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A General Diversification Theorem: A Note

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THE MOTIVATION FOR diversification is well known. Since the pioneering work of Markowitz [5] and Tobin [9], the analytic tools have been available to show that zero or negative correlation of the risky asset returns provides the strongest incentive to diversify. Samuelson and Brumelle (see [7] Theorem III and [2] Theorem four) have provided successively more general models which show that nonpositive interdependence of asset returns yields an incentive to diversify (and is free of the two-parameter restriction found in the analyses of Markowitz and Tobin). However, positive interdependence (or positive correlation in a two-parameter model) is a common relation between risky assets. If two risky assets are characterized by positive interdependence, there will still be an incentive to diversify if the investor can alter his portfolio to preserve the same mean income and reduce the riskiness of his portfolio income. It is easy to demonstrate the conditions which ensure this in the context of Tobin's model where it is well known that positive correlation is consistent with positive diversification. An analogous result has not been demonstrated for a more general portfolio model which includes a safe asset.[1] Such a diversification theorem is provided in the next section. The theorem provides the conditions on the joint distribution function which suffice to show that the optimal portfolio of any risk averse investor will be positively diversified if it exists. The theorem is sufficiently robust to allow for positive interdependence of the risky assets and to characterize the extent of the positive interdependence. In two corollary results, it is shown that if the conditions in the diversification theorem are satisfied then by positively diversifying the investor is able to reduce the weight in the tails of his portfolio income distribution and reduce the variance of the portfolio income distribution.

I. Model and Theorem

To construct the portfolio model, let R_j, $j = 1, 2$ be a random variable denoting the return per dollar invested in risky asset j. Let R_0 denote the return per dollar invested in a safe asset. Similarly let x_j, $j = 0, 1, 2$, denote the dollar investment in asset j. If we let W represent the investor's initial wealth then the amount invested in the safe asset is $x_0 = W - (x_1 + x_2)$. Note that x_0 may be negative, thus allowing for borrowing, or greater than W, thus allowing for short sales in

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[1] There are a few cases in the literature which include the possibility that the asset returns are positively correlated; for example, see Samuelson's Theorem II, Hadar and Russell's Theorem five, and Brumelle's Theorem three in [7], [3], and [2], respectively. However, these models do not incorporate a safe asset; nor do they directly characterize the extent of the positive correlation consistent with positive diversification.

the risky assets. Then the investor's portfolio income is a random variable

$$\begin{aligned} Y &= WR_0 + x_1(R_1 - R_0) + x_2(R_2 - R_0) \\ &= V + x_1 Z_1 + x_2 Z_2 \\ &= V + xZ \end{aligned}$$

where $V = WR_0$, $Z_j \equiv R_j - R_0$, $j = 1, 2$, $x = (x_1, x_2)$, and $Z = (Z_1, Z_2)$ is a random vector; note that Z_j is the random return on asset j in excess of the return on the safe asset and so may be positive or negative.[2] Given a utility function u and a joint distribution function F, the investor selects the portfolio $x = (x_1, x_2) \in \mathbb{R}^2$ to maximize the expected utility function H: $\mathbb{R}^2 \rightarrow \mathbb{R}$ where

$$H(x) = \int_{\mathbb{R}^2} u(V + xz)\, dF(z).$$

We assume that $u' > 0$, $u'' < 0$, $E\,|\,Z_j\,| < \infty$, $j = 1, 2$, and $E\,|\,u(V + xZ)\,| < \infty$, so that H is clearly a well-defined concave function.[3]

We say that the portfolio of risky assets $x = (x_1, x_2)$ is diversified if x_1 and x_2 are both nonzero. Similarly the portfolio of risky assets is positively diversified if x_1 and x_2 are both positive and negatively diversified if x_1 and x_2 are both negative.[4]

