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Tax Effects in Term Structure Estimation

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ABSTRACT

This study is a refinement and an extension of an earlier study by McCulloch of tax effects in the regression equation for term structure estimation. This study includes tests for tax effects and heteroskedasticity, a reconsideration of the need for an instrumental variable, and a search for the capital gains tax rate in addition to the ordinary-income tax rate. There are two major findings: (1) statistically significant tax-induced bias in the non-tax-adjusted equation and (2) evidence that the capital gains tax is misspecified in the tax-adjusted equation.

IT IS WELL-KNOWN (e.g., see [4], [2; pp. 727–30], and [25]) that the yield curve is an imperfect surrogate for the term structure of interest rates, and various methods have been developed to estimate the term structure "directly" by regressing prices on payments in a sample of default-free securities.[1] Since investors are concerned with the after-tax payments on securities, it is desirable to incorporate tax effects in term structure estimation. McCulloch [22] and Schaefer [26, 29] have employed different approaches to the tax-effect problem. McCulloch employed the regression approach and searched for a single tax rate, T, that "... best explains the whole structure of prices ...,"[2] i.e., that minimized the standard error of the regression (SER). Schaefer used linear programming to find term structures specific to various tax clienteles.

The linear programming (LP), or tax clientele, approach is consistent with a formal model of equilibrium in the default-free bond market. Schaefer [26, 27] demonstrated that if there are at least two tax brackets in a market in which spanning is assumed, then there is no equilibrium unless there are other frictions. A prohibition on short sales is a tractable friction which may also reflect actual conditions such as the asymmetric tax treatment of long and short positions.

In the resulting equilibrium, investors will hold those bonds which have net present values equal to zero. Investors' portfolio decisions are modeled as a linear program in which the cost of acquiring a portfolio is minimized subject to the constraints that after-tax cash flows attain a desired level and that holdings of all bonds are non-negative. The dual of this problem maximizes the present value of target cash flows subject to the constraints that present values of bonds are less than or equal to prices and that discount factors are non-negative and

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[1] Several methods of term-structure estimation have been advanced including four different continuous-function methods ([6], [20], [22], [32]) and one discrete discount factor method [5].

[2] [22, p. 825].

monotonic non-increasing. For a chosen tax rate, the dual solution is the term structure implicit in the bonds held by investors in that tax bracket.

An empirical weakness of the LP methodology is that, by design, no negative "residuals" (net present values) are permitted. The result is a tendency to overstate the clientele effect and to bias downward the estimated discount function.[3] Thus, on empirical grounds, a regression approach would seem to be preferred in order to avoid this bias. Unfortunately, the single-tax-rate regression equation may be theoretically deficient. If equilibrium is a restricted, tax-clientele equilibrium, it is not clear what the estimated tax rate and the estimated discount function represent. Even if interpreted as some kind of complex averages of the clientele tax rates and discount functions, their significance for pricing any particular bond or for calculating meaningful spot or forward rates is unclear.

In spite of this serious caveat, the regression approach to term structure estimation is regularly used (e.g., see [3], [15], and [32]), and the tax specification used by McCulloch is common in the literature (e.g., see [16], [17], [18], [23], and [31]). The first purpose of this paper is to reexamine the consequences of omitting taxes from the regression equation. Previously used tax-effects tests, here more rigorously interpreted, detect statistically significant tax bias. The second purpose is to reexamine the tax specification. The need for an instrumental variable is reevaluated, and the capital gains tax rate is not restricted to be one-half the ordinary tax rate. Evidence is found of misspecification of the equation due to the treatment of the capital gains tax.

In the next two sections, term structure estimation with and without taxes is described, and the differences between this study and McCulloch's are made clear. Section III contains a description of the data and the empirical results. Section IV is a summary.

I. Estimating the Term Structure

Define R_t as the before-tax opportunity rate applied by the market to value a default-free payment occurring at time t. The before-tax discount factor for time t is then $d_t = (1 + R_t)^{-t}$. For a sample of N default-free, fixed payment securities, the non-tax-adjusted equation for estimation of the discount factors is

$$P_j + A_j = \sum_{t=1}^{M_j} d_t X_{j,t} + \epsilon_j, \qquad j = 1, 2, \cdots, N, \tag{1a}$$

where P_j is the quoted price, A_j is the accrued coupon, M_j is the number of payment periods until maturity, ϵ_j is the disturbance, and $X_{j,t}$ is the payment each period. For a bill, the payments (in dollars per \$100 of principal) are defined by

$$X_{j,t} = 100 \qquad \text{for} \quad t = M_j. \tag{1b}$$

[3] Consider two identical bonds whose present values are determined by a true discount function for a particular tax bracket. If, purely randomly, one bond is overpriced and the other underpriced, then the overpriced bond will not appear in the primal solution. The estimated discount function will be based on the lower priced bond and thus biased downward. The tendency of the approach to fit to extremes of the data was noted in [26, p. 429].

