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Option Pricing Bounds in Discrete Time

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ABSTRACT

Upper and lower bounds are derived for call options traded at discrete intervals. These bounds are independent of assumptions on the stock price distribution other than a restriction satisfied by the stock being "non-negative beta." The development of the bounds relies on the single-price law and arbitrage arguments. Both single-period and multiperiod results are produced, and put option bounds follow by extension. The bounds exist as equilibrium values given a consensus on stock price distribution; they are also valid for empirical studies, being adjustable for dividends and commissions.

FOLLOWING THE IMPETUS of the Black-Scholes pricing model, the majority of option analyses have been in continuous time. Such approaches have relied on the formation of perfect, riskless hedges involving the option, its underlying security, and a riskless asset; since the hedge must be continuously revised and maintained, while actual trading opportunities are discrete, this crucial assumption limits the accuracy and applicability of the Black-Scholes and related formulae. In fact, the Black-Scholes formula can only be considered as "a valid approximation to the discrete-time solution" (Merton [8, p. 663]), while the closeness of that approximation has not been ascertained in general.[1] Earlier discrete models such as Boness [3] and Samuelson [12] have been succeeded by those of Brennan [5], Cox et al. [6], and Rubinstein [11], in which distributional restrictions are imposed on share returns; usually, the discrete formula converges to the Black-Scholes expression under appropriate limit conditions.

Using the Rubinstein [11] approach, we derive upper and lower bounds for option prices with both a general price distribution and discrete trading opportunities. The bounds are functions of the share and exercise prices, the riskless rate of interest, the distribution of the share price change per period, and the number of periods of expiration of the option. The lower bound is tighter than that of Merton [7], derived from more general arbitrage considerations, while being less exact than the results already mentioned for restricted distributions. Thus the bounds form a range of prices in equilibrium derivable for any general distribution; they have the further advantage of being adjustable for dividends and commissions. The generality of assumptions and adjustable range provide important advantages for empirical studies.

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[1] See [6], [9], and [10] for pertinent comments on this point.

I. Call Option Bounds

We adopt the following notation:

S	the initial price of the underlying asset,
X	the exercise price of the option,
Y	the random price change (ΔS) per period,
$F(Y)$	the probability distribution of Y for $Y \in [-S, \infty)$,
i	the riskless rate of interest per period, and
$C(S, X, i)$	the single-period European call price.

The basic assumptions of the analysis are those of Rubinstein[2] [11, pp. 408–11]: (i) the single-price law of markets, (ii) non-satiation, (iii) perfect, competitive, and Pareto-efficient financial markets, (iv) rational time-additive tastes, and (v) weak aggregation, or the existence of an average investor. We further assume the absence of taxes or dividends.

Under these assumptions, Rubinstein determined that the current price of any asset, such as the stock price S, is given by the expression

$$S = V_0[S + Y] = \frac{1}{1+i}\{S + E(Y) + \mathrm{cov}[U'(c), Y]/E[U'(c)]\} \tag{1}$$

where $V_0[\cdot]$ is the current value of proceeds received at the end of the period and $U'(c)$ is the marginal utility for consumption of the average investor. Expression (1) remains valid for intermediate trading, in the absence of dividend payments. To arrive at a specific formula, Rubinstein makes the further assumption of a bivariate lognormal distribution for price $S + Y$ and $U'(c)$; this is analogous to assumptions in [5], while in [6] a binomial distribution of Y is required. In our case, we need only to restrict the distribution of Y in its relation to that of $U'(c)$; basically this is a restriction on investors' consumption proclivities relative to changes in share prices. Specifically, we assume: (vi) The normalized conditional expected marginal utility for consumption is non-increasing in the price change Y. Analytically, this is equivalent to: $\bar{Z}(y) \cong E[U'(c) \mid Y = y]/E[U'(c)]$ is a non-increasing function of y.

Using the definition of $\bar{Z}(y)$, Rubinstein originally expressed result (1) as $S = \frac{1}{1+i} E[(S + Y)\ \bar{Z}(Y)]$, and we further note that the consumption-related operator $\bar{Z}(y)$ has the properties $\bar{Z}(y) > 0$ and $E[\bar{Z}(y)] = 1$. Furthermore, (vi) implies that $\mathrm{cov}[U'(c), h(Y)] \le 0$ for all non-decreasing functions $h(\cdot)$ so that $V_0[h(Y)] \le \frac{1}{1+i} E[h(Y)]$; thus, also, all contingent claims with terminal wealth $h(Y)$ are assets with non-negative consumption betas. Correlating consumption betas with market betas (see Breeden [4]), (vi) excludes from consideration "negative beta" securities, whose current values exceed their expected returns.[3] The practical consequences of this exclusion are minimal, since it is known (see [2]) that such securities are rarely, if ever, encountered in practice. Assumption

[2] The same assumptions were also used by Brennan [5].

