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ABSTRACT

This paper presents a new methodology, quasilinear estimation, for efficiently estimating economic variables reflected in the prices of corporate securities. For example, ex ante bankruptcy costs are not directly observable, however, if these costs are sufficiently large, then current security prices are affected and bankruptcy costs can be indirectly measured. When bankruptcy costs and other relevant parameters are known, there are many numerical solution techniques that can be used to determine security prices. One technique, the method of lines, is compatible with quasilinear estimation, which has been employed extensively in the physical sciences for the estimation of coefficients in differential equation models. We demonstrate that quasilinear estimation is a potentially reliable and efficient technique for the estimation of corporate bankruptcy costs and the asset variance from security prices.

BEGINNING WITH THE SEMINAL WORK of Black and Scholes [3] it has been known that the valuation of many corporate liabilities can be represented as the solution to a parabolic partial differential equation (p.d.e.). Unfortunately, closed form solutions exist for only a limited number of interesting securities (e.g., the pure discount bond), but much attention has been paid to numerical methods for solving these equations. In this paper we are interested in solving the inverse of this problem; that is, given observed values of a security and the form of the p.d.e., are there efficient methods for estimating the parameters that appear in the p.d.e.? We present a methodology termed "quasilinear estimation" which we employ to estimate bankruptcy costs implicit in the prices of corporate debt.

In recent years the view has been advanced that the firm's optimal capital structure is determined by a trade-off between tax subsidies arising from the deductibility of interest and bankruptcy costs.[1] However, both the tax advantages from debt financing and the significance of bankruptcy costs have been questioned. In the absence of bankruptcy costs, Miller [20] has argued that the demand for debt is determined at the aggregate level. In this case there is no optimal capital structure for the firm. Haugen and Senbet [10] argue that in a market where the cost of trading securities is small, rational investors will choose informal reorganization to avoid the costs associated with a formal bankruptcy; thus the impact of bankruptcy costs on the value of the firm as a whole must be bounded by the transaction cost required by bondholders the buy out the stockholders' position or vice versa. If both Miller and Haugen and Senbet are correct, then neither tax subsidies nor bankruptcy costs are relevant in making

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[1] See [5], [9], [15], [16], [21], [22] for discussions of optimal capital structure in the presence of bankruptcy costs and tax subsidies.

the capital structure decision, and financial theorists must develop another story to explain the systematic differences in capital structure across industries. Of course, empirical estimates of the size of the tax subsidy and bankruptcy costs would help resolve the debate on the significance of these factors.

Our objective is to develop a methodology for estimating ex ante bankruptcy costs.[2] We propose two alternative bankruptcy cost models. The first model treats the bankruptcy cost as a deadweight cost that occurs if the firm's asset value is less than the face value of maturing debt. The second model treats bankruptcy cost as an avoidable cost. In this case the bankruptcy cost is assumed to have no impact on the value of the firm as a whole, but the presence of bankruptcy costs can still significantly impact bond and equity prices. In either model, bankruptcy may be viewed as occurring only at the maturity date, or alternatively as an event that may occur at any point during the firm's life. Section I discusses, in more detail, the formulation of those alternative bankruptcy models. Section II addresses the question of how large bankruptcy costs must be in order to detect a significant impact on bond prices. In Section III we describe the method of quasilinear estimation and demonstrate with simulated bond data that it is potentially a computationally efficient and reliable technique for estimating bankruptcy costs and the variance of the firm's assets. Our conclusions are in Section IV.

I. Alternative Models of Bankruptcy and Bankruptcy Costs

Merton [18] derived a partial differential equation for valuing pure discount corporate bonds.[3] Under his assumptions, the value of corporate debt is determined by a differential equation of the following form:

$$\frac{\partial D}{\partial \tau} = \frac{1}{2}\frac{\partial^2 D}{\partial A^2}\sigma^2 A^2 + \frac{\partial D}{\partial A} rA - rD \tag{1}$$

$$D(A, 0) = \mathrm{MIN}(F, A) \qquad \tau = 0 \tag{2a}$$

$$\lim_{A\to\infty} D(A, \tau) = Fe^{-rt} \qquad \tau > 0 \tag{2b}$$

$$\lim_{A\to 0} D(A, \tau) = 0 \qquad \tau > 0 \tag{2c}$$

where $D \equiv$ value of debt
$F \equiv$ face value of debt
$A \equiv$ assets of firm
$\sigma^2 \equiv$ variance rate of firms assets
$r \equiv$ risk free rate

Merton shows that the solution to (1) can be expressed in terms of the Black-Scholes analytic solution for a call option. However, when coupons, dividends,

[2] Baxter [1] and Van Horne [22] report direct bankruptcy costs to be about 20% of firm value. However, both samples consist primarily of very small companies.

[3] See [18] for details. Note that applying Merton's formation involves assuming that all liabilities of the firm—even the claim on the bankruptcy costs—are traded or have their value determined exogenously. While this assumption is violated for the bankruptcy costs, we are not overly concerned as the same differential equation could be obtained with more compatible assumptions, (see [4]).

sinking fund payments, callability, conversion or bankruptcy costs are present, Merton's model in (1) is no longer valid. Fortunately more general models can be readily developed.

