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The Harmonic Mean and Other Necessary Conditions for Stochastic Dominance

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ABSTRACT

In this paper a systematic procedure is developed to determine necessary conditions for all degrees of stochastic dominance. The previously known necessary conditions are specified as to which degrees of dominance they belong, and two new necessary conditions, a ranking of harmonic means and a ranking of algebraic combinations of the first three moments, are derived.

SINCE THE DEVELOPMENT OF stochastic dominance ranking models for discriminating between risky outcomes, the principal reported use of that tool has been the determination of the efficient set of security portfolios using time series data of returns. Comparisons of entire empirical distributions point-by-point for the first two degrees and between points of occurrence for the third and higher degrees have proved to be extremely consuming of computer time and capacity. The existing stochastic dominance algorithms that have some claim to the relative efficiency utilize the known necessary conditions for stochastic dominance, including the least value comparison, the ranking of arithmetic means, the ranking of geometric means, the ranking of semivariances, and the comparison of algebraic combinations of the arithmetic mean and variance, in a preliminary screening process to determine if a stochastic dominance ranking is possible before the point-by-point comparison is begun. The more necessary conditions for stochastic dominance are identified: the more efficient the algorithms can become, and the more practical the tool will be for comparison of risky alternatives. In this paper a general method for developing necessary conditions is outlined, and this method is used to prove that a ranking of harmonic means and a ranking of algebraic combinations of the arithmetic mean, variance, and third central moment are necessary conditions for stochastic dominance rankings of any degree.

Since the development of the first three degrees of stochastic dominance by Quirk and Saposnik [17], Hadar and Russell [6], Hanoch and Levy [8], and Whitmore [20], a number of papers have been published and presented at professional meetings utilizing the stochastic dominance tool to discriminate among investment portfolios. The papers primarily concerned with the development of an efficient algorithm for performing the tests include [1], [3], [4], [9], [12], and [16]. Other papers placing more emphasis on the empirical results are [15] and [19].

At the present stage of development, the stochastic dominance tool is used to

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discriminate between two mutually exclusive investment opportunities when the statistical distributions of returns from those two opportunities are known. Methods to form optimum portfolios of securities by the use of stochastic dominance procedures have not yet been accomplished, although Hadar and Russell's results on optimality of diversification [7] and Levy and Kroll's results with combinations of a risky portfolio and a risk-free security [14] seem to be fruitful steps in this development. Another serious problem with the validity of empirical tests using stochastic dominance is the sampling characteristics problem, as pointed out by Kroll and Levy [13]. In stochastic dominance empirical studies, the observed points have been used as estimates of the population values of returns, with no procedure available for assessing the statistical reliability of those estimates.

In this paper these unsolved problems are left unsolved, and the decision situation is confined to the case of two mutually exclusive opportunities. The returns are measured as holding period returns, one plus the rate of return, and those returns are finite. The two statistical distributions used in the derivations are continuous, although the results obtained also hold for discrete distributions.

The holding period returns are X, which has the range

$$a \le X \le b$$

In the most general case $\underline{a}$ would be zero and $\underline{b}$ positive infinity, although the $\underline{a}$ and $\underline{b}$ values will be restricted below.

The probability density function of the first alternative is $f(X)$, and the probability density function of the second alternative is $g(X)$. The notational convention followed here is that the first integral of the probability density function, the distribution function, is $F_1(X)$, the integral of $F_1(X)$ is $F_2(X)$, and so forth. The sequence of integrals for the first alternative is

$$f(X),\ F_1(X),\ F_2(X),\ \cdots,\ F_k(X),\ \cdots$$

and the sequence for the second alternative is

$$g(X),\ G_1(X),\ G_2(X),\ \cdots,\ G_k(X),\ \cdots$$

The stochastic dominance ranking conditions will be stated uniformly here as the first alternative is preferred to the second and the $\underline{f}$ distribution is preferred to the $\underline{g}$ distribution.