[3] It is possible, however, to derive bounds on options for such securities by changing assumption (vi) to make $\bar{Z}(y)$ a *non-decreasing* function of y.

(vi), however, is *more restrictive* than non-negativity of the consumption beta of the underlying asset; in particular, it implies that all call options on the security also have non-negative consumption betas.[4] On the other hand, (vi) permits any form of distribution for Y.

In deriving option bounds, we shall compose portfolios and establish the relationship between their initial and end-of-period values. This process is analogous to, but significantly different from, the hedging created in other analyses. We assume that an investor has wealth S to invest initially with the choice of portfolios: (A) one share of stock at price S, (B) one call option on the stock at price[5] C and $S - C$ invested in the riskless asset, and (C) S/C call options on the stock[6] (at price C). The terminal values of the three portfolios are, respectively, (A) $S + Y$, (B) $h_1(Y) \cong (S - C)(1 + i) + \max\{0, S + Y - X\}$, and (C) $h_2(Y) \cong S/C \max\{0, S + Y - X\}$. These three values are depicted as functions of $S + Y$ in Figure 1. Using these portfolios we can establish the following proposition for the bounds on the call price.[7]

PROPOSITION: *Under assumptions* (i)–(vi), *the single-period call option price lies in equilibrium within the bounds*:

$$\max\left\{0,\, S + \frac{1}{1+i}\left[-X + \int_0^X F(w - S)\, dw\right]\right\}$$

$$\le C(S, X, i) \le S + \frac{S}{S + E[Y]}\left[-X + \int_0^X F(w - S)\, dw\right]. \quad (2)$$

Proof: To establish the lower bound we compare portfolios (A) and (B). By the single-price law, $V_0[S + Y] = V_0[h_1(Y)]$, which by (1) implies

$$\int_{-S}^{\infty} (S + y)\bar{Z}(y)\, dF(y) = \int_{-S}^{\infty} h_1(y)\bar{Z}(y)\, dF(y).$$

Referring to Figure 1, we reexpress this as:

$$\int_{-S}^{y_1} [h_1(y) - (S + y)]\bar{Z}(y)\, dF(y) - \int_{y_1}^{\infty} [(S + y) - h_1(y)]\bar{Z}(y)\, dF(y) = 0.$$

Since $\bar{Z}(y)$ is non-increasing, the first integral decreases and the second increases if $\bar{Z}(y)$ is replaced by $\bar{Z}(y_1)$; substituting, rearranging, and simplifying, we have $E[S + Y] \ge E[h_1(Y)]$ or

$$S + E[Y] \ge (S - C)(1 + i) + \int_{X-S}^{\infty} (S + y - X)\, dF(y).$$

[4] However, for the special distributions used in [5], [6], and [11], it can be shown that (vi) is equivalent to the non-negativity of the consumption beta of the underlying security.

[5] Hereafter, we neglect the arguments of C.

[6] Earlier versions of this paper had derived a different bound by using instead of (C) a portfolio including a single short put, and then involving the put-call parity result of [7]. Portfolio (C), however, which was first suggested by H. Levy, results in a tighter bound and avoids the use of put-call parity.

[7] The proposition also follows using arbitrage considerations from assumptions (i), (ii), and (vi) alone.

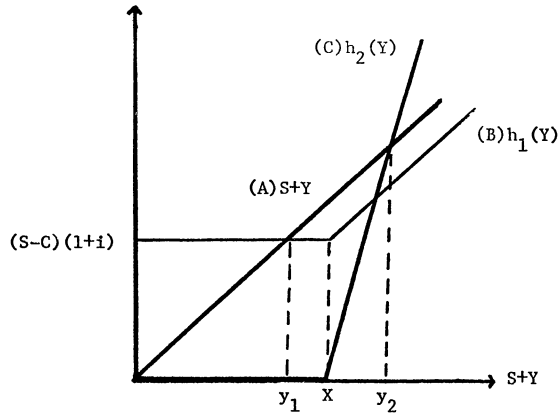

Figure 1. Terminal Values of Portfolios (A), (B), and (C) for Single-Period Analysis

Replacing with $w \cong S + y$, simplifying, and solving for C, we have the lower bound in (2).