For a bond, the payments are defined by

$$X_{j,t} = C_{j,t} \qquad \text{for} \quad t = 1, 2, \cdots, M_j - 1;$$

$$X_{j,t} = 100 + C_{j,t} \qquad \text{for} \quad t = M_j; \tag{1c}$$

where $C_{j,t}$ is the portion of the annual coupon paid each period.

The discount factors may be estimated, in principle, by ordinary least squares (OLS) applied to (1a); this is the method of Carleton and Cooper [5]. However, severe multicollinearity arises unless the sample provides at least one maturity payment (principal) at each payment date,[4] and, in practice, this prevents accurate estimation of discrete discount factors for the U.S. Government Bond market beyond about five years maturity. Furthermore, if the term structure is viewed as a continuum, and if there is a need for rates applicable to payment dates other than those in the sample, then it is desirable to estimate a continuous discount function.

Continuous approximations of the discount function have the general form

$$d_t = \textstyle\sum_{k=0}^{L} b_k f_k(t), \tag{2}$$

where the b_k are coefficients to be estimated and the $f_k(t)$ are chosen functions. To ensure $d_0 = 1$, the form

$$d_t = 1 + \textstyle\sum_{k=1}^{L} b_k f_k(t), \tag{3}$$

where $f_k(0) = 0$, $k = 1, 2, \cdots, L$, is used.[5]

Substitution of (3) into (1a) results in the non-tax-adjusted regression equation for continuous approximation of the discount function,

$$P_j + A_j - \textstyle\sum_{t=1}^{M_j} X_{j,t} = \sum_{k=1}^{L} b_k \sum_{t=1}^{M_j} X_{j,t} f_k(t) + \epsilon_j, \qquad j = 1, 2, \cdots, N. \tag{4}$$

In this paper, the choice of L, the number of functions, and $f_k(t)$, the functional form, follows McCulloch [22]. That is, L is the square root of N and $f_k(t)$ is a cubic spline function.[6]

II. Tax-Effect Specification

A tax-adjusted specification may be represented by

$$P_j + A_j = \textstyle\sum_{t=1}^{M_j} d_t^T X_{j,t}^T + \epsilon_j^T, \qquad j = 1, 2, \cdots, N, \tag{5a}$$

where the "T" superscript denotes after-tax quantities.

[4] See [5] and [12, pp. 8–10] for amplification. It is possible to minimize the squared errors by mathematical programming and avoid these difficulties with the closed-form solution.

[5] In his regression version, Schaefer [24] applied inequality constraints and quadratic programming to ensure non-negative forward rates or monotonically declining d_t. The necessity of these is apparently related to the high order polynomial employed by Schaefer, because in unconstrained OLS estimations (see [12]), only Schaefer's method had a tendency to produce negative forward rates. Schaefer's polynomials have desirable properties only when weights are restricted to be non-negative.

[6] An n-degree spline consists of n-degree polynomials joined at "knot" points in such a manner that the spline is continuous with $n - 1$ continuous derivatives. In [20], a quadratic spline was used, but this caused scalloped forward-rate curves due to the discontinuous second derivative. Even so, quadratic spline forward rates were used to estimate liquidity premia in [21]. See [12], [15], [30], and [32] for other characteristics and criticisms of the cubic spline.

In approximate accordance with U.S. tax laws, the conventional tax-adjustment equation taxes bill discount at rate T, deflates coupons by $(1 - T)$, amortizes bond premia to maturity or first call date at T, and taxes bond discount at a capital gains rate of δT at maturity. Typically, δ is assigned a value of ½.

Thus the tax-adjusted payments are different for bills, discount bonds, and premium bonds. For a bill,

$$X_{j,t}^T = 100 - T(100 - P_j) \qquad \text{for} \quad t = M_j. \tag{5b}$$

For a discount bond,

$$\begin{aligned} X_{j,t}^T &= C_j - T(C_j - A_j) && \text{for} \quad t = 1; \\ X_{j,t}^T &= C_j(1 - T) && \text{for} \quad t = 2, \cdots, M_j - 1; \\ X_{j,t}^T &= 100 - \delta T(100 - P_j) + C_j(1 - T) && \text{for} \quad t = M_j. \end{aligned} \tag{5c}$$