The stochastic dominance ranking values are derived by comparing the expected utility from the first alternative to the expected utility from the second alternative. Using the notation $U(X)$ for the utility function the first alternative is preferred if

$$\int_a^b f(X)U(X)\,dX \ge \int_a^b g(X)U(X)\,dX$$

These integrals can be integrated by parts $\underline{n}$ times (with $U(X)$ and its derivatives of $\underline{u}$ and $f(X)$ or $g(X)$, and the integrals as $\underline{dv}$ in the standard formula), and the signs of the utility function derivatives, nonpositive for the even derivatives and nonnegative for the odd derivatives, can be used to derive the nth degree stochastic dominance conditions for the first alternative to be preferred to the second:

$$F_k(b) \leq G_k(b) \qquad k = 2, 3, \cdots, n - 1$$

and

$$F_n(X) \leq G_n(X) \qquad \text{for all } X$$

For discrete distributions the theorem of the mean can be applied repeatedly to derive analogous ranking conditions, as shown in [6].

In the case of finite range distributions, if the least value with nonzero probability density for the first alternative is less than the least value for the second alternative, then the required inequality

$$F_n(X) \leq G_n(X)$$

does not hold for all X, and the first alternative cannot dominate the second alternative by any degree of stochastic dominance. The ordering of least values in the two distributions is therefore a necessary condition for any degree of stochastic dominance, as was pointed out by the original developers of the stochastic dominance tools.

If any degree of stochastic dominance exists between the two investment alternatives, then certain mathematical relationships exist between the probability integrals $F_k(X)$ and $G_k(X)$. These relationships can be used in the derivation of necessary conditions, which can be used to eliminate dominated alternatives. Because the cumulative probability functions are positive-valued monotonically increasing functions over all X the ordering

$$F_k(X) \leq G_k(X) \qquad \text{for all } X \text{ and any } k$$

results in the higher degree ordering

$$F_{k+1}(X) \leq G_{k+1}(X) \qquad \text{for all } X$$

An ordering of the two integrals over all X for some degree $\underline{n}$ implies the same ordering of the two integrals for all higher degrees of integrals. nth degree stochastic dominance therefore implies all higher degree stochastic dominance, as shown in [11].

An ordering of lower partial moments is known to be a necessary condition for certain degrees of stochastic dominance. The second-order lower partial moments, the semivariances, of the two distributions must be ordered for first-, second-, or third-degree stochastic dominance to exist. The semivariance of a distribution with reference value $\underline{c}$ can be derived by integration by parts of the mathematical function to be

$$SV_f(c) = 2F_3(c)$$

If first-degree stochastic dominance exists between the two distributions then

$$F_k(X) \leq G_k(X) \qquad \text{for all } X \text{ with } k = 1, 2, \cdots$$

and therefore

$$SV_f(c) \le SV_g(c)$$

If second-degree stochastic dominance exists then

$$F_k(X) \le G_k(X) \qquad \text{for all } X \text{ with } k = 2, 3, \cdots$$

and

$$SV_f(c) \le SV_g(c)$$

With third-degree stochastic dominance

$$F_k(X) \le G_k(X) \qquad \text{for all } X \text{ with } k = 3, 4, \cdots$$

and the semivariances must be ordered.

The ordering of semivariances is not a necessary condition for fourth or higher degrees of stochastic dominance to exist. The inequality

$$SV_f(c) = 2F_3(c) > 2G_3(c) = SV_g(c)$$

does not prevent the stochastic dominance inequality

$$F_k(X) \le G_k(X) \qquad \text{for all } X \text{ with } k = 4, 5, \cdots$$

from holding. It is therefore possible that a semivariance ranking procedure could lead to the selection of a less preferred investment alternative.

The lower partial moment (LPM) of order $\underset{\sim}{k}$ with reference value $\underset{\sim}{c}$ is defined by Bawa and Lindenberg [2] and Fishburn [5] as

$$\mathrm{LPM}_f^k(c) = \int_a^c (c - X)^k f(X)\, dX$$

Using integration by parts, all the lower partial moments can be expressed in terms of the repeated integrals of the probability density function:

$$\mathrm{LPM}_f^k(c) = k!\, F_{k+1}(c)$$

If nth degree stochastic dominance exists between the two alternatives then the $n - 1$'st order lower partial moments must be ordered. Since lower degree stochastic dominance implies higher degree, the sets of lower partial moments beginning with the $n - 1$'st are ordered:

$$\mathrm{LPM}_f^k(c) \le \mathrm{LPM}_g^k(c) \qquad k = n - 1, n, n + 1, \cdots$$

Those lower partial moment rankings are necessary for the nth degree stochastic dominance. It is not necessary, however, for the $n - 2$'nd and lesser order lower partial moments to be ordered. The results derived above for the semivariance, the second-order lower partial moment, and fourth-degree stochastic dominance are an example of this general case.