The necessary and sufficient conditions for positive diversification are easily derived for a suitably modified version of the mean-variance model originally presented by Markowitz and Tobin. For example, Tobin showed that the investor holds the risky assets in fixed proportions. If $x^* = (x_1^*, x_2^*)$ is an optimal portfolio then this implies $x_2^* = k(\mu, \Sigma)x_1^*$ where $k(\mu, \Sigma)$ is a constant which depends on the parameters of the bivariate normal distribution (i.e., $Z = (Z_1, Z_2) \sim N(\mu, \Sigma)$ where $\mu = (\mu_1, \mu_2) = (EZ_1, EZ_2)$ is a vector of means and Σ is the variance-covariance matrix). Of course $k(\mu, \Sigma)$ is the ratio of risky asset holdings x_2/x_1 which minimizes the variance of portfolio income subject to a constant, but arbitrarily chosen, mean portfolio income. Therefore, if the expected net returns on the risky assets are positive and k is positive, the investor can reduce the variance of portfolio income by moving from an undiversified portfolio to a positively diversified portfolio. In this case direct calculation yields

$$k(\mu, \Sigma) = \frac{\sigma_{11} - (\mu_1/\mu_2)\sigma_{12}}{(\mu_1/\mu_2)\sigma_{22} - \sigma_{12}}$$

[2] The standard convention of letting the upper case letters Y and Z_j, $j = 1, 2$, denote random variables while the lower case letters y and z_j denote realizations of the random variables is followed here. Thus, the vector (z_1, z_2) is a realization of the random vector (Z_1, Z_2).

[3] The assumptions $E\,|\,Z_j\,| < \infty$, $j = 1, 2$ and $E\,|\,u(V + xZ)\,| < \infty$ make H well-defined while the concavity of H follows from the concavity of u, i.e., for $0 \le t \le 1$ and $x', x'' \in \mathbb{R}^2$,

$$\begin{aligned} H(tx' + (1-t)x'') &= Eu(V + tx'Z + (1-t)x''Z) \\ &\ge tEu(V + x'Z) + (1-t)Eu(V + x''Z) \\ &= tH(x') + (1-t)H(x''). \end{aligned}$$

[4] The standard notation for vector inequalities is used here. A portfolio $x = (x_1, x_2)$ is positively diversified if $x > 0$ which means $x_1 > 0$ and $x_2 > 0$. Similarly a portfolio is negatively diversified if $x < 0$ which means $x_1 < 0$ and $x_2 < 0$.

where σ_{12} is the covariance of the asset returns and σ_{jj} is the variance of the return on asset $j = 1, 2$. Given a bivariate normal distribution $Z \sim N(\mu, \Sigma)$ we know that the asset returns are independent if and only if the covariance is zero. Hence, we may note that independence and a positive expected net return on each asset (i.e., $\sigma_{12} = 0$ and $\mu_j > 0, j = 1, 2$) suffice to make $k > 0$ and so guarantee positive diversification. It is also clear that a negative covariance and positive expected net returns on the assets (i.e., $\sigma_{12} < 0$ and $\mu_j > 0$, $j = 1, 2$) guarantee positive diversification. Although a nonpositive covariance is one of the sufficient conditions, it is not a necessary condition. If the covariance is positive then the conditions for positive diversification become

$$\frac{\sigma_{12}}{\sigma_{22}} < \frac{\mu_1}{\mu_2} < \frac{\sigma_{11}}{\sigma_{12}}$$

and positive expected net returns on the assets.

In the context of a more general portfolio model (i.e., without the assumption that $Z \sim N(\mu, \Sigma)$) without a safe asset, Samuelson showed that any risk averse investor will hold a positively diversified portfolio if the asset returns are independently distributed and have a common mean. Brumelle has since provided a nice generalization of Samuelson's result by showing that the independence assumption may be replaced by an appropriately defined notion of negative interdependence. The following theorem provides a generalization of Samuelson's Theorem III and Brumelle's Theorem four in the context of a model which includes a safe asset. It shows that independence or negative interdependence are special cases of a stronger result which includes positive interdependence. Given the existence of an optimal portfolio, the theorem provides sufficient conditions for the risk averse investor to select a positively diversified portfolio.[5] It also shows that when the model includes a safe asset, the condition $ER_1 = ER_2$ is no longer a necessary condition for positive diversification, i.e., as was the case in the Samuelson and Brumelle models. For simplicity, we assume that Z is absolutely continuous so that there is a well-defined joint density function f on $\mathbb{R}^2$, i.e.,

$$F(z_1, z_2) = \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} f(t_1, t_2)\, dt_1\, dt_2.$$

Also let F_j denote the marginal distribution of Z_j and let S_j denote the support of F_j.