For a premium bond,

$$\begin{aligned} X_{j,t}^T &= C_j - T(C_j - A_j) + T\,\frac{P_j - 100}{M_j} && \text{for} \quad t = 1; \\ X_{j,t}^T &= C_j(1 - T) + \frac{T(P_j - 100)}{M_j} && \text{for} \quad t = 2, \cdots, M_j^c - 1; \\ X_{j,t}^T &= 100 + C_j(1 - T) + T\,\frac{(P_j - 100)}{M_j} \\ &\quad + T(P_j - 100)\,\frac{(M_j - M_j^c)}{M_j} && \text{for} \quad t = M_j^c; \end{aligned} \tag{5d}$$

where M_j^c is the earliest call date. This premium bond specification ignores any option value associated with the call feature and assumes that a callable premium bond will be called at the earliest call date. For a premium, non-callable bond, set $M_j^c = M_j$ in (5d). The continuous approximation, tax-adjusted regression specification is then

$$P_j + A_j - \textstyle\sum_{t=1}^{M_j} X_{j,t}^T = \sum_{k=1}^{L} b_k^T \sum_{t=1}^{M_j} X_{j,t}^T f_k(t) + \epsilon_j^T, \qquad j = 1, 2, \cdots, N. \tag{6}$$

At this point, some refinements in this study that distinguish it from McCulloch's [22] may be made clear.

(1) Coupon accrual is not omitted. McCulloch assumed continuous coupon payments and set $A_j = 0$.

(2) Tests for heteroskedasticity are performed. McCulloch divided each observation by one-half the bid-ask spread based on assumed heteroskedasticity due to the use of the mean of bid and ask for P_j.

(3) The non-tax-adjusted Equation (4) is tested for tax effects through residual analysis. McCulloch reported no tax-effects tests.

(4) No instrumental variable is used in the tax-adjusted specification. Because price appears in both the dependent and independent variables in (5a), the assumption of the independence of regressors and disturbances necessary for

unbiasedness and consistency in OLS is violated. McCulloch used principal value, 100, as an instrument for P_j in the regressors. This instrument removes most of the tax effect that the specification is designed to capture.

(5) The capital gains tax parameter, δ, is treated as a parameter to be estimated rather than assumed equal to ½. The assumption, $\delta = ½$, is common but perhaps unduly restrictive particularly due to the heavy participation of commercial banks and securities dealers which pay ordinary taxes on gains ($\delta = 1.0$).[7]

III. Data and Results

Data on U.S. Treasury bills, notes, and bonds were taken from the CRSP Government Bond Tape, which was available through 1979. Forty samples were collected consisting of cross-sectional observations for the last trading day of the first month in each quarter (January, April, July, and October) from 1970 through 1979.

Some results from [12] are reported here. Data in that study consisted of U.S. Treasury note and bond data from the *Wall Street Journal* and the *U.S. Treasury Bulletin.*

Treasury bonds used in the payment of estate taxes (flower bonds) were omitted from all samples.

A. Tests for Heteroskedasticity

To test for heteroskedasticity in the non-tax-adjusted regression (4), the absolute value of the residual was regressed on bid-ask spread and the related variables maturity, duration, and issue size. This test, developed by Glejser [9], has the form

$$|\grave{\epsilon}_j| = \alpha + \beta(V_j) + u_j,$$

where $\grave{e}_j$ is the residual, and V_j is a variable suspected of contributing to heteroskedasticity. If β is significantly different from zero, then the hypothesis of homoskedasticity is rejected.[8]

Table I shows the results of the Glejser test before and after division by spread—McCulloch's correction,[9,10] Before the correction, evidence of hetero-

[7] This information may be found in the tax code and related literature. For example, see [8, p. 1634 and p. 3419] and [1]. See [7] and [10] for more complete treatments of the effect of capital gains taxation on security prices. For an alternative empirical treatment, namely, $T_g = 0$, see [19].

[8] Note that the distribution of u_j is unlikely to be normal. Nevertheless, a t test is asymptotically valid. See [13, pp. 146–7 and pp. 298–9] for a discussion. Another test for heteroskedasticity, rank correlation of $|\grave{\epsilon}_j|$ and V_j as suggested in [11, p. 219], was found to reinforce the result of the Glejser test in a few samples and to fail to detect heteroskedasticity in many samples where the parametric test detected it.

[9] In all cases, the relationship between $|\grave{\epsilon}|$ and the suspected variables was positive. Thus, the t-statistics for the Glejser test parameters are inflated.

[10] The SERs in these corrected regressions are, in some cases, many times smaller than in [22] (e.g., 11.51 in 1965 is reported on p. 827). For a more direct comparison, in [22, p. 827] an SER value of 3.31 was reported for July 31, 1973. In this study, the SER for July 1973 of 0.9634 is approximately three times smaller; two times smaller would be appropriate because McCulloch divided by one-half the spread. Possible causes include neglect of coupon accrual and the inclusion of some flower bonds [22, p. 817] in McCulloch's samples.