Our primary objective is to develop a model that can be used for the empirical estimation of bankruptcy costs. The model we propose includes coupons, sinking fund payments, dividends, and bankruptcy costs. We introduce the first three cashflow in continuous form as in Jones, Mason, and Rosenfeld [13], and obtain the following differential equation:[4]

$$\frac{\partial D}{\partial \tau} = \frac{1}{2}\frac{\partial^2 D}{\partial A^2}\sigma^2 A^2 + \frac{\partial D}{\partial A}[rA - C - SF - DIV] + [C + SF] - rD \tag{3}$$

where

$$C \equiv \text{coupon payment to bondholders}$$
$$SF \equiv \text{sinking fund payment to bondholders}$$
$$DIV \equiv \text{dividend payment to stockholders}$$

Let the original face value of the debt be 100 and the original term of the bond be T years. If there is a sinking fund that retires the issue at a continuous rate of s per unit time, then the face value when there is τ years to go is $F(\tau) = 100[1 - (T - \tau)s]$. If the firm has the option to sink at the lesser of par or market, the sinking fund payment when there is τ years to go is $SF(\tau) = \text{minimum} \left(s100,\ s100\,\frac{D(\tau)}{F(\tau)}\right)$.[5] The coupon is equal to the coupon rate, c, times the current face value, $C(\tau) = cF(\tau)$. The dividend payment is equal to the dividend rate d times the value of the firm's equity $DIV(\tau) = dE(\tau)$, where $E(\tau)$ denotes the value of the common stock.

There are several alternative ways in which bankruptcy costs can be introduced. First consider bankruptcy costs as a deadweight cost incurred at maturity if the firm value is less than the face value of debt.[6] In this case the bondholders receive the following distribution $\bar{D}$ at maturity ($\tau = 0$)

$$\bar{D} = \begin{cases} F & A > F \\ A - B & A < F \end{cases} \tag{4}$$

where $B \equiv$ bankruptcy costs.

The bankruptcy cost is typically taken as a fixed cost or a function of firm value. In our simulation below we will consider the case of a bankruptcy cost that is proportional to the value of the firm.

Haugen and Senbet [10] argue that in a frictionless capital market, bondhold-

[4] Another numerical problem in dealing with callable bonds is noted by Jones, Mason, and Rosenfeld [13]. When a firm has multiple bond issues that are callable, the value of any particular bond depends on whether or not other bonds have been previously called. With two callable bonds the computational effort required to calculate bond values is doubled. With n bonds computation increases by a factor of 2^{n-1}.

[5] See [12] for a discussion of the valuation of the sinking fund.

[6] See references in footnote 1 for some treatments of bankruptcy as a deadweight cost.

ers, stockholders or outsiders could acquire all the shares of the firm and thereby avoid bankruptcy and bankruptcy costs. However, they also mention that in the case where the face value of debt exceeds the market value of the ongoing firm, the bondholders would be willing to pay stockholders an amount up to the cost of bankruptcy to avoid bankruptcy.[7] Hence, the existence of bankruptcy costs can produce a wealth transfer from bondholders to stockholders motivated by the bondholder's desire to avoid bankruptcy costs. The size of this wealth transfer is not limited to the size of transaction costs.

To illustrate the "avoidable bankruptcy model" suppose that bankruptcy costs are proportional to firm value, $B = \gamma A$. If bankruptcy occurs, bondholders receive $\min(A(1 - \gamma), F)$. If no bankruptcy occurs bondholders receive $\min(A, F)$. It follows that bondholders will be willing to pay stockholders an amount up to $\bar{P} \equiv [\min(A, F) - \min(A(1 - \gamma), F)]$ to avoid bankruptcy. If $A \leqq F$, this maximum payment is equal to the bankruptcy cost itself. If $A \geqq F/(1 - \gamma)$, there will be no payment since bondholders will always receive the face value in liquidation. If $F < A < F/(1 - \gamma)$, then the maximum payment is positive but less than the bankruptcy cost. Suppose further that bondholders can avoid bankruptcy costs by a payment of a fraction λ of the maximum payment $\bar{P}$. In this case, the distribution $\bar{D}$ at maturity ($\tau = 0$) to bondholders is $\min(A, F) - \lambda P$ and is equal to:[8]

$$\bar{D} = \begin{cases} A - \lambda(\gamma A) & A < F \\ A - \lambda[F - A(1 - \gamma)] & F \leq A < F/(1 - \gamma) \\ F & F/(1 - \gamma) \leq A \end{cases} \tag{5}$$

With either of these stories about bankruptcy costs, we can consider "random event bankruptcy." Instead of assuming that bankruptcy can occur only at the maturity date of debt, we can imagine that firms may be forced into bankruptcy, say by an inability to roll over short term debt, at any time. Rather than attempting to explicitly model the process that might produce such a "crisis," we can introduce a "random event bankruptcy." In particular, we assume that at each instant of time there is a probability Pdt that a crisis will occur, where P is a monotonically decreasing function of the ratio of firm value to face value, $P \equiv P(A/F)$. If a crisis occurs, then the value of debt changes to D^B where D^B is equal to $\bar{D}$ as given in (4) for the deadweight bankruptcy, and $D^B = \mathrm{Min}(D, \bar{D})$ where $\bar{D}$ is given in (5) for the avoidable bankruptcy and D is the cost of buying back the debt at market value. For the random event bankruptcy model, the differential equation in (3) is modified as follows:[9]

$$\frac{\partial D}{\partial \tau} = \frac{1}{2}\frac{\partial^2 D}{\partial A^2}\sigma^2 A^2 + \frac{\partial D}{\partial A}[rA - C - SF - DIV] + [C + SF] - rD + \frac{P}{1 - P}(D^B - D) \tag{6}$$

[7] See footnote 4 of [10].