DIVERSIFICATION THEOREM. *If an optimal portfolio $x^* = (x_1^*, x_2^*)$ exists, $\mu_j > 0, j = 1, 2$ and $\mu_i E(Z_j \mid Z_i = z_i) - \mu_j z_i$ is decreasing for all $z_i \in S_i$, $i, j = 1, 2$, then $x_j^* > 0$ for $j = 1, 2$.*

Proof: Let $D_i H(x)$ denote the partial derivative of H with respect to x_i. *Let* $v = (-\mu_2, \mu_1) \in \mathbb{R}^2$ and let $D_v H(x)$ denote the derivative of H in the direction v. If

[5] See [1] and [4] for the necessary and sufficient conditions for the existence of an optimal portfolio. From [1] and [4] it is clear that if the lim $u'(y) > 0$ as $y \to -\infty$, the lim $u'(y) = 0$ as $y \to +\infty$, and $P(xZ < 0) > 0$ for all $x \in \mathbb{R}^2$, $x \neq 0$, then an optimal portfolio exists.

$x_1 > 0$ then $D_v H(x_1, 0) > 0$ since

$$
\begin{aligned}
D_v H(x_1, 0) &= -\mu_2 D_1 H(x_1, 0) + \mu_1 D_2 H(x_1, 0) \\
&= -\mu_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u'(V + x_1 z_1) z_1 f(z_1, z_2)\, dz_2\, dz_1 \\
&\quad + \mu_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u'(V + x_1 z_1) z_2 f(z_1, z_2)\, dz_2\, dz_1 \\
&= \mu_1 \int_{-\infty}^{\infty} u'(V + x_1 z_1) f_1(z_1) \left(\int_{-\infty}^{\infty} z_2 g(z_2 \mid z_1)\, dz_2 \right) dz_1 \\
&\quad - \mu_2 \int_{-\infty}^{\infty} u'(V + x_1 z_1) z_1 \left(\int_{-\infty}^{\infty} f(z_1, z_2)\, dz_2 \right) dz_1 \\
&= \mu_1 \int_{-\infty}^{\infty} u'(V + x_1 z_1) E(Z_2 \mid Z_1 = z_1) f_1(z_1)\, dz_1 \\
&\quad - \mu_2 \int_{-\infty}^{\infty} u'(V + x_1 z_1) z_1 f_1(z_1)\, dz_1 \\
&= \int_{-\infty}^{\infty} u'(V + x_1 z_1)[\mu_1 E(Z_2 \mid Z_1 = z_1) - \mu_2 z_1] f_1(z_1)\, dz_1 \\
&= Eu'(V + x_1 Z_1) E[\mu_1 E(Z_2 \mid Z_1) - \mu_2 Z_1] \\
&\quad + \mathrm{Cov}(u'(V + x_1 Z_1), \mu_1 E(Z_2 \mid Z_1) - \mu_2 Z_1) \\
&= \mathrm{Cov}(u'(V + x_1 Z_1), \mu_1 E(Z_2 \mid Z_1) - \mu_2 Z_1) \\
&> 0.
\end{aligned}
$$

The third equality follows by noting that $f(z_1, z_2) = g(z_2 \mid z_1) f_1(z_1)$ where f_1 is the marginal density of Z_1 and g is the conditional density of Z_2. The seventh equality follows by noting that $E(E(Z_2 \mid Z_1)) = \mu_2$. Finally, the inequality follows due to the concavity of u (i.e., $u'' < 0$). Conversely, if $x_1 < 0$ then $D_v H(x_1, 0) < 0$. Similarly, letting $v = (\mu_2, -\mu_1)$ it follows that $D_v H(0, x_2) > 0$ if $x_2 > 0$ and $D_v H(0, x_2) < 0$ if $x_2 < 0$. Then, due to the concavity of H, if the optimal portfolio x^* is not equal to $(0, 0)$ then x^* is either positively or negatively diversified (i.e., $x^* > 0$ or $x^* < 0$). Next consider the portfolio $(0, 0)$. Let the direction $v = (v_1, v_2)$ satisfy $v \geq 0$ (i.e., $v_1 \geq 0$, $v_2 \geq 0$, and $v \neq (0, 0)$). Then note that