Table I
Heteroskedasticity Tests

No Heteroskedasticity Correction		
Average SER	0.1296	
Number of Tests Significant at level	0.05	0.01
Regression: $\lvert\hat{\epsilon}\rvert$ on Spread	39	39
$\lvert\hat{\epsilon}\rvert$ on Maturity	38	37
$\lvert\hat{\epsilon}\rvert$ on Duration	39	38
$\lvert\hat{\epsilon}\rvert$ on Issue Size	18	8
After Division by Spread		
Average SER	0.6743	
Number of Tests Significant at level	0.05	0.01
Regression: $\lvert\hat{\epsilon}\rvert$ on Spread	14	9
$\lvert\hat{\epsilon}\rvert$ on Maturity	15	6
$\lvert\hat{\epsilon}\rvert$ on Duration	12	7
$\lvert\hat{\epsilon}\rvert$ on Issue Size	13	4

Number of samples = 40.

skedasticity with respect to spread, maturity, and duration was pervasive. After division by spread, the number of samples significantly heteroskedastic was reduced by approximately 60%. There remained more samples (12 to 15 out of 40) indicating heteroskedasticity than would be expected by chance at the 5% level. It is possible that further improvement could be achieved by non-linear methods (e.g., see [6] and [14, pp. 416–17]), but the use of such methods would introduce a third parameter, in addition to T and δ to be determined in the tax-adjusted specification. In view of the large reduction in heteroskedasticity achieved by the simple correction, it was decided to forego the additional complexity. All further empirical results were obtained from Equations (4) or (6) after division by spread.

B. Tests for Tax Effects

To test for tax effects in the non-tax-adjusted regression (4), the residual, $\hat{\epsilon}_j$, was regressed on coupon and on price in each of the 40 samples. Similar tests have been used in the term-structure-estimation literature.[11] An intuitive argument for the tests is that the omission of taxes overstates the regressors and results in overestimated prices or negative residuals. The effect would increase with coupon and price.

However, the results provided little support for this argument. In the residual-coupon regressions, there were 21 positive coefficients and 19 negative. At the 5% level of significance, 7 of the positive coefficients and 3 of the negative coefficients were significant. In the residual-price regressions, there were 18 positive coefficients and 22 negative. At the 5% level, 6 of the positive coefficients and 7 of the negative coefficients were significant. Thus, a greater number of

[11] Schaefer [24] hypothesized a negative residual-coupon relationship and found this graphically in one sample. Carleton and Cooper [5] regressed residual on discount from par and found significance (at 5%) in one of four samples. The sign was neither hypothesized nor reported.

significant coefficients were found than would be expected if the residuals were unbiased, but the sign of the coefficients exhibited no pattern.

In an additional test for a sign tendency in the residuals, the coefficients in the residual-coupon ($\hat{\beta}_c$) and residual-price ($\hat{\beta}_p$) regressions were correlated with the following sample characteristics: the percentage of discount bonds in the sample, $\%D$, defined as

$$\%D = \frac{\#\text{ discount bonds}}{\#\text{ all bonds and bills}} \times 100,$$

mean coupon, mean price, and mean yield. The correlations were computed for all 40 samples and for the 14 samples for which either $\hat{\beta}_c$ or $\hat{\beta}_p$ was significant at the 5% level. Only one correlation was significant at the 5% level. $\hat{\beta}_p$ was negatively correlated with mean coupon across all 40 samples, a finding consistent with the intuitive argument. A statistically less significant negative correlation (the significance level was 6.4%) existed between $\hat{\beta}_c$ and mean coupon in the 14 samples, another consistency with the intuitive argument.

A final sign test was applied. The residual signs were classified by premium and discount bonds and, separately, by bills and bonds. The χ^2 two-way contingency table tests were significant at the 5% level in only two of the 40 samples in each case, the number that would be expected if the sign differences in the residuals were unrelated to security type.

This empirical evidence of statistically significant but ambiguously signed bias in the residuals indicated the need for further analysis of the effects of the omission of taxes.

There are two tax-related explanations for these results. First, if the McCulloch, tax-adjusted regression (5a) is the correct specification, but OLS is applied to the non-tax-adjusted regression (1a), then it can be shown that the expected value of the residual is given by

$$E(\hat{\epsilon}_j) = (\underline{X}_j^T - \underline{X}_j\underline{K})\,\underline{d}^T, \tag{7}$$

where $\underline{X}_j^T$ and $\underline{X}_j$ are the $1 \times M_N$ vectors of after- and before-tax payments on security j, $\underline{K}$ is the $M_N \times M_N$ vector equal to $(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{X}^T$, $\underline{X}^T$ and $\underline{X}$ are the $N \times M_N$ matrices of after- and before-tax payments on the N securities in the sample, and $\underline{d}^T$ is the $M_N \times 1$ vector of after-tax discount factors.[12]

