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Negative Cash Flows, Duration, and Immunization: A Note

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THE BOND IMMUNIZATION LITERATURE has developed two well-known results regarding duration; namely, that duration is no greater than a bond's term to maturity and that duration determines the holding period for which portfolio value is immunized against particular types of interest rate changes. This note demonstrates that these results may not hold if portfolio assets generate both cash inflows and cash outflows. Cash outflows could occur if the setting is expanded to include taxes, short-selling, margin-buying, or financial futures contracts. In each of these cases, negative cash flows are directly associated with the asset portfolio rather than with an independent liability. A taxable zero coupon bond example serves to illustrate the effects of negative cash flows on duration and their consequences for immunization. In particular, it is shown that after-tax duration does not determine an immunized holding period for the investor in a single zero coupon bond but does determine an immunized holding (borrowing) period for the issuer.[1]

Duration can be defined as:

$$D(r) = \textstyle\sum_{t=1}^{n} tw_t(r) \tag{1}$$

where r is the yield to maturity and $w_t(r) = C_t e^{-rt}/P(r)$ is the contribution of each cash flow C_t to the price or present value $P(r)$ of a cash flow stream. Since $D(r) = -P'(r)/P(r)$, it is a measure of price changes resulting from parallel shifts in a flat yield curve. Hicks [6], Macaulay [9], Samuelson [11], and Redington [10] all recognized the relevance of duration in describing the sensitivity of asset and liability values to a general movement in interest rates. Redington coined the term "immunization" to signify that net worth (assumed zero initially) would not be adversely affected by interest rate changes and showed that an overall portfolio is locally immunized when 1) its assets and liabilities (fixed insurance or pension fund payments) have equal durations and 2) the spread or variance of inflow timing around asset duration is not less than the variance of outflow timing around liability duration. These conditions imply that for small changes in interest rates, net worth cannot decrease. This in turn implies that assets will be at least large enough to meet the scheduled liability outflows, i.e., that the liabilities are immunized.[2]

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[1] While they do not analyze the duration of the taxable zero coupon bond specifically, Fisher and Weil [3] note that provisions must be made for tax payments before the zero is "unconditionally attractive for immunization purposes."

[2] Fong and Vasicek [4] later extended Redington's results by deriving necessary and sufficient conditions for immunizing a fixed multiple liability stream against large equal changes in all interest rates.

In a series of papers beginning with Fisher and Weil [3], the relationship between duration and immunization was investigated in the context of nontaxable bonds and nonflat yield curves. If the yield curve is subject only to parallel shifts, the duration measure is given by Equation (1) where r denotes a vector of rates determined from the yield curve rather than a single rate.[3] Since $w_t \geq 0$ for all t and $\sum w_t = 1$, nontaxable bond duration can never exceed the bond's term to maturity. In their formal analysis, Fisher and Weil defined the liability stream as a single lump sum equal to the end-of-holding-period portfolio value (reinvested coupons and market value) anticipated on the basis of forward rates embedded in the initial yield curve.[4] Since the duration of this liability is simply H, the length of the holding period, Redington's first-order condition is satisfied if the initial duration of the bond portfolio also equals H. Fisher and Weil then showed that, since all bond cash flows are positive, this terminal liability is immunized against any *large* change in rates; i.e., that $D = H$ is sufficient for global immunization (duration determines an immunized holding period). Immunization of this liability implies that the holding period return of a bond cannot fall short of the bond's initially promised return, where the "promised" return is simply that which will obtain if the initial forward rates are exactly realized as future spot rates. Note that immunization requires equality of D and H over time; therefore, with the passage of time, it may be necessary to rebalance the portfolio (adjust the bond portfolio so that its duration equals the remaining holding period).

It should be obvious that the nontaxable bond example is a special case of the more general problem addressed by Redington and that its main result ($D = H$ is sufficient for immunization) cannot be presumed to hold when asset portfolios themselves generate negative cash flows. Recent research obscures this point. Hessel and Huffman [5], for example, considered bonds which are subject to tax outflows and claimed that the nontaxable bond results still obtain. But their results follow immediately from the fact that net cash flows were constructed to be positive in all time periods. This may not be the case since, among other considerations, an original-issue discount is taxable as ordinary income over the life of a corporate bond rather than as a capital gain at the end of a holding period as Hessell and Huffman assumed. Similarly, other researchers, such as Kolb and Gay [8], have suggested that financial futures contracts can be combined with bonds in such a way that the resulting portfolio is immunized for a holding period equal to its duration. Kolb and Gay, however, failed to consider the second-order conditions for this problem and therefore overlooked the possibility that the nontaxable bond results may not apply to all futures/bond portfolios. A

[3] The specification of duration depends on the form of interest rate change assumed. For discussion of the duration measures implied by particular nonparallel yield curve shifts see Bierwag [1] or Khang [7].