$$
\begin{aligned}
D_v H(0, 0) &= v_1 D_1 H(0, 0) + v_2 D_2 H(0, 0) \\
&= v_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u'(V) z_1 f(z_1, z_2)\, dz_1\, dz_2 \\
&\quad + v_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u'(V) z_2 f(z_1, z_2)\, dz_1\, dz_2
\end{aligned}
$$

$$= u'(V)[v_1\mu_1 + v_2\mu_2]$$

$$> 0$$

since $u' > 0$ and $\mu_j > 0, j = 1, 2$. Therefore, $x = (0, 0)$ is not optimal and further, due to the concavity of H, no $x < 0$ is optimal. Q.E.D.

Notice that the mean portfolio income, $\mu_Y = V + x_1\mu_1 + x_2\mu_2$, does not change when the investor moves in the direction $(-\mu_2, \mu_1)$ or $(\mu_2, -\mu_1)$. To interpret the theorem let Y_1 and Y_2 be random variables denoting portfolio income. For example, let $Y_1 = V + x_1Z_1$ so that it is the portfolio income from an undiversified portfolio of the form $(x_1, 0)$ where $x_1 > 0$. Next let $Y_2 = V + x_1Z_1 + x_2Z_2$ where $x_1, x_2 > 0$ and this positively diversified portfolio is obtained by moving from the portfolio $(x_1, 0)$ in the direction $v = (-\mu_2, \mu_1)$. Note that in this case the mean portfolio incomes are the same (i.e., $\mu_{Y_1} = \mu_{Y_2}$). Now Y_2 is less risky that Y_1 in the Rothschild-Stiglitz sense (see [6]) if $\mu_{Y_2} = \mu_{Y_1}$ and $Eu(Y_2) > Eu(Y_1)$, and due to the Rothschild-Stiglitz theorem we know that these conditions are equivalent to Y_1 being a mean-preserving spread of Y_2 (i.e., Y_1 having more weight in the tails of its distribution that Y_2). Therefore, the theorem shows that it is always possible to reduce the riskiness of the income distribution by moving from an undiversified to a positively diversified portfolio; equivalently, for every undiversified portfolio there exists a diversified portfolio which generates the same mean portfolio income and an income distribution with less weight in its tails. The conditions $\mu_iE(Z_j \mid Z_i = z_i) - \mu_jz_i$ decreasing for z_i for $i, j = 1, 2$, ensure that the investor can decrease the weight in the tails of the portfolio income distribution by diversifying. To see this, suppose the investor holds an undiversified portfolio $x = (x_1, 0)$ where $x_1 > 0$. Then for a particular realization of the random variable $Z_1 = z_1$ and a portfolio move in the direction $v = (-\mu_2, \mu_1)$, we may interpret μ_2z_1 as the loss and $\mu_1E(Z_2 \mid Z_1 = z_1)$ as the expected gain from the move. If the expected gain is greater than the loss for small values of z_1 and if the expected gain is less than the loss for large values of z_1 then expected portfolio income (i.e., conditioned on z_1) is greater than before for small z_1 and less than before for large z_1. This implies that the positively diversified portfolio yields a portfolio income distribution with less weight in its tails and the implication is demonstrated directly in the following corollary to the diversification theorem.

COROLLARY 1. *If the investor holds an undiversified portfolio of the form* $(x_1, 0)$ *where* $x_1 > 0$, *if* $\mu_j > 0, j = 1, 2$, *and if* $\mu_1E(Z_2 \mid Z_1 = z_1) - \mu_2z_1$ *is a decreasing function for all* $z_1 \in S_1$ *then by positively diversifying the portfolio the investor can obtain a portfolio income distribution with less weight in its tails.*