The components of the coefficient of $\underline{d}^T$ in (7) are coupons, prices, and principle payments. Thus, unless the coefficient is equal to zero, the residual is expected to be related to price and coupon. But the sign of the relationship cannot be specified *a priori*. For example, for bills, discount bonds, and some premium bonds (those for which $C_j > (P_j - 100)/M_j^c$) every element of $\underline{X}_j^T$ is less than every element of $\underline{X}_j$; thus, if the elements of $\underline{K}$ were all greater than or equal to one, $E(\hat{\epsilon}_j)$ would be negative. However, every element of $\underline{X}^T$ is less than every element of $\underline{X}$, so every element of $\underline{K}$ is less than one, and the sign of $E(\hat{\epsilon}_j)$ is

[12] In this vector notation, the superscripts " ′ " and " -1 " denote transpose and inverse, respectively. To derive (7), define $\hat{\epsilon}_j = P_j + A_j - \underline{X}_j\hat{\underline{d}}$ where $\hat{\underline{d}} = (\underline{X}'\underline{X})^{-1}\underline{X}'(\underline{P} + \underline{A})$ and $\underline{P} + \underline{A}$ are the $N \times 1$ vectors of quoted prices and accruals. Substitute the assumed correct specification (5a) for $P_j + A_j$ and $\underline{P} + \underline{A}$ and take expectations.

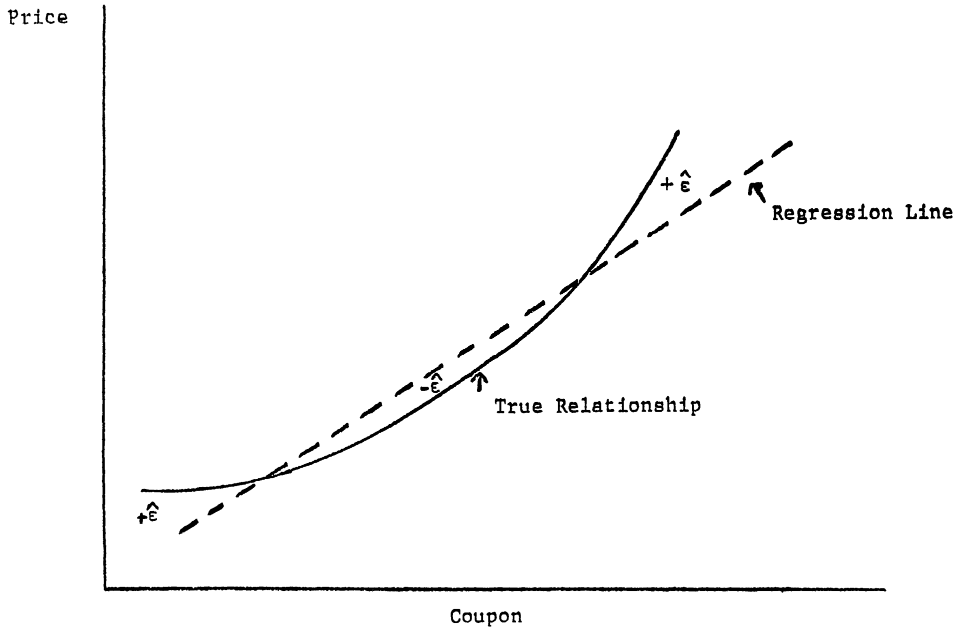

Figure 1. Linear Regression Through Convex Data

ambiguous.[13] Thus, regressions of $\grave{\epsilon}_j$ on coupon and price can be positively, negatively, or not significant in various samples.

The second explanation of the empirical results relies on Schaefer's [28] argument, disputed by Livingston [18], that the price-coupon relationship may be convex in a world of tax clienteles with short-sale restrictions. A linear regression of price on coupon, such as (1a) and (4), through convex price-coupon data could result in both positive and negative residuals as illustrated in Figure 1.

Thus, these tests appear to confirm the existence of tax effects but do not distinguish between the McCulloch-Livingston and the Schaefer tax-effect models.

C. Tax-Rate Search

The tax-adjusted regression, as described in Equation (6), was performed iteratively as T was varied from 0.0 to 1.0 in increments of 0.1. For each value of $T > 0.0$, δ was varied from 0.0 to 1.0 in increments of 0.01.

