{-a_i w_i}$. Where a is the coefficient of absolute risk aversion which is constant. It is well known that with the negative exponential utility function $(1/t_i)$ in our equation will equal a_i. Making this substitution, we get Lintner's equation (10), page 362.

The use of the negative exponential function is more important in Lintner's derivation in that it leads to a second result not found in Rabinovitch and Owen. The negative exponential utility function allows individual utilities to be aggregated and thus allows the assertion of a market utility function. Furthermore, we know that this market utility function will be a negative exponential function. This means that we can "suppose that the market as a whole acts as a single composite 'price-taker'.[8] Thus, there is some market portfolio which must be efficient. Thus, from Roll's results we can write:

$$r_{mj} = r_F + \frac{\bar{r}_m - r_F}{\operatorname{var}(r_m)}\operatorname{cov}(r_{mj}, r_m) \tag{36}$$

[7] Lintner's result, were presented long before Rabinovitch and Owen's. We have presented Rabinovitch and Owen's results first because the Lintner results are a special case of these more general results.

[8] Lintner, page 357.

where the subscript m standing alone indicates the market's expectations about the market portfolio while the double subscript mj indicates the market's belief about security j. Switching this to price form yields Lintner's equation (5m) (i), page 358.

Before closing this section we should say a few words about Sharpe's derivation of equilibrium with heterogeneous expectations. If one takes equation (36) and aggregates across all investors, one gets with the appropriate change in notation the equilibrium result presented by Sharpe in Appendix D, page 292.

REFERENCES

1. Brennan, Michael, "Investor Taxes, Market Valuation and Corporate Financial Policy," *National Tax Journal* (December 1970).
2. Chen, Andres and Boness, James, "Effects of Uncertain Inflation on the Investment and Financing Decision of a Firm," *Journal of Finance*, Vol. 30, No. 2, (May, 1975), 469–485.
3. Friend, Irwin, Landskroner, Yoram, and Losq Etienne, "The Demand for Risky Assets Under Uncertain Inflation," *Journal of Finance*, Vol. 31, No. 5, (December 1976), pp. 1287–1299.
4. Grauer, Frederick, Litzenburger, Robert and Stehle, Richard, "Sharing Rules and Equilibrium in an International Capital Markets Under Uncertainty," *Journal of Financial Economics*, Vol. 3, (June 1976), pp. 233–256.
5. Lintner, John, "The Aggregation of Investors Diverse Judgements and Preferences in Purely Competitive Markets," *Journal of Financial and Quantitative Analysis*, Vol. 4, No. 4, (December 1969), pp. 347–400.
6. Lintner, John, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, (February 1965).
7. Mayer, David, "Non-marketable Assets and Capital Market Equilibrium Under Uncertainty" in Jenson, Mike, *Studies in the Theory of Capital Markets* (Praeger: New York), 1972.
8. Roll, Richard, "A Critique of Asset Pricing Theory's Tests," *Journal of Financial Economics*, (March 1977), pp. 129–176.
9. Rabinovich, Ramon and Owen, Joel, "Non-homogeneous Expectations and Information in the Capital Asset Pricing Model," *Journal of Finance*, Vol. 33, No. 2, (May 1979), pp. 575–587.
10. Mossin, Jan, "Equilibrium in a Capital Asset Model," *Econometrica*, (October 1966).
11. Senbet, Lemma, "International Capital Market Equilibrium and the Multi-National Firm Financing and Investment Policies," *Journal of Financial and Quantitative Analysis*, Vol. 14, No. 3, (September 1979), pp. 455–477.
12. Senbet and Chen, Andrew, "International Purchasing Power Risk and Capital Market Equilibrium," Working Paper, Ohio State University, 1977.
13. Sharpe, William, *Portfolio Theory and Capital Markets* (McGraw-Hill: New York) 1970.
14. Solnick, Bruno, "An Equilibrium Model of the International Capital Market," *Journal of Economic Theory*, Vol. 8, (August 1974), pp. 500–524.