Proceeding analogously for the upper bound, we use portfolios (A) and (C). $V_0[S + Y] = V_0[h_2(y)]$ leads to:

$$\int_{-S}^{y_2} [(S + y) - h_2(y)]\bar{Z}(y)\, dF(y) - \int_{y_2}^{\infty} [h_2(y) - (S + y)]\bar{Z}(y)\, dF(y) = 0.$$

Replacing $\bar{Z}(y)$ by $\bar{Z}(y_2)$ decreases the first integral and increases the second, so that $E[S + Y] \leq E[h_2(Y)]$; finally we have:

$$S + E[Y] \leq S/C \int_{X-S}^{\infty} (S + y - X)\, dF(y),$$

which yields the upper bound in (2). Q.E.D.

If we make the added assumption that the return on the stock is a random variable distributed independently of the current price, the properties of the bounds can be interpreted more easily.[8] Let the wealth relative be V on $[0, \infty)$ with distribution $G(v)$ and expectation $\bar{V}$; then the end-of-period stock price $S + Y$ becomes SV. Expression (2) is then

$$\max\left\{0, S\left[1 - \frac{1}{1+i}\left(\frac{X}{S} - \int_0^{X/S} G(v)\, dv\right)\right]\right\} \cong \max\{0, J(S)\}$$
$$\leq C(S, X, i) \leq S\left[1 - \frac{1}{\bar{V}}\left(\frac{X}{S} - \int_0^{X/S} G(v)\, dv\right)\right]. \quad (2a)$$

From this expression, it is immediate that the properties of option prices noted by Merton [7] are satisfied by both bounds. In particular, the bounds are linear homogeneous in (S, X), non-decreasing and convex in S, and non-increasing and convex in X.

[8] This is a common assumption; for instance see Merton [7].

We have explicitly noted that the lower bound is at least zero. This is necessitated by the fact that the derived expression can be negative[9] for small S, since $J'(S) = 1 - \frac{1}{1+i} \int_0^{X/S} [G(X/S) - G(v)]\, dv$ and $\lim_{S\to 0} J'(S) = 1 - \frac{\bar{V}}{1+i} < 0$; note also that $\lim_{S\to\infty} J'(S) = 1$. Then there exists a value S_1, the solution of $J(S) = 0$, where $1 + i = \int_0^{X/S_1} [1 - G(v)]\, dv$. Figure 2 depicts the upper and lower bounds as well as Merton's lower bound $S - \frac{X}{1+i}$. It can be easily demonstrated that the three bounds converge as S tends to ∞. Finally we note that the width of the interval between the bounds, ignoring the non-negativity is

$$S\left[\frac{1}{1+i} - \frac{1}{\bar{V}} \int_0^{X/S} [1 - G(v)]\, dv\right],$$

which can be reexpressed as

$$S\left[\frac{\bar{V} - (1+i)}{1+i}\right] \frac{1}{\bar{V}} \int_0^{X/S} [1 - G(v)]\, dv.$$

This is the discounted risk-premium on the stock, multiplied by a factor less than one. This factor goes to one as $S \to 0$ and to zero as $S \to \infty$.

II. Extensions and Conclusions

The derived bounds have been developed from consideration of the portfolios which are purchased at the beginning of a single period and held to the end. As such, the results might appear to rely on the prohibition of trading during the period. In fact, however, since Rubinstein's valuation formula (1) is valid for intermediate trading, the analysis permits the bounds to stand even with trades occurring during the period, and we can reproduce the result by progressively subdividing the trading interval. We shall now demonstrate that, because of the negativity of $J(S)$ on $[0, S_1]$, the bounds can be made effectively tighter through subdivision.

As a general procedure, the time to expiration is progressively subdivided into m equal length intervals for which the stock return per subinterval is $G(v)$, which by m-fold multiplicative convolution equals the distribution for the entire period (previously $G(v)$); the interest rate i is adjusted correspondingly. It is then assumed that the returns are independent and identically distributed random variables in each subinterval as in [6] and [11], with the expected return over the entire period being $\bar{V}^m$. We refer to the call price with n of the subintervals remaining as $C^n(S, X, i)$ and calculate C^n recursively from C^{n-1}. For illustration, we limit our demonstration to $m = 2$, and we call each subinterval a trading period.