The need for an instrumental variable was explored in [12], and these results are reported here. In some samples, the tax search was conducted using no instrument (that is, keeping P in the regressors), McCulloch's instrument (100), price on the trading day prior to the sample date (P_{-1}), and price five trading days prior to the sample date (P_{-5}). An instrumental variable should be correlated

[13] Only premium bonds with unusually large amortization tax shields, such that $C_j < (P_j - 100)/M_j^c$, would have after-tax payments greater than before tax payments. The presence of such bonds in a sample complicates the analysis but the sign of $E(\grave{\epsilon}_j)$ remains unpredictable.

with P and uncorrelated with ϵ. In practice, these two requirements are very difficult to satisfy simultaneously. The use of P_{-1} and P_{-5} was designed to detect sensitivity to the choice of an instrument, even through neither is likely to be uncorrelated to ϵ. Table II shows the unrestricted SER-minimizing rates (T^{**}, δ^{**}) and the restricted SER-minimizing rate (T^*, $\delta = 0.5$) in four samples when no instrument was used versus when the previous day's price, P_{-1}, and 100 were used. In three of these samples the use of 100 reduced the values of T^{**}, δ^{**}, and T^*. In order to observe the effect of an instrument less correlated with P_j, the price five days prior, P_{-5}, was used. The results are not shown here, because estimated tax rates were identical and standard errors were negligibly different when no instrument, P_{-1}, and P_{-5} were used. Since 100 is clearly an unsatisfactory instrument, and little difference was observed when P, P_{-1}, and P_{-5} were used, no instrument was used in this study.

The behavior of SER as T and δ were varied was quite reasonable in some samples. Table III shows an example of this for October 1975. SER decreased with T to 0.10 and decreased with δ to a minimum for each value of T. However, in many samples, δ drove to 1.0 for every value of T. While some participants are subject to $\delta = 1$, it seems unlikely that this value pervades the market. An analysis of the estimated tax parameters follows.

Table II

Effect of Instrumental Variables, 100 and P_{-1}

Date	Instrument	$T = 0$ SER	T^{**}, δ^{**}, SER	T^*, SER
April 1, 1974	None	2.1880	0.7, 1.0, 1.0384	0.2, 1.9078
	P_{-1}	2.1880	0.7, 1.0, 1.0377	0.2, 1.9447
	100	2.1880	0.1, 0.3, 2.0073	0.1, 2.0219
January 2, 1976	None	1.9391	0.2, 0.1, 1.1572	0.2, 1.5264
	P_{-1}	1.9391	0.2, 0.1, 1.4543	0.2, 1.5204
	100	1.9391	0.2, 0.0, 1.5807	0.2, 1.7596
October 5, 1977	None	1.6977	0.3, 0.7, 1.4615	0.2, 1.4684
	P_{-1}	1.6977	0.3, 0.7, 1.4572	0.2, 1.4652
	100	1.6977	0.0, 0.0, 1.6977	0.0, 1.6977
June 1, 1978	None	1.5520	0.2, 1.0, 1.3771	0.1, 1.5105
	P_{-1}	1.5520	0.2, 1.0, 1.3782	0.1, 1.5109
	100	1.5520	0.0, 0.0, 1.5520	0.0, 1.5520

P_{-1} = price one day prior to observation date. T^{**}, δ^{**} = ordinary and capital gains tax rates that minimize the standard error of the regression (SER). T^* = ordinary tax rate that minimizes SER with $\delta = 0.5$.

Source: Jordan [12].

Table III

Behavior of SER with T, δ for October 1975

	δ					
T	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.7419	—	—	—	—	—
0.1	0.7023	0.7010	0.7016	0.7039	0.7080	0.7139
0.2	0.7351	0.7158	0.7035	0.6986	0.7012	0.7114

Note: Values of $T > 0.20$ are not shown because behavior is similar.

Table IV shows estimated tax rates and sample characteristics for the 40 samples. Values of T^{**} and δ^{**} are quite variable across the samples, as shown in the second column of Table IV. The range of T^{**} is 0.0 to 1.0 with a mean of 0.45. The most frequently occurring value (10 cases) is 1.0 and the next most frequently occurring values (9 cases each) are 0.1 and 0.2. The range of δ^{**} is 0.0 to 1.0 with a mean of 0.72. The most frequently occurring value (17 cases) is 1.0 and the next most frequently occurring value (8 cases) is 0.0.

The third and fourth columns show values of the SER minimizing T when δ is constrained to 0.5 (T^*) and 0.0 (T^0). Values of T^* and T^0 are much less variable and, *a priori*, more reasonable.

The remaining columns of Table IV contain variables that appear to be related to the behavior of T^{**} and δ^{**}. These are SERB, the standard error of the regression before tax adjustment; $\bar{Y}$, the mean yield to maturity in the sample, an indication of the level of interest rates; $\bar{Y}_{10} - \bar{Y}_1$, the average 10-year yield to maturity minus the average 1-year yield to maturity, an indication of the slope of the yield curve; and $\%D$, the percentage of bonds in the sample priced below par.