A general procedure for deriving necessary conditions for all degrees of stochastic dominance can be based on a relationship shown in [11]. If any degree of stochastic dominance exists, then

$$F_k(b) \le G_k(b) \qquad k = 1, 2, \cdots$$

This ordering of each degree integral pair evaluated at the endpoint value $\underline{b}$, which at this stage can be either finite or infinite, forms a set of necessary conditions for all degrees of stochastic dominance. This set of necessary conditions is usually ignored, since direct evaluation of the $F_k(b)$ and $G_k(b)$ is performed as part of the comparison of the entire distributions.

The necessary ordering of $F_k(b)$ and $G_k(b)$ can be used to rederive some of the well-known necessary conditions for stochastic dominance. For example, the arithmetic means of the two distributions where $\underline{b}$ is finite were derived in [10] as $b - F_2(b)$ and $b - G_2(b)$. The well-known necessary condition that the arithmetic means must be ordered can be attained by substitution.

The necessary condition involving an algebraic combination of arithmetic means and variances that Whitmore [20] derived for third-degree stochastic dominance is necessary for <u>any</u> degree of stochastic dominance. His condition was

$$\sigma_g^2 - \sigma_f^2 + (E_f - E_g)(2b - E_f - E_g) \geq 0$$

Using the derivations in [10] the arithmetic means and variances for finite range distributions can be expressed in terms of the $F_k(b)$:

$$E_f = b - F_2(b)$$

$$\sigma_f^2 = 2F_3(b) - [F_2(b)]^2$$

and the substitution can be made in Whitmore's inequality with the simplified result

$$F_3(b) \leq G_3(b)$$

Since this inequality must hold for any degree of stochastic dominance, Whitmore's condition is necessary for all degrees of stochastic dominance with finite distributions, not just third degree.

A new necessary condition involving algebraic combinations of the arithmetic means, variances, and third central moments is easily derived. The third central moment in terms of $F_k(b)$ is

$$C_{3f} = -6F_4(b) + 6F_3(b)F_2(b) - 2[F_2(b)]^3$$

The third central moment, the variance, and the arithmetic mean can be combined in an algebraic expression to yield

$$(E_f - b)^3 + 3(E_f - b)\sigma_f^2 + C_{3f} = -6F_4(b)$$

Since $F_4(b) \leq G_4(b)$ must hold for any degree of stochastic dominance,

$$(E_f - b)^3 + 3(E_f - b)\sigma_f^2 + C_{3f} \geq (E_g - b)^3 + 3(E_g - b)\sigma_g^2 + C_{3g}$$

is the necessary condition stated in the form of algebraic combinations of the arithmetic means, variances, and third central moments of the two distributions.

The expression of the arithmetic means and central moments in terms of $F_k(b)$ requires that $\underline{b}$ be finite. The proofs above are therefore applicable only to finite range distributions, and alternative proof of the necessary conditions in the case of infinite range distributions such as the log-normal will be required.

Jean [11] derived the geometric means of the two distributions in terms of infinite series with terms composed of the $F_k(b)$ and $G_k(b)$ and proved that a ranking of geometric means is a necessary condition for any degree of stochastic dominance with finite range positive distributions. A similar procedure can be used to prove the necessity of a ranking of harmonic means.

The harmonic mean of a distribution is the reciprocal of the expected value of the reciprocal of X, and is defined only when the random variable assumes strictly positive values. The distributions are further restricted here to the finite range. The harmonic mean of the first distribution is

$$\frac{1}{H_f} = E\left[\frac{1}{X}\right] = \int_a^b \frac{1}{X} f(X)\, dX \qquad \text{with } a > 0$$

To develop an infinite series expansion of the reciprocal of the harmonic mean, the integral on the right can be integrated by parts an indefinite number of times with $\frac{1}{X}$ and its derivatives u and $f(X)$ and its integrals as dv in the standard formula:

$$\frac{1}{H_f} = \frac{1}{b} + \sum_{k=1}^{k=\infty} \frac{k!}{b^{k+1}} F_{k+1}(b)$$

Before this series is used to prove necessity of the harmonic means ranking, the convergence of the series must be investigated.