[8] The payoff function in the avoidable bankruptcy model is similar but not identical to that Chen's [9] partial transfer of ownership model.

[9] In the derivation of (7) it is assumed that the random event of bankruptcy is completely unsystematic risk. This treatment is similar to Merton's [19] modelling of the jump process in the pricing of options.

There are many other features that might be included in this model. For example, stochastic interest rates could be introduced.[10] One could also include asset growth and new debt or stock financing. However, it is not appropriate to introduce the asset growth rate or the capital structure changes without introducing the firm's optimal control of these decision variables.[11] Since bankruptcy costs have never been estimated directly from bond data, it seems best to first experiment with simpler models. If simple models fail than one might consider introducing stochastic interest rates and/or optimally controlled asset growth and capital structure.

II. Bankruptcy Costs and Bond Prices

In order to detect the presence of bankruptcy costs it is necessary that the bankruptcy cost be large enough to have a significant impact on bond prices. One question that arises is at what firm value (relative to face value) can we hope to detect bankruptcy costs of say, 5%, or 10%, or 20%? Alternatively, how large must bankruptcy costs be to detect the cost if the firm debt ratio is 30% or 50% or 80%?

To answer these questions we construct simulations of the three bankruptcy models. For all three models the following coefficients are used:

$$\begin{aligned}
c &\equiv \text{coupon rate} = .15 \\
s &\equiv \text{sinking fund rate} = .05 \\
d &\equiv \text{dividend rate} = .05 \\
r &\equiv \text{riskless rate} = .12 \\
\sigma^2 &\equiv \text{variance rate of firm assets} = .20 \\
T &\equiv \text{term of debt} = 1, 3, 5, 10, 20 \text{ years} \\
B &\equiv \text{bankruptcy costs} = \gamma A \\
\gamma &\equiv \text{bankruptcy cost parameter} = 0, .5, .10, .20, .40
\end{aligned} \tag{7}$$

In the avoidable bankruptcy model we must also specify the payment that bondholders make to stockholders to avoid bankruptcy:

$$\lambda[\min(A - F) - \min(A(1 - \gamma) - F)] = \text{payment to stockholders} \tag{8}$$

$$\lambda = \text{fraction of maximum payment} = .5$$

We must also modify boundary condition (2b) for our bankruptcy models. As asset value becomes large relative to the face value of debt, the value of debt approaches that of a riskless bond with an identical coupon rate and sinking fund rate. Therefore the boundary condition becomes:[12]

$$\lim_{A\to 0} D(A, \tau) = 0 \qquad \tau > 0 \tag{9a}$$

$$\lim_{A\to \infty} D(A, \tau) = D^*(A, T) \qquad \tau > 0 \tag{9b}$$

[10] However, computation time increases geometrically with the number of state variables. Theoretically there is no difficulty in adding interest rates to the bond pricing model.

[11] See Brennan and Schwartz [7].

[12] The sinking fund payment on risky debt is $\min(sF, sD)$; the firm has an option to sink at market or par. We ignore this option in the calculation of the value of the riskless debt in (9b). For our numerical results, the riskless debt always sells at a premium to par, so it is always optimal to sink

where

$$D^*(A, \tau) = \int_0^\tau [cF(t) + sF(T)]e^{-r(\tau-t)}\,dt + F(0)e^{-r\tau}$$

$$= [sF(T) + cF(0)]\,\frac{1 - e^{-r\tau}}{r} + csF(T)\left[\frac{1}{r}\left(\tau - \frac{1 - e^{-r\tau}}{r}\right)\right] + F(0)e^{-r\tau}$$

and

$$F(t) = F(T)[1 - s(T - t)] = \text{face value of remaining debt for } t \text{ years to go}$$

$$\tau = \text{time to go}$$

$$T = \text{original term of debt}$$

To solve the bond's differential equation coupled with the appropriate terminal and boundary conditions, we employ the method of lines. This replaces the derivatives $\partial^2D/\partial A^2$ and $\partial D/\partial A$ in (3) or (6) with finite difference approximations. This discretizes the partial differential equation along the "A dimension" but not the time dimension. The result is a system of ordinary differential equations. For example, applying the method of lines to the differential equation in (3) produces is system of ordinary, coupled differential equations can be accurately solved with a number of efficient and reliable techniques.[13]

$$\frac{\partial D^i}{\partial \tau} = \left\{\frac{1}{2}\left[\frac{D^{i+1} - 2D^i + D^{i-1}}{\Delta^2}\right]\sigma_A^2 A^{i^2} + \left[\frac{D^{i+1} - D^{i-1}}{2\Delta}\right](rA^i - (C^i - SF^i - DIV^i) + (C^i + SF^i) - rD^i\right\} \quad i = 1, 2, \cdots, n \qquad (12)$$

where

D^i = value of debt at firm value A^i
D^0 = 0 (See equation (9a))
D^{n+1} = D^* (See equation (9b))
Δ = grid size for discretization of A

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the riskless debt at par rather than market value. In this case, we can solve for the value of riskless debt in (9b) analytically by use of integration by parts. If riskless debt sold at a discount from par, then "$sF(t)$" in (9b) must be replaced by $MIN(sF(t), D^*(t))$ and a numerical solution is required.