The indication in Table IV is that the methodology of searching over both T and δ is unreliable. This may be due to the difficulty of uniquely determining both parameters. If the model fits well for values T' and δ' then it will also fit well for T'' and δ'' if $(1 - T')/(1 - \delta' T') = (1 - T'')/(1 - \delta'' T'')$. For example, in Table III the SER for $(T', \delta') = (0.10, 0.20)$ is nearly the same as for $(T'', \delta'') = (0.20, 0.80)$. The ratio, $(1 - T)/(1 - \delta T)$ is 0.918 and 0.952 for these two cases.

This problem would tend to produce a positive correlation between T^{**} and δ^{**} across samples. That is, if a particular pair of values, and thus a particular ratio, were approximately correct for all 40 samples, then higher values of δ^{**} would necessitate higher values of T^{**} to keep the ratio constant.[14] In fact, across the 40 samples, the correlation coefficient between T^{**} and δ^{**} is 0.65, significant at the 0.001 level. This effect may also account for the lower mean value of T^* and the still lower mean of T^0.

However, the identification problem does not fully explain the behavior of T^{**} and δ^{**}. There is also a systematic relationship between the parameters and sample characteristics.

The correlations of T^{**} and δ^{**} with various sample characteristics are shown in Table V. Clearly, T^{**} and δ^{**} are higher in samples characterized by higher mean yields ($\bar{Y}$), more negatively sloping yield curves ($\bar{Y}_{10} - \bar{Y}_1$), poorer fits of the non-tax-adjusted equation (SERB), and greater proportions of discount bonds ($\%D$). This set of correlations could be due to a clientele effect. If higher tax-bracket investors are attracted to discount bonds, then the estimated single tax rate may be higher when more discount bonds are available and more high-bracket investors are in the government bond market. Larger proportions of discount bonds are associated with higher mean yields, negatively sloping yield curves, and poorer fits of the non-tax-adjusted equation. But this explanation for the correlations does not explain the *a priori* unreasonably large values of T^{**} (say $T^{**} \geq 0.6$). These samples are marked by the symbol # in Table IV.

[14] The argument only holds for a ratio less than one, i.e., for $\delta < 1$.

Table IV

Estimated Tax Rates and Sample Characteristics

Date						SERB	$\bar{Y}$	$\bar{Y}_{10} - \bar{Y}_1$		
Year	Month	T^{**}	δ^{**}	T^* ($\delta = 0.5$)	T^0 ($\delta = 0$)	(\$/\$100)	(%)	(%)	%D	
1970	1	0.6	1.0	0.1	0.1	0.77	8.03	0.19	36	#
	4	0.1	0.9	0.1	0.0	0.55	7.25	0.54	30	
	7	0.5	1.0	0.0	0.0	0.54	6.80	0.89	25	
	10	0.4	1.0	0.1	0.0	0.72	6.24	0.99	24	
1971	1	0.2	0.9	0.2	0.1	0.82	4.47	1.37	6	
	4	0.2	0.8	0.1	0.1	0.65	4.59	1.84	8	
	7	0.2	0.8	0.2	0.1	0.76	5.87	1.22	21	
	10	0.3	0.6	0.3	0.3	0.79	4.66	1.68	3	
1972	1	0.1	0.8	0.1	0.1	0.48	4.05	1.99	2	
	4	0.2	0.0	0.2	0.2	0.65	4.37	1.86	2	
	7	0.1	0.0	0.1	0.1	0.56	4.55	1.56	5	
	10	0.2	0.4	0.2	0.1	0.53	5.34	0.93	20	
1973	1	0.1	0.8	0.1	0.1	0.49	5.98	0.43	27	
	4	0.4	1.0	0.2	0.1	0.62	6.56	0.34	32	
	7	1.0	1.0	0.1	0.1	0.96	8.41	−1.01	42	#
	10	1.0	1.0	0.4	0.2	0.78	7.25	−0.96	29	#
1974	1	0.6	0.9	0.1	0.0	0.33	7.41	−0.72	29	#
	4	1.0	1.0	0.1	0.0	0.52	8.83	−1.27	43	#
	7	1.0	1.0	0.1	0.0	0.67	8.31	0.02	41	#
	10	1.0	1.0	0.1	0.0	0.67	7.91	−0.24	33	#
1975	1	0.4	0.9	0.2	0.1	0.46	6.25	1.21	22	
	4	0.5	1.0	0.1	0.0	0.69	6.50	1.85	29	
	7	0.2	0.9	0.1	0.0	0.61	7.09	0.92	39	
	10	0.2	0.6	0.2	0.1	0.74	6.38	1.71	19	
1976	1	0.1	0.0	0.1	0.1	0.57	5.70	2.28	10	
	4	0.0	0.0	0.0	0.0	0.75	5.97	1.83	7	
	7	0.1	0.0	0.1	0.1	0.56	6.20	1.65	8	
	10	0.1	0.0	0.1	0.1	0.74	5.68	1.47	0	
1977	1	0.1	0.0	0.1	0.1	0.91	5.79	1.68	9	
	4	0.0	0.0	0.0	0.0	0.77	5.78	1.70	9	
	7	0.1	0.5	0.1	0.1	1.04	6.25	1.17	17	
	10	0.5	0.9	0.3	0.1	0.93	6.93	0.59	35	
1978	1	0.2	1.0	0.1	0.0	0.86	7.21	0.67	45	
	4	0.2	0.9	0.1	0.0	0.72	7.48	0.83	56	
	7	0.9	1.0	0.1	0.0	1.11	7.97	0.15	63	#
	10	1.0	1.0	0.1	0.1	1.23	9.18	−0.57	69	#
1979	1	1.0	1.0	0.1	0.0	1.03	9.57	−1.05	66	#
	4	1.0	1.0	0.1	0.0	1.09	9.75	−0.80	70	#
	7	1.0	1.0	0.1	0.0	0.76	9.36	−1.02	61	#
	10	1.0	1.0	0.1	0.0	1.14	12.00	−2.18	70	#
Mean		0.45	0.72	0.15	0.07	0.74	6.85	0.65	29.1	
Std. Dev.		0.37	0.39	0.16	0.07	0.21	1.71	1.12	20.8	