As shown in [11] the upper bound for $F_{k+1}(b)$ is expressed in terms of the upper and lower limits of the distribution, b and a,

$$F_{k+1}(b) \leq \frac{(b-a)^k}{k!}$$

The terms in the harmonic mean series are all positive so the lower bound for the series is zero. The upper bound for the series is

$$\frac{1}{H_f} \leq \frac{1}{b} + \sum_{k=1}^{k=\infty} \frac{k!}{b^{k+1}} \frac{(b-a)^k}{k!} = \frac{1}{b}\left[\sum_{k=0}^{k=\infty} \left[\frac{b-a}{b}\right]^k\right]$$

which may be recognized as a geometric series with factor $\left[\frac{b-a}{b}\right]$. Since the distributions have strictly positive values of X, the factor in the geometric series is less than one, and that series converges. Therefore, the series for the harmonic mean converges, and is a legitimate representation of the harmonic mean.

If any degree of stochastic dominance exists between the two alternatives such that the first alternative is preferred to the second, then

$$\frac{1}{H_f} = \frac{1}{b} + \sum_{k=1}^{k=\infty} \frac{k!}{b^{k+1}} F_{k+1}(b) \leq \frac{1}{b} + \sum_{k=1}^{k=\infty} \frac{k!}{b^{k+1}} G_{k+1}(b) = \frac{1}{H_g}$$

The result of any degree of stochastic dominance between strictly positive finite range distributions is an ordering of the harmonic means,

$$H_f \geq H_g$$

so the ranking of the harmonic means is a necessary condition for all degrees of stochastic dominance.

The derivations of the moments, the geometric mean, and the harmonic mean shown above assumed a continuous distribution with probability density function $f(X)$. To derive the same results for a discrete distribution, such as an empirical set of data, the cumulative probability is treated as a step function, with the steps occurring at the points of the distribution. The integration to derive the formulas for the moments and means in this case will be piecewise, but the same results will be obtained.

Full use of the listed set of necessary conditions should speed up the determination of an efficient set of portfolios by stochastic dominance. To give an indication of the relative effectiveness of the various necessary conditions listed above, two checks were made of randomly selected stock portfolios. In the first, 100 portfolios were used, giving a possible total of 4,950 pairwise comparisons that might have to be made in deriving the efficient set of portfolios for a given degree of stochastic dominance. The 100 portfolios were first ordered by arithmetic means, and each of the other necessary conditions was tested separately against the arithmetic mean for conflicts of ordering. When a conflict exists no stochastic dominance can exist. Of the 4,950 pairs the least value condition eliminated 2,071 pairs from further tests, the semivariance condition eliminated 1,879 pairs, the Whitmore combination of the arithmetic mean and variance eliminated 1,624 pairs, the algebraic combination of the arithmetic mean, variance, and third moment derived above eliminated 2,774 pairs, the geometric mean condition eliminated 606 pairs, and the harmonic mean condition eliminated 1,120 pairs. From these results it would appear that the harmonic mean condition is more powerful than the geometric mean condition, and that the three moment combination is more powerful than any other individual condition.

With the same 100 portfolios, the seven necessary conditions were tested simultaneously. Of the 4,950 possible pairs 4,120 were eliminated, leaving only 830 to be tested by further computations. Assuming that a higher degree of stochastic dominance than third degree might be used, the semivariance condition was eliminated and the remaining six necessary conditions were tested simultaneously. This time 4,041 pairs were eliminated from the 4,950, leaving 909 for further stochastic dominance test computations.

To provide a check on these numerical results, a second set of 500 portfolios was tested using the same procedures. For the 124,750 possible pairs that can be formed, the separate comparisons of the necessary condition with the arithmetic mean condition yielded:

Condition	Elimination	Percentage
Least value	65,065	52.2%
Semivariance	54,262	43.5%
Whitmore	30,570	24.5%
Three moment	70,934	56.9%
Geometric mean	6,502	5.2%
Harmonic mean	15,969	12.8%

All seven necessary conditions considered simultaneously eliminated 108,622 pairs (87.1%), leaving 16,128 (12.9%) for further stochastic dominance calculations. When the semivariance condition was removed the remaining six conditions considered simultaneously eliminated 105,464 pairs (84.5%), leaving 19,286 (15.5%) for further computations.