A second consideration is that corporate debt does not become riskless until the firm asset value is equal to infinity. However, as demonstrated in [17], for all practical purposes corporate debt becomes riskless when $A/F = 10$, i.e., when firm asset value is ten times the face value at maturity. In the calculation of *yield differentials* for bonds with and without bankruptcy costs, we report yield differentials for $A/F \in [0, 4]$ which are based on the riskless boundary imposed at $A/F = 32$.

[13] We employ a combination of the Runge-Kutta method and the Adams-Moulton method to solve (12). These techniques are discussed in detail in [8]. The difference between the method of lines and the numerical techniques employed by Brennan and Schwartz is that Brennan and Schwartz also discretize $\partial D/\partial\tau$ which results in a system of algebraic equations. As we will see, quasilinear estimation is developed specifically for ordinary differential equations.

In the first simulation we examine the impact of bankruptcy costs in the deadweight bankruptcy cost model. This requires solving the differential equation in (3) coupled with (4) and (9). Table I reports the yield differential between risky corporate bonds with and without bankruptcy costs. For example, when bankruptcy cost is 5%, the yield differential is 4.11% on a one-year bond when the asset to face ratio is .5 and .58% when A/F is 2.0. We feel bankruptcy costs are casually "detectable" when the yield differential is about .5% or greater, since a .5% differential is roughly equivalent to a difference of one quality rating, e.g., the yield differential between "AAA" bonds and "AA" bonds is about .5%.

Table I shows that a deadweight bankruptcy cost of 5% is detectable only for one-year bonds when $A/F \leq 2.0$. A 10% bankruptcy cost is detectable for one year and three bonds when $A/F \leqq 2.0$. A 20% bankruptcy cost is detectable for one, three, and five-year bonds when $A/F \leq 2.0$. Finally a 40% bankruptcy cost is detectable for one, three, and five-year bonds for $A \leqq 4.0$. The presence of bankruptcy costs, even as large as 40%, have little impact on bonds when the time until maturity is ten or more years or when the asset to face ratio is more than four. Furthermore, when bankruptcy costs are 20% or less, they are detectable only at very low asset to face ratios. We also observe that the doubling of the bankruptcy cost tends to roughly double the yield differential for every maturity and asset to face ratio. This occurs because the probability of the asset value falling to the bankruptcy range is independent of the bankruptcy cost. Therefore a doubling of bankruptcy costs approximately doubles the present value of potential bankruptcy costs.

We also examined the avoidable bankruptcy model. We find that when the payoff to stockholder is $\lambda\%$ of the maximum payoff, the yield differential for the avoidable bankruptcy model is roughly $1/\lambda\%$ of the yield differential in the deadweight bankruptcy cost model. This suggests that it will be very difficult or

Table I

Yield Differentials for Deadweight Cost Bankruptcy Model

Years Until Maturity	Asset to Face Ratio	Yield Without Bankruptcy Costs	Yield Differential With Cost Bankruptcy Model (in % per annum)			
			5%	10%	20%	40%
1 Year	0.5	10.00%	4.11%*	8.34%*	17.40%*	38.46%*
	1.0	36.03	3.07*	6.24*	12.88*	27.59*
	2.0	13.92	.58*	1.17*	2.36*	4.97*
	4.0	12.03	.02	.03	.07	.15
3 Years	0.5	57.06	.52*	1.05*	2.13*	4.38*
	1.0	29.81	.70*	1.42*	2.89*	6.02*
	2.0	16.77	.39	.79*	1.60*	3.28*
	4.0	12.76	.11	.21	.43	.87
5 Years	0.5	50.89	.11	.22	.44	.89*
	1.0	28.57	.23	.46	.94*	1.91*
	2.0	13.52	.20	.41	.82*	1.68*
	4.0	12.93	.10	.20	.39	.80*

* Yield differential is more than .5%.

impossible to empirically differentiate between a deadweight model with a bankruptcy cost of $X\%$ and an avoidable bankruptcy model with an avoidable cost of $\frac{1}{\lambda} X\%$.

Finally we examined the yield differentials in the random event bankruptcy model which is described by the differential equation in (6) coupled with (9) and an avoidable bankruptcy cost.[14] The introduction of early bankruptcy produces several unique effects. First, the addition of early bankruptcy increases the yield differentials for all maturities and all asset to face ratios. The relative increase in the yield differential is especially large for the lowest asset to face ratios and longest term bonds. Second, when early bankruptcy is possible, bankruptcy costs even as low as 5% are detectable for bonds with maturities as long as 20 years when the asset to face ratio is low.

Having examined the alternative models, we make the following observations. First, the deadweight cost model and the avoidable bankruptcy model are empirically indistinguishable. Second, when bankruptcy occurs only at maturity, bankruptcy costs are casually detectable in the deadweight cost model or the avoidable bankruptcy model only for very short-term bonds and only for very low asset to face ratios. Third, the random event or early bankruptcy model has distinctive effect on yield differential and is detectable over a much greater range of maturities and asset to face ratios.