[4] This liability can be interpreted as 1) an explicit cash outflow or 2) an implicit target. Under the first interpretation, the Fisher-Weil (F-W) problem is equivalent to that of Redington or Fong and Vasicek; net overall portfolio value (assets minus liabilities) is initially zero and the "anticipated" net return is zero. Under the second interpretation, net portfolio value is initially positive and the anticipated return is positive. Thus the F-W approach can describe two slightly different aspects of immunization.

futures (or forward) contract consists of both an asset and a liability. A long position implies an outflow at the delivery date and subsequent inflows from the delivered instrument; a short position implies an inflow at the delivery date and subsequent outflows equal to the cash flows of the delivered instrument.[5] Thus, portfolios containing futures positions will include liabilities or implicit negative cash flows. In much the same way, short-selling in general requires that the short sale be covered at an uncertain price in the future, and margin-buying requires that interest payments be made. It is important to recognize that in any situation which gives rise to explicit or implicit negative cash flows, $D = H$ is not a sufficient condition for immunization.

The particular aspect of the immunization problem which we consider, therefore, is whether duration defines an immunized holding period for portfolios subject to negative cash flows. A terminal lump-sum liability is defined as in Fisher and Weil. For portfolios with positive value at initial rates r_0, the necessary and sufficient conditions for immunization against parallel shifts in a flat yield curve can be stated as follows:

I. $D(r_0) = H$ where H is the holding period.

IIa. $D'(r_0) \leq 0$ where $D'(r)$ is the first derivative of portfolio duration $(-P'(r)/P(r))$ with respect to r.

IIb. $D'(r) \leq 0$ for all $r \geq 0$.

Conditions I and IIa are necessary and sufficient for terminal portfolio value to have a local minimum at the initial rate r_0; Conditions I and IIb are necessary and sufficient for terminal portfolio value to have a global minimum at r_0. It is obvious that IIa and IIb are satisfied if all portfolio cash flows are positive as in the nontaxable bond case. However, if an explicit net outflow is incorporated into the portfolio cash flow stream, the portfolio present value equation must include a negative term. Depending on the timing of this outflow relative to inflows, duration $(-P'/P)$ may be increased or decreased. Furthermore, the presence of a negative term implies that the present value equation has at least one root e^{-r*}. If r^* is positive, global immunization is impossible since if rates jump to r^*, terminal value must be zero and thus fall short of the terminal liability. Whether r^* is positive or negative, the sign of $D'(r)$ over the domain of positive rates is no longer obvious. Since duration switches from $+\infty$ to $-\infty$ at any r^* where $P(r)$ changes sign, duration must be increasing around such r^*. Even if r^* is negative, D' may be positive for some or all positive rates. Thus, IIa and/or IIb may not be satisfied when the portfolio present value equation contains negative cash flow terms.

Since there are a variety of situations in which negative cash flows could be associated with the asset portfolio, we choose one example—the taxable zero

[5] The futures case is complicated by the requirement that positions be marked to the market. This daily cash settlement practice results in actual cash inflows if the value of a position increases but actual cash outflows if the position decreases. These cash settlements can be viewed as adjustments to the basic liabilities associated with either long or short positions as discussed above, but their full implications for immunization require a separate, specific analysis.

coupon bond. The analysis is straightforward and suggests the results in more complicated situations (e.g., futures). We stress that the taxable zero coupon bond illustrates the effects of one specific configuration of positive and negative cash flows on D and D'.

Section 1232 of the Internal Revenue Code requires that a portion of the original issue discount (OID) on corporate bonds be included in investor income each year over the bond's life. Likewise, issuers may deduct a portion of the OID in each year of the bond's life. For bonds issued between May 29, 1969 and July 1, 1982, the taxable (deductible) portion of the OID is computed on a straight-line basis.[6] Given a positive tax rate, the investor's cash flow stream from a zero is a series of negative cash flows followed by a positive cash flow (which may involve capital gains taxes) at maturity; the borrower's cash flow stream is a series of positive cash flows (tax savings) followed by a negative cash flow at maturity.