Defintions:

SERB = standard error of regression (SER) before tax adjustment
T^{**}, δ^{**} = SER-minimizing ordinary (T) and capital gains (δ) tax rates
T^* ($\delta = 0.5$) = SER-minimizing T when δ is set at 0.5
T^0 ($\delta = 0.0$) = SER-minimizing T when δ is set at 0.0
$\bar{Y}$ = mean yield in the sample
$\bar{Y}_{10} - \bar{Y}_1$ = mean 10-year yield minus mean 1-year yield, an indication of yield curve slope
%D = proportion of discount bonds in sample
= indicates samples for which $T^{**} \geq 0.6$

Table V

Pearson Correlations among Related Variables

	δ^{**}	SERB	$\bar{Y}$	$\bar{Y}_{10} - \bar{Y}_1$	$\%D$
T^{**}	0.65	0.42	0.82	−0.86	0.78
	(0.001)	(0.004)	(0.001)	(0.001)	(0.001)
δ^{**}		0.18	0.56	−0.63	0.87
		(0.141)	(0.001)	(0.001)	(0.001)

Significance levels in parentheses.
For definitions, see Table IV.

When the 13 samples for which $T^{**} \geq 0.6$ are considered separately from the other 27 samples, it is clear that the two groups are quite different with respect to sample characteristics. For example, the hypotheses that the mean values of T^{**}, δ^{**}, SERB, $\bar{Y}$, $\bar{Y}_{10} - \bar{Y}_1$, and $\%D$ are equal in the two groups are rejected at the 0.01 level by t tests. More importantly, the behaviors of T^{**} and δ^{**} are very different between the two groups.

In the 27 "reasonable-T^{**}" samples, δ^{**} is quite variable. The range is 0.0 to 0.9 with a mean of 0.58. The most frequently occurring values (six cases each) are 0.9 and 0.0. But, in these samples, T^{**} remains in the reasonable range even for large δ^{**}. The range of T^{**} is 0.0 to 0.5 with a mean of 0.21 and modes of 0.1 and 0.2. This behavior is consistent with the identification problem. That is, in samples of relatively few discount bonds, δ would be relatively unimportant and, given the identification problem, could take on almost any value.

But in the 13 "unreasonable-T^{**}" samples of relatively many discount bonds (roughly 30% or more) where δ should be an important parameter, δ always drives to 1.0 or 0.9 unless it is constrained. This evidence suggests misspecification of the price equation for discount bonds due to the treatment of capital gains. This is not particularly surprising given the *ad hoc* nature of the capital gains specification.

IV. Conclusion

The regression equation for term structure estimation is heteroskedastic with respect to bid-ask spread, maturity, and duration. Division by bid-ask spread substantially corrects the problem, although more refinement is possible.

The omission of taxes biases the equation. In regressions of residual on coupon and residual on price, tax-effect tests found in the literature, significant (at 5%) coefficients were found in more than one-fourth of the samples. Although the signs of the coefficients were not uniform, it was shown that these results are consistent with tax effects.

The iterative OLS search over parameters T and δ is hampered by the difficulty of simultaneously identifying both T and δ. However, T and δ were clearly most poorly determined in samples containing relatively large numbers of discount bonds, an indication that the capital gains tax effect is not properly captured in the equation.

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