At the beginning of each trading period, the same portfolios, (A) stock and (B) option with riskless asset, are available, where, in (B), the option considered

[9] The negative portion of the lower bound can be rationalized by noting that if S is sufficiently small then the expected pay-off on portfolio (B) is less than that on (A) unless the investor is actually paid to buy the call (so that he can invest more than S in the riskless asset).

expires at the end of the last period. Initially ($m = n = 2$) (A) has value S which grows to Sv at $n = 1$ while (B) becomes $(S - C^2)(1 + i) + C^1(Sv, X, i)$. $\bar{Z}(v)$ is the normalized conditional expected marginal utility for consumption at the end of the first period and is non-increasing in v by (vi). Figure 3, depicting the values of (A) and (B) as functions of Sv at $n = 1$, reveals that the proof of the proposition (for $m = 1$) follows by analogy. The equality of the current values of the two portfolios and (vi) require that the expected value of (A) must exceed that of (B). Therefore,

$$\begin{aligned} S\bar{V} &\geq (S - C^2)(1 + i) + E[C^1(SV, X, i)] \\ &\geq (S - C^2)(1 + i) + E[\max\{0, J(SV)\}] \end{aligned}$$

and

$$\max\left\{0, S\left(\frac{\bar{V}}{1 + i} - 1\right) + \frac{1}{1 + i}\int_{S_1/S}^{\infty} J(Sv)\, dG(v)\right\}$$

$$\cong \max\{0, J^2(S)\} \leq C^2(S, X, i) \qquad (3)$$

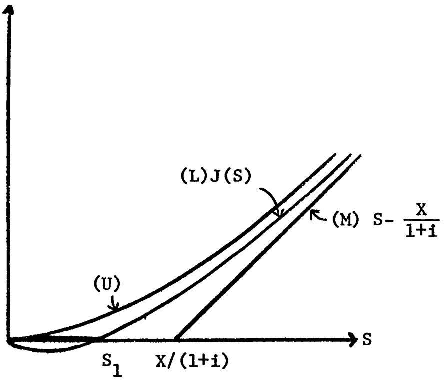

Figure 2. Upper (U), Lower (L), and Merton's (M) Bounds for the Call Price

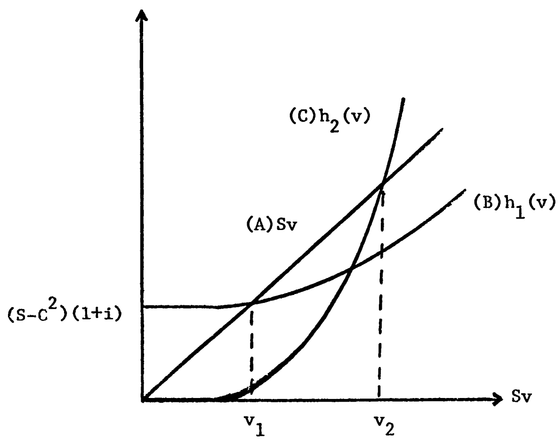

Figure 3. Terminal Values of Portfolios (A), (B), and (C) for Multiperiod Analysis

Since the integrand in Expression (3) is everywhere positive while $J(Sv) < 0$ on $[0, S_1)$, the bound will be lowered if S_1/S is replaced by 0 as the limit of integration. Substitution for $J(Sv)$ will yield a result identical to our single-period bound, as claimed, provided the probability measure on $[0, S_1)$ is non-zero; using S_1/S as the limit provides a tighter bound. Continuing to an n-fold subdivision yields a progressively tighter bound as each C^n is derived from a C^{n-1} bound on the reduced interval $[S^{n-1}/S, \infty)$.

The upper bound derivation for n period is similar but simplified since there is no corresponding S^1 to complicate the bound. Thus, while the lower bound is raised by subdivision, the upper bound is unaffected.

Put bounds can also be derived directly from the call bounds using the put-call parity relation $P = C - S + X/(1 + i)$. Alternatively, portfolios of puts can be created and valued analogously to the analysis of (A), (B), and (C). In this case puts, which are non-increasing functions of the stock price, must be held short.

In contrast to the Black-Scholes result which requires only the estimate of the stock price variance, our bounds require the entire distribution. Other generalizations of Black-Scholes, such as [9], suffer from similar amplifications of data requirements. If a simple distribution, such as a uniform or trinomial, is used for each subperiod, then two estimated parameters are sufficient to compute our bounds in the multiperiod case. These bounds may then be adjusted for the inclusion of fixed dividends and trading commissions.

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