III. Quasilinear Estimation of Bankruptcy Costs

In section II it was demonstrated that the existence of bankruptcy costs can have a significant impact on bond prices. We now address the issue of how to estimate bankruptcy costs from bond data. Other model parameters, in particular the firm's variance rate, may be difficult to exogenously estimate. Therefore we develop a multivariate procedure to simultaneously estimate both the asset variance and bankruptcy costs. Quasilinear estimation was developed for fitting ordinary differential equation to data. This method has been applied extensively to problems that arise in physical problems and has been demonstrated to be computationally efficient and reliable.[15] However, there is no guarantee that

[14] In the random event bankruptcy model we must specify the instantaneous probability of "crisis." We assume that this probability is a monotonically decreasing function of the ratio (A/F). Further it seems plausible that the probability is close to 1 when $(A/F) = 0$ and tends to 0 as (A/F) becomes sufficiently large. The hyperbolic tangent function can be used to approximate these qualitative characteristics.

$$\begin{aligned} pdt &= p(A/F)\, dt \equiv \text{instantaneous probability of crisis} \\ p &= \tfrac{1}{2}[1 - \tanh(a(A/F) - b)] \\ a &= 2 \\ b &= 5 \end{aligned}$$

by choosing $b = 2$ and $a = 2$ the function is approximately equal to 1 for $A/F = 0$, equal to .5 for $A/F = 1$, and approximately equal to 0 for $A/F = 3$. The shape of this probability function resembles a "flattened reverse-s" with asymptotes of 1 and 0. Certainly many other functions could be used to generate equally plausible probability curves.

[15] See [2] and [11] for a discussion of some applications of quasilinear estimation to physical problems and economic problems; [14] describes application of this method to the Black-Scholes model.

quasilinear estimation will work satisfactorily with the bankruptcy cost estimation problem.

To assess the usefulness of quasilinearization we propose the following experiment. First we generate bond prices with pre-determined values for the bankruptcy parameter and the firm's variance rate σ^2. Then, starting with different values of γ and σ^2 we employ quasilinear estimation to see if we can recover the original values of γ and σ^2. As a further test we corrupt the bond data with a random noise and attempt to recover γ and σ^2. We want to answer three questions. Can both γ and σ^2 be recovered without bias? How fast, computationally, can recovery be made? How does the presence of noisy data affect the recovery of parameters?

In this section, we focus on the deadweight bankruptcy model which is described by the differential equation (3) coupled with boundary conditions (4) and (9). Model coefficients are given in (7). The values $\gamma = .10$ (or $(1 - \gamma) = .90$) and $\sigma^2 = .20$ (or $\sigma = .4472$) are taken as true values. Using these coefficients bond prices are generated for the deadweight bankruptcy model. As we observed in the previous section, bankruptcy costs are detectable in the deadweight bankruptcy model when the time to maturity is short and when the asset to face ratio is low. In each of the following experiments we simulate fifteen bond prices using the true parameter values. Five bond prices are chosen at $A/F = .90$, $A/F = 1.00$ and $A/F = 1.10$ and equally partitioned over time until maturity.

Our first experiment is to see if we can simultaneously recover the original parameters γ and σ^2 with quasilinearization. We start with initial guesses of γ and σ^2 that vary from the true values by 10%, 30%, and 50%. Tables II and III report the rate of convergence to the true values. By the fourth iteration both γ and σ^2 have converged to four significant digits. However, three iterations are needed for two digits of accuracy. The rate of convergence is much quicker for one-year bonds as opposed to five and ten-year bonds, since bankruptcy costs have a more significant effect on short-term bonds. Convergence on the ten-year bond is very encouraging since the yield differential with and without bankruptcy costs on a ten-year bond is only .0002 or 2 basis points (when $A/F = 1.0$, $\gamma = .10$, $\sigma^2 = .20$).

Table II

Convergence of $(1 - \gamma)$
(True value $(1 - \gamma) = .90$)

Time To Go	Error in Initial Guess	Initial Guess $(1-\gamma)^{(0)}$	Iteration 1 $(1-\gamma)^{(1)}$	Iteration 2 $(1-\gamma)^{(2)}$	Iteration 3 $(1-\gamma)^{(3)}$	Iteration 4 $(1-\gamma)^{(4)}$
	10%	.8100	.9034	.9000	.9000	.9000
1	30%	.6300	.9356	.9010	.9800	.9000
	50%	.4500	.9010	.9018	.9000	.9000
5	30%	.6300	.8640	.8704	.8956	.9000
	50%	.4500	.9036	.8870	.8971	.9000
10	30%	.6300	.9350	.8973	.8999	.9000

Table III

Convergence of σ
(True value $\sigma = .4472$)

Time To Go	Error in Initial Guess	Initial Guess $(\sigma)^{(0)}$	Iteration 1 $(\sigma)^{(1)}$	Iteration 2 $(\sigma)^{(2)}$	Iteration 3 $(\sigma)^{(3)}$	Iteration 4 $(\sigma)^{(4)}$
	10%	.4919	.4496	.4472	.4472	.4472
1	30%	.5814	.4756	.4486	.4472	.4472
	50%	.6708	.5569	.4521	.4472	.4472
5	30%	.5814	.2615	.4932	.4400	.4472
	50%	.6708	.5760	.4971	.4479	.4472
10	30%	.5814	.2989	.4842	.4442	.4472