These numerical results are of course specific to the particular data sets used, but do give an indication of the possible results of incorporating a test of the known necessary conditions in a stochastic dominance algorithm. Use of the necessary conditions at an early stage in the algorithm should save over 80% of the point-by-point comparisons otherwise necessary.

REFERENCES

1. Vijay S. Bawa. "Optimal Rules for Ordering Uncertain Prospects." *Journal of Financial Economics* 2 (March 1975), 95–121.
2. ——— and Eric B. Lindenberg. "Capital Market Equilibrium in a Mean-Lower Partial Moment Framework." *Journal of Financial Economics* 5 (November 1977), 189–200.
3. ———, Eric B. Lindenberg, and Lawrence C. Rafsky. "An Efficient Algorithm to Determine Stochastic Dominance Admissible Sets." *Management Science* 25 (July 1979), 609–22.
4. Roger P. Bey. "Estimating the Optimal Stochastic Dominance Efficient Set with a Mean-Semivariance Algorithm." *Journal of Financial and Quantitative Analysis* 14 (December 1979), 1059–70.
5. Peter C. Fishburn. "Mean-Risk Analysis with Risk Associated with Below-Target Returns." *American Economic Review* 67 (1977), 116–26.
6. Josef Hadar and William R. Russell. "Rules for Ordering Uncertain Prospects." *American Economic Review* 59 (March 1969), 25–34.
7. ———. "Stochastic Dominance and Diversification." *Journal of Economic Theory* 3 (September 1971), 288–305.
8. Giora Hanoch and Haim Levy. "The Efficiency Analysis of Choices Involving Risk." *Review of Economic Studies* 36 (July 1969), 335–46.
9. Billy P. Helms, William H. Jean, and Hassan Tehranian. "An Algorithm for *N*th Degree Stochastic Dominance." Paper presented at the Financial Management Association meetings, 1982.
10. William H. Jean. "Comparison of Moment and Stochastic Dominance Ranking Methods." *Journal of Financial and Quantitative Analysis* 10 (March 1975), 151–61.
11. ———. "The Geometric Mean and Stochastic Dominance." *Journal of Finance* 35 (March 1980), 151–58.
12. Richard B. Kearns and Richard C. Burgess. "An Effective Algorithm for Estimating Stochastic Dominance Efficient Sets." *Journal of Financial and Quantitative Analysis* 14 (September 1979), 547–52.
13. Yoram Kroll and Haim Levy. "Sampling Errors and Portfolio Efficient Analysis." *Journal of Financial and Quantitative Analysis* 15 (September 1980), 655–88.
14. Haim Levy and Yoram Kroll. "Efficiency Analysis with Borrowing and Lending: Criteria and Their Effectiveness." *Review of Economics and Statistics* 56 (February 1979), 125–30.
15. R. Burr Porter and Roger P. Bey. "An Evaluation of the Empirical Significance of Optimal Seeking Algorithms in Portfolio Selection." *Journal of Finance* 29 (December 1974), 1479–90.
16. R. Burr Porter, James R. Wart, and Donald L. Ferguson. "Efficient Algorithms for Conducting Stochastic Dominance Tests on Large Numbers of Portfolios." *Journal of Financial and Quantitative Analysis* 8 (January 1973), 71–82.
17. James P. Quirk and Rubin Saposnik. "Admissibility and Measurable Utility Functions." *Review of Economic Studies* 29 (March 1962), 140–46.
18. Robert C. Scott and Philip A. Horvath. "On the Direction of Preference for Moments of Higher Order than the Variance." *Journal of Finance* 35 (September 1980), 915–19.
19. Hassan Tehranian. "Empirical Studies in Portfolio Performance Using Higher Degrees of Stochastic Dominance." *Journal of Finance* 35 (March 1980), 159–72.
20. George A. Whitmore. "Third-Degree Stochastic Dominance." *American Economic Review* 60 (June 1970), 457–59.