Proof: First we want to show that there exists a $z_1^* \in S_1$ such that $\mu_1E(Z_2 \mid Z_1 = z_1) > \mu_2z_1$ as $z_1 < z_1^*$ and $\mu_1E(Z_2 \mid Z_1 = z_1) < \mu_2z_1$ as $z_1 > z_1^*$. Suppose not. Then either $\mu_1E(Z_2 \mid Z_1 = z_1) > \mu_2z_1$ for all $z_1 \in S_1$ or $\mu_1E(Z_2 \mid Z_1 = z_1) < \mu_2z_1$ for all $z_1 \in S_1$. However, note that $\mu_1E(Z_2 \mid Z_1 = z_1) > \mu_2z_1$ for all $z_1 \in S_1$ implies $\mu_1\mu_2 > \mu_2z_1$ for all $z_1 \in S_1$ which implies $\mu_1\mu_2 > \mu_1\mu_2$, a contradiction. An equivalent argument shows that $\mu_1E(Z_2 \mid Z_1 = z_1)$ cannot be less than μ_2z_1 for all $z_1 \in S_1$. Then by the continuity of $E(Z_2 \mid Z_1 = z_1)$ it follows that there exists a $z_1^* \in S_1$

such that $\mu_1 E(Z_2 \mid Z_1 = z_1^*) = \mu_2 z_1^*$. Since $\mu_1 E(Z_2 \mid Z_1 = z_1) - \mu_2 z_1$ is decreasing for all $z_1 \in S_1$, it follows that $\mu_1 E(Z_2 \mid Z_1 = z_1) > \mu_2 z_1$ as $z_1 < z_1^*$ and $\mu_1 E(Z_2 \mid Z_1 = z_1) < \mu_2 z_1$ and $z_1 > z_1^*$.

Next consider the portfolio income distribution. Recall that porfolio income is the random variable Y where

$$Y = V + xZ.$$

Hence

$$P(Y \leq y) = P(xZ \leq s)$$

where $s \equiv y - V$ and

$$P(xZ \leq s) = \int_{-\infty}^{\infty} \int_{-\infty}^{h(x)} f(z_1, z_2)\, dz_1\, dz_2$$

where

$$h(x, z_2) = \frac{s - x_2 z_2}{x_1}.$$

Note that if $x_2 = 0$ then we obtain

$$\begin{aligned} P(x_1 Z_1 \leq s) &= P\left(Z_1 \leq \frac{s}{x_1}\right) \\ &= F_1\left(\frac{s}{x_1}\right) \end{aligned}$$

where F_1 is the marginal distribution of Z_1.

Now to show that the probability distribution $P(xZ \leq s)$ loses weight in the left tail due to diversification. Let s be such that

$$\frac{s}{x_1} = z_1 < z_1^*$$

and differentiate in the direction $v = (-\mu_2, \mu_1)$. Then

$$D_v P(xZ \leq s) = -\mu_2 \int_{-\infty}^{\infty} f(h, z_2) D_1 h\, dz_2 + \mu_1 \int_{-\infty}^{\infty} f(h, z_2) D_2 h\, dz_2$$

where $D_j h$ is the partial derivative of h with respect to x_j, $j = 1, 2$, i.e.,

$$D_1 h = \frac{-(s - x_2 z_2)}{(x_1)^2}$$

and

$$D_2 h = -\frac{z_2}{x_1}.$$

Evaluating the derivative at $x = (x_1, 0)$ yields

$$\begin{aligned} D_v P(x_1 Z_1 \le s) &= \mu_2 \int_{-\infty}^{\infty} f\left(\frac{s}{x_1}, z_2\right) \frac{s}{(x_1)^2}\, dz_2 - \mu_1 \int_{-\infty}^{\infty} f\left(\frac{s}{x_1}, z_2\right) \frac{z_2}{x_1}\, dz_2 \\ &= \frac{1}{x_1}\left\{\mu_2 \int_{-\infty}^{\infty} f(z_1, z_2) z_1\, dz_2 - \mu_1 \int_{-\infty}^{\infty} f(z_1, z_2) z_2\, dz_2\right\} \\ &= \frac{1}{x_1}\left\{\mu_2 \int_{-\infty}^{\infty} z_1 f(z_1, z_2)\, dz_2 - \mu_1 \int_{-\infty}^{\infty} z_2 g(z_2 \mid z_1) f_1(z_1)\, dz_2\right\} \\ &= \frac{1}{x_1}\{\mu_2 z_1 - \mu_1 E(Z_2 \mid Z_1 = z_1)\} f_1(z_1) \\ &< 0 \end{aligned}$$

since

$$\frac{s}{x_1} = z_1 < z_1^*$$

yields $\mu_1 E(Z_2 \mid Z_1 = z_1) > \mu_2 z_1$.