Table IV

Estimated Values with 1% Error

Parameter	True Value	Average Estimated Value	Standard Deviation of Estimated Values	Minimum Estimated Value	Maximum Estimated Value
$(1-\gamma)$	.9000	.9018	.0084	.8907	.9154
σ	.4472	.4464	.0173	.4186	.4687

In our next experiment we introduce a random error in the bond data. The random error is intended to reflect either the existence of noisy bond data or a random difference between model bond prices and actual bond prices. We select 5 bond prices at each of the following asset to face ratios of .9, 1.0, 1.1, and 1.2 and equally partitioned over the time until maturity. Each bond price is perturbed by an independently generated random error, $\tilde{\varepsilon} \sim N(0, 1)$. Since we posit a geometric random walk for the asset value of the firm, we introduce the error in a multiplicative manner. The perturbed bond price is defined as the simulated bond price times $(1 + \lambda\tilde{\varepsilon})$, where λ is chosen as .01 and .05 for two separate experiments. When λ is .05, a bond might be perturbed as much as 15%, i.e., three standard deviations from the mean. For each λ we repeat the experiment ten times. Tables IV and V report the results of these experiments.

When the error is 1%, the average estimates of $(1-\gamma)$ and σ for ten trials are .9018 and .4464, respectively, which agree with the true values to two digits. The average estimates for a 5% error are almost as good. This suggests that the quasilinear technique will produce unbiased estimates of the parameter values. However, for specific trials, the deviation from the true parameter values can be much larger. The maximum and minimum estimated values for a 1% error differ from true values by 1.0% to 1.7% for $(1-\gamma)$ and by 4.8% to 6.4% for σ. When the error is 5% the maximum and minimum estimates deviate from the true values by 3.5% to 8.8% for $(1-\gamma)$ and by 24.2% to 28.2% for σ. A 5% error produces minimum-maximum range that is roughly five times that produced by a 1% error. It might also be noted that the minimum and maximum correspond roughly to the average estimate plus or minus two standard deviations. However,

Table V

Estimated Values with 5% Error

Parameter	True Value	Average Estimated Value	Standard Deviation of Estimated Values	Minimum Estimated Value	Maximum Estimated Value
$(1-\gamma)$	.9000	.9130	.0400	.8688	.9792
σ	.4472	.4471	.0742	.3210	.5557

we should avoid further speculation about confidence intervals until more extensive simulations are conducted.

IV. Conclusion

Bankruptcy costs may have a significant impact on bond prices. We have seen that quasilinear estimation may be a viable technique for estimating bankruptcy costs and other model parameters. Having demonstrated the usefulness of this technique with simulated data, the next step is to examine real bond data.

However, the use of real data creates several new but hopefully resolvable problems. First, the asset value of the firm is not directly observable unless all the liabilities of the firm are tradeable.[16] Second, most firms have multiple debt issues. Jones, Mason, and Rosenfeld [13] develop numerical procedures for treating multiple debt issues, however, they also observe rapidly increasing computation costs. Thus, we may want to limit ourselves to firms with only a few debt issues. Third, if bankruptcy costs exist, they are probably detectable only for firms where the ratio of asset value to face value is relatively low and the time to go is short. Finally, we may have to exclude firms having callable convertible bonds due to the complexity of satisfactorily modeling these features. Thus, the potential sample for estimating bankruptcy costs is restricted to firms with tradeable, non-callable, non-convertible debt, with only a few issues outstanding, with relatively low asset to face ratios and a short time until maturity. Should this restricted sample prove to be more than an empty set, we feel that quasilinear estimation will be useful in measuring ex ante bankruptcy costs.

Appendix: Quasilinear Estimation

To illustrate the method of quasilinear estimation, consider the deadweight bankruptcy cost model described by the differential equation (3) and boundary conditions (4) and (9). Suppose that γ and σ^2 are unknown and must be estimated from observed bond data D_t^* available at times $t = t_1, t_2 \cdots t_n$. First, we transform the differential equation (3) into a system of ordinary differential equations as shown in (12). Let $\underline{D} \equiv [D^1, D^2 \cdots D^n]$ denote the model values of debt at $\underline{A} \equiv$

[16] Jones, Mason, and Rosenfeld [13] proposed that the asset value of the firm can be estimated endogenously by determining an asset level that produces a model stock price that is equal to the actual stock price. However, if model stock prices are systematically more or less than actual stock prices, then such an endogenous estimate of firm assets may magnify the bias between model bond prices compared with actual bond prices.

$[A^1, A^2 \cdots A^n]$. The terminal condition (3) is re-written as:

$$\underline{D}(0) = \underline{a} + (1 - \gamma)\underline{b} \tag{B1}$$

where

$$\underline{a}' = [a_1, a_2 \cdots a_n]$$

$$\underline{b}' = [b_1, b_2 \cdots b_n]$$

$$a_i = \begin{cases} F & A^i \geq F \\ 0 & A^i < F \end{cases} \qquad b_i = \begin{cases} 0 & A^i \geq 0 \\ A^i & A^i < F \end{cases}$$

The system of differential equations in (12) is re-written as:

$$\frac{\partial D^i}{\partial \tau} \equiv f^i \equiv f^i(\underline{D}; \gamma, \sigma^2) \qquad i = 1, 2, \ldots, n \tag{B2}$$

or

$$\frac{\partial \underline{D}}{\partial \tau} = \underline{f}(\underline{D}; \gamma, \sigma^2)$$

where

$$f^i \equiv \frac{1}{2}\left[\frac{D^{i+1} - 2D^i + D^{i-1}}{\Delta^2}\right](A^i)^2\sigma^2$$

$$+ \left[\frac{D^{i+1} - D^{i-1}}{2\Delta}\right](rA^i - C^i - SF^i - DIV^i) + (C^i + SF^i) - rD^i$$

and where D^0 and D^{n+1} are given in (9).

To estimate γ and σ^2 we employ the "least squares" criteria. Let $D^i(t; \gamma, \sigma^2)$ denote the solution to (B2) subject to (B1) and (9). We want to find γ and σ^2 to satisfy the following criteria:[17]

$$MINIMIZE_{(\gamma, \sigma^2)}\ SS \equiv \textstyle\sum_t (D^{i*}(t; \gamma, \sigma^2) - D_t^*)^2 \tag{B3}$$

where

$D^{i*} \equiv$ model bond price at asset level A^{i*}
$D_t^* \equiv$ observed bond price at time t
$i^* \equiv$ index of observed firm asset value at time t

[17] It may also be necessary to impose upper and lower bounds on γ and σ^2. Since $D(t; \gamma, \sigma^2)$ could be numerically evaluated, standard non-linear least squares procedures or non-linear programming could be employed. Unfortunately, the computation time of each evaluation of D is typically high, and non-linear procedures require numerous evaluations to determine approximate derivatives for choosing the direction of "steepest descent." Quasilinear estimation provides alternative computationally efficient method. This method replaces D in (B3) with a sequence of approximations that are linear in γ and σ^2. The sequence has the property of quadratic convergence toward the original non-linear problem.

Let γ_0 and σ_0^2 denote initial guesses of γ and σ^2. Let $D_0^i(t)$ denote the solution to (B2) using γ_0 and σ_0^2. To produce better estimates of γ and σ^2 we introduce $D_1^i(t)$ which is determined from a linearization of the original system. Consider a first order Taylor expansion of $\underline{f}(\underline{D}_1; \gamma_1, \sigma_1^2)$ about $\underline{f}(\underline{D}_0; \gamma_0, \sigma_0^2)$

$$\underline{f}_1(\underline{D}_1; \gamma_1, \sigma_1^2) = \underline{f}_0 + \underline{\underline{A}}(\underline{D}_1 - \underline{D}_0) + \underline{B}(\gamma^1 - \gamma^0) + \underline{C}(\sigma_1^2 - \sigma_0^2) \qquad \text{(B4)}$$

where

$$\underline{f}_0 \equiv \underline{f}_0(\underline{D}_0; \gamma_0, \sigma_0^2)$$

$$\underline{\underline{A}} \equiv \begin{bmatrix} \frac{\partial f_0^1}{\partial D^1} \frac{\partial f_0^1}{\partial D^2} \cdots \frac{\partial f_0^1}{\partial D^n} \\ \\ \\ \frac{\partial f_0^n}{\partial D^1} \frac{\partial f_0^n}{\partial D^2} \cdots \frac{\partial f_0^n}{\partial D^n} \end{bmatrix} \quad \underline{B}' \equiv \begin{bmatrix} \frac{\partial f_0^1}{\partial \gamma} \frac{\partial f_0^2}{\partial \gamma} \cdots \frac{\partial f_0^n}{\partial \gamma} \end{bmatrix} \equiv 0$$

$$\underline{C}' \equiv \begin{bmatrix} \frac{\partial f_0^1}{\partial \sigma^2} \frac{\partial f_0^2}{\partial \sigma^2} \cdots \frac{\partial f_0^n}{\partial \sigma^2} \end{bmatrix}$$

Note that $\underline{B} \equiv 0$ since γ does not enter the differential equation and hence this term can be dropped from (B4). Now introduce the auxiliary functions $\underline{P}(t)$, $\underline{Q}(t)$, $\underline{R}(t)$ which are the solutions to the following systems of ordinary differential equations:

$$\frac{\partial \underline{P}}{\partial \tau} = \underline{\underline{A}}\underline{P} + [\underline{f}_0 + \underline{\underline{A}}\underline{D}_0\underline{B}\gamma_0 - \underline{C}\sigma_0^2] \qquad \text{subject to } \underline{P}(0) = \underline{a}$$

$$\frac{\partial \underline{Q}}{\partial \tau} = \underline{\underline{A}}\underline{Q} + \underline{B} \qquad \text{subject to } \underline{Q}(0) = \underline{b}$$

$$\frac{\partial \underline{R}}{\partial \tau} = \underline{\underline{A}}\underline{R} + \underline{C} \qquad \text{subject to } \underline{R}(0) = \underline{0} \qquad \text{(B5)}$$

All of the coefficients depend on $\underline{D}_0$, γ_0, and σ_0^2 but are treated as constants in the determination of $\underline{P}$, $\underline{Q}$, and $\underline{R}$. Define $\underline{D}_1$ as the solution to:

$$\frac{\partial \underline{D}_1}{\partial \tau} = \underline{f}_0 + \underline{\underline{A}}(\underline{D}_1 - \underline{D}_0) + \underline{C}(\sigma_1^2 - \sigma_0^2) \qquad \text{subject to } D_1(0) = \underline{a} + (1 - \gamma_1)\underline{b}$$