Finally, to show that the probability distribution loses weight in its right tail note that

$$P(xZ \ge s) = \int_{-\infty}^{\infty} \int_{h}^{\infty} f(z_1, z_2)\, dz_1\, dz_2$$

and for $v = (-\mu_2, \mu_1)$

$$D_v P(xZ \ge s) = -\mu_2\left\{-\int_{-\infty}^{\infty} f(h, z_2) D_1 h\, dz_2\right\} + \mu_1\left\{-\int_{-\infty}^{\infty} f(h, z_2) D_2 h\, dz_2\right\}.$$

Then selecting s such that

$$\frac{s}{x_1} = z_1 > z_1^*$$

and evaluating the derivative at $x = (x_1, 0)$ yields

$$\begin{aligned} D_v P(x_1 Z_1 \ge s) &= \frac{1}{x_1}\left\{-\mu_2 \int_{-\infty}^{\infty} z_1 f(z_1, z_2)\, dz_2 + \mu_1 \int_{-\infty}^{\infty} z_2 f(z_1, z_2)\, dz_2\right\} \\ &= \frac{1}{x_1}\{-\mu_2 z_1 + \mu_1 E(Z_2 \mid Z_1 = z_1)\} f_1(z_1) \\ &< 0 \end{aligned}$$

since $\mu_1 E(Z_2 \mid Z_1 = z_1) < \mu_2 z_1$ for $z_1 > z_1^*$. Q.E.D.

Corollary 1 shows the critical role played by the condition $\mu_1 E(Z_2 \mid Z_1 = z_1) - \mu_2 z_1$ decreasing for all $z_1 \in S_1$. If the condition holds then the corollary shows that the investor can increase his expected portfolio income given small z_1 and decrease it given large z_1. Diversification away from the portfolio $(x_1, 0)$ in the direction $v = (-\mu_2, \mu_1)$ achieves this result and so yields a portfolio income distribution with less weight in its tails. An analogous argument may be used to show that a positively diversified portfolio exists which exhibits less weight in the tails of its distribution than the undiversified portfolio $(0, x_2)$.

The diversification theorem clearly holds if $E(Z_j \mid Z_i = z_i)$ is constant or decreasing for all $z_i \in S_i$, $i, j = 1, 2$ (i.e., the independence or negative interdependence cases, respectively). Of course it should also be noted that the theorem does exclude some joint distributions by limiting attention to those cases in which an optimal portfolio exists. For example, suppose $Z_2 = c - Z_1$ where $c > 0$; this example satisfies all of the conditions of the theorem except for existence. In this case, note that there exists an $x = (x_1, x_2)$ such that $P(x_1 Z_1 + x_2 Z_2 > 0) = P(x_1 R_1 + x_2 R_2 > (x_1 + x_2) R_0) = 1$. Hence, there is a portfolio which will guarantee a larger return than an equivalent investment in the safe asset. In this case, there is an arbitrage opportunity and $v = (1, 1)$ is an arbitrage direction since

$$D_v H = c \int_{\mathbb{R}^2} u' \, dF > 0,$$

i.e., by moving in the direction $v = (1, 1)$ the investor can continue to increase expected utility indefinitely. More generally, it must be true that $P(xZ < 0) > 0$ for all $x \neq 0$, since otherwise an arbitrage opportunity will exist.[6]