By construction of $\underline{P}$, $\underline{Q}$ and $\underline{R}$ in (B5) it follows that:

$$\underline{D}_1(t) = \underline{P}(t) + \gamma_1\underline{Q}(t) + \sigma_1^2\underline{R}(t)$$

Hence, $\underline{D}_1$ is linear in γ_1 and σ_1^2. To determine γ_1 and σ_1^2, we solve an ordinary least squares problem with standard regression techniques:

$$MINIMIZE\ ((\gamma_1,\sigma_1^2)\ SS = \textstyle\sum_t \{(P^{i*}(t) + \gamma_1 Q^{i*}(t) + \sigma_1^2 R^{i*}(t) - D_t^*)\}^2$$

Having obtained γ_1, σ_1^2 and $\underline{D}_1(t)$ we then use these values to update $\underline{\underline{A}}$ and $\underline{B}$, determine new $\underline{P}$, $\underline{Q}$, and $\underline{R}$, and determine new estimates of γ_2, σ_2^2, and $\underline{D}_2$. The sequence of linearized solutions $\underline{D}_1$, $\underline{D}_2$, $\underline{D}_3$ $\cdots$ converges quadratically toward the solution of the original problem. Finally it is noted that while we have employed ordinary least squares, this method of quasilinearization is also compatible with generalized least squares and systems of simultaneous differential equations.

REFERENCES

1. Baxter, Nevins D. "Leverage, Risk of Ruin, and the Cost of Capital," *Journal of Finance*, (September, 1967), pp. 395–404.
2. Bellman, Richard, and Robert E. Kalaba. *Quasilinearization and Nonlinear Boundary—Value Problems*, New York, N.Y.: American Elsevier Publishing Co., Inc., 1965.
3. Black, Fischer, and Myron Scholes. "The Pricing of Options and Corporate Securities," *Review of Economics and Statistics*, (February, 1973), pp. 50–62.
4. Brennan, Michael J. "The Pricing of Contingent Claims in Discrete Time Models," *Journal of Finance*, (March, 1979), pp. 53–68.
5. Brennan, Michael J., and Eduardo S. Schwartz. "Corporate Income Taxes, Valuation, and the Problem of Optimal Capital Structure," *Journal of Business*, (January, 1978a), pp. 103–114.
6. Brennan, Michael J., and Eduardo Schwartz. "Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis," *Journal of Financial and Quantitative Analysis*, (September, 1978b), pp. 461–474.
7. Brennan, Michael J., and Eduardo S. Schwartz. "Optimal Financial Policy and Firm Valuation," *Journal of Finance*, (July 1984).
8. Carnahan, B., H. A. Luther, and J. O. Wilkes. *Applied Numerical Methods*, New York, N.Y.: John Wiley and Sons, 1969.
9. Chen, Andrew H. "A Theory of Corporate Bankruptcy and Optimal Capital Structure," in *Handbook of Financial Economics*, edited by James H. Bicksler, North-Holland Publishing Company, 1979.
10. Haugen, Robert A., and Lemma W. Senbet. "The Insignificance of Bankruptcy Costs to the Theory of Optimal Capital Structure," *Journal of Finance*, (May, 1978), pp. 383–393.
11. Helms, Jan, Harriet Kasiwada, Robert E. Kalaba, John Niedercorn. "Fitting Differential Equation Models to Observed Economic Data—I. Quasilinearization," *Applied Mathematics and Computation*, Vol. 4, No. 2, (1978), pp. 139–146.
12. Ho and Singer. "Sinking Fund Provisions on Corporate Debt as a Delivery Option," working paper, New York University, N.Y., (June, 1982).
13. Jones, Philip E., Scott P. Mason, and Eric Rosenfeld. "Contingent Claims Valuation of Corporation Liabilities: Theory and Empirical Tests," *Journal of Finance*, (July, 1984).
14. Kalaba, Robert E., Terence C. Langetieg, and Nayef Zubi; "Quasilinearization and the Estimation of Models Arising in the Pricing of Contingent Claims," mimeo, University of Southern California (1982).
15. Kim, Han E. "A Mean-Variance Theory of Optimal Capital Structure and Corporate Debt Capacity," *Journal of Finance*, (March, 1978), pp. 45–63.
16. Kraus, Alan, and Robert Litzenberger. "A State-Preference Model of Optimal Financial Leverage," *Journal of Finance*, (September, 1973), pp. 911–922.
17. Mason, Scott P. "Essays on Continuous Time Finance," dissertation, Massachusetts Institute of Technology, 1979.
18. Merton, Robert C. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, (May, 1974), pp. 449–470.
19. Merton, Robert C. "Option Pricing When Underlying Stock Returns are Discontinuous," *Journal of Finance*, (May, 1974).
20. Miller, Merton. "Debt and Taxes," *Journal of Finance*, (May, 1977), pp. 261–275.
21. Scott, James. "A Theory of Optimal Capital Structure," *Bell Journal of Economics and Management*, (Spring, 1976), pp. 33–54.
22. Van Horne, James. "Corporate Bankruptcy and Bankruptcy Costs," Research Paper No. 205, Standard Graduate School of Business, 1976.
23. Warner, Jerold B. "Bankruptcy Costs. Some Evidence," *Journal of Finance*, (May, 1977), pp. 337–348.