More importantly the diversification theorem clearly holds if $E(Z_2 \mid Z_1 = z_1)$ and $E(Z_1 \mid Z_2 = z_2)$ are increasing (i.e., the positive interdependence case) at rates less than μ_2/μ_1 and μ_1/μ_2, respectively.[7] Since $\mathrm{Cov}(Z_1, Z_2) = \mathrm{Cov}(Z_1, E(Z_2 \mid Z_1))$, it follows that $E(Z_1 \mid Z_2 = z_2)$ increasing for all $z_2 \in S_2$ implies a positive covariance and correlation between Z_1 and Z_2. Hence, the theorem provides conditions sufficient for the risk averse investor to select a positively diversified portfolio of positively correlated assets. Of course, if $Z \sim N(\mu, \Sigma)$ then

$$E(Z_2 \mid Z_1 = z_1) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1} (z_1 - \mu_1)$$

where ρ is the correlation coefficient and it may be shown by direct calculation that the conditions in the theorem become the conditions for positive diversification, given positive correlation, in the Tobin model. To complete the analogy within the Tobin model, it is interesting to note that even if Z does not have a joint normal distribution but the conditions in the theorem hold then the conditions for positive diversification which we obtained from the Tobin model

[6] See [4] for a discussion of this condition.

[7] If the safe asset is removed from the model then, as Brumelle has shown in his Corollary two, $\mu_{R_1} = \mu_{R_2}$ is a necessary condition for positive diversification. In this case, the diversification theorem shows that $\mu_{R_1} = \mu_{R_2} > 0$ and $\mu_{R_1}(E(R_j \mid R_i = r_i) - r_i)$ decreasing in r_i suffice to make positive diversification optimal; here $E(R_j \mid R_i = r_i)$ may be increasing but at a rate less than one for $i, j = 1, 2$.

must still hold. We note this result, for the case in which the assets are positively correlated, in the following corollary.

COROLLARY 2. *If the hypotheses of the diversification theorem hold and* $E(Z_j \mid Z_i = z_i)$ *is increasing for all* $z_i \in S_i$, $i, j = 1, 2$, *then*

$$\frac{\sigma_{12}}{\sigma_{22}} < \frac{\mu_1}{\mu_2} < \frac{\sigma_{11}}{\sigma_{12}}.$$

Proof: Let $U_2 = \mu_1 Z_2 - \mu_2 Z_1$ and $U_1 = \mu_2 Z_1 - \mu_1 Z_2$. By hypothesis, $E(U_2 \mid Z_1 = z_1)$ is decreasing for all $z_1 \in S_1$ and so it follows that

$$EU_2 Z_1 \leq EU_2 EZ_1.^{8}$

Since $EU_2 = 0$, this inequality may be rewritten as

$$\begin{aligned} EU_2 Z_1 &= \mu_1 EZ_1 Z_2 - \mu_2 EZ_1^2 \\ &= \mu_1(\sigma_{12} + \mu_1 \mu_2) - \mu_2(\sigma_{11} + \mu_1^2) \\ &= \mu_1 \sigma_{12} - \mu_2 \sigma_{11} \\ &< 0. \end{aligned}$$

Note that $E(Z_2 \mid Z_1 = z_1)$ increasing for all $z_1 \in S_1$ yields $\sigma_{12} > 0$ and so the above inequality may be equivalently expressed as

$$\frac{\mu_1}{\mu_2} < \frac{\sigma_{11}}{\sigma_{12}}.$$

Similarly $E(U_1 \mid Z_2 = z_2)$ decreasing for all $z_2 \in S_2$ yields

$$\frac{\mu_1}{\mu_2} > \frac{\sigma_{12}}{\sigma_{22}}. \qquad \text{Q.E.D.}$$

The corollary shows that the Tobin conditions for positive diversification given positive correlation hold in a more general setting than the mean-variance model. Of course, the converse of the corollary does not hold since, for example, $\mathrm{Cov}(U_2, Z_1) < 0$ does not imply that $E(U_2 \mid Z_1 = z_1)$ is decreasing for all $z_1 \in S_1$. The stronger conditions of the diversification theorem (i.e., such as $E(U_2 \mid Z_1 = z_1)$ decreasing) are needed because the model is more general than Tobin's. However, the corollary shows that the conditions in the diversification theorem have a semantic content closely related to that of Tobin's conditions, since they show that an investor can reduce the riskiness of portfolio income Y by selecting a positively diversified portfolio while Tobin's conditions show that the investor can reduce the variance of portfolio income σ_Y^2 by selecting a positively diversified portfolio. Alternatively, the corollary shows that a reduction in risk implies a reduction in variance.

[8] See Brumelle's Theorem five or Shea's Lemma 2.1 in [2] and [8] for a proof of this result.

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