The price of any taxable zero or nonzero coupon bond, using continuous discounting and a flat after-tax yield curve, is

$$P(r) = C(r) + M(r) \tag{2}$$

where r is the after-tax yield, $C(r) = \sum_{t=1}^{T} c_t e^{-rt}$ is the present value of after-tax cash flows (coupons), and $M(r) = Me^{-rT}$ is the present value of after-tax proceeds at the maturity date T. It is assumed that the tax rate is positive and known with certainty and that all c_t have the same sign. It is also assumed that the final tax payment is due at maturity.

The duration measure is then

$$D = -(C' + M')/(C + M). \tag{3}$$

This duration can also be expressed as the weighted average of the duration of after-tax maturity value (T) and the duration of other after-tax cash flows, denoted D_c, where $D_c = -C'/C$. Since the weights $M/(C + M)$ and $C/(C + M)$ sum to 1, Equation (3) can be rewritten as follows:

$$D = T - (C/(C + M))(T - D_c). \tag{4}$$

Since D_c cannot exceed T, it follows from Equation (4) that taxable bonds with positive prices have:

$$D \gtreqless T \quad \text{as} \quad c_t < 0, = 0, > 0 \quad \text{for all} \quad t.$$

Therefore, the after-tax duration of a nonzero coupon bond is less than maturity unless the bond also has a taxable OID large enough to cause c_t to be negative. However, the after-tax duration of a zero coupon bond generally exceeds the bond's term to maturity. Only when there is a single tax payment due at maturity does the zero's duration equal its term to maturity.[7] It is important to note that

[6] 1982 tax amendments require that, for bonds issued after July 1, 1982, the annual taxable portion of the OID be computed on an interest accrual rather than straight-line basis. IRS regulations, however, are as yet unwritten.

[7] If the final tax payment is deferred beyond maturity, duration will be smaller than indicated by Equation (4). In the case that the zero has less than a year to maturity and there is a single tax payment due after maturity ($D_c > T$), then after-tax duration will be smaller than the term to maturity.

pre-tax duration indicates the volatility of bond price with respect to changes in pre-tax yields. Since after-tax duration depends on the investor's tax bracket, after-tax duration may not be uniquely defined in the bond market.

If all net cash flows are positive, a portfolio is globally immunized (subject to rebalancing and assumptions as to the form of yield curve shift) for a holding period equal to its duration. It is easy to see that this cannot be true for the taxable zero with $D > T$ since there exists a real finite positive rate r^* at which present value goes to zero. The absence of even local immunization is obvious if we invoke Redington's second-order conditions; when tax rates are certain, the tax payments plus the terminal liability define a fixed overall liability stream which is more spread out around liability duration than is the single inflow at maturity around its duration. In terms of IIa and IIb, it is easily shown that for the zero with $D \neq T$, $D'(r)$ is positive for *all* r (unless $r = r^*$, in which case $D'(r^*)$ is undefined). This implies that for a holding period equal to the after-tax duration, terminal portfolio value has a global maximum at $r = r_0$. Therefore, setting $D(r_0) = H$ is a minimax strategy rather than the maximin strategy it is for nontaxable bonds (or portfolios with only positive cash flows). Given any parallel yield curve shift, an investor holding a single zero will earn a lower after-tax return over the holding period than was "promised." If an investor desires immunization, he must combine the taxable zero with other assets whose cash inflows are such that the terminal value of the combined portfolio takes on a global minimum at $r = r_0$. (IIa is necessary and IIb is sufficient.)

However, from the issuer's point of view, a single taxable zero is immunized if $D(r_0) = H$. If we merely redefine the cash flows as cash *out*flows, duration and its first derivative are of course the same as for the investor. But now the terminal value function is an outflow or cost function which takes on its global maximum at initial rates r_0. Thus, for $D(r_0) = H$, the issuer's after-tax cost of borrowing cannot exceed the "promised" cost.[8] The problem of rebalancing over time still exists for the borrower; that is, tax savings must be reinvested so as to maintain the equality between duration and the remaining holding (borrowing) period.

In conclusion, it is important to recognize situations in which selecting a portfolio whose duration equals the desired holding period is not sufficient for immunization. A portfolio may have its duration extended or shortened by incorporating assets having explicit or implicit negative cash flows, but at the same time, the derivative of portfolio duration may become positive for some or all interest rates. If D' is positive for all $r \geq 0$, the duration of a given portfolio does not determine a holding period (for the investor) which is immunized in either the local or global sense. Even if D' is negative at initial rates, the existence of negative cash flows may cause D' to be positive for some rates. In this case only local immunization is guaranteed by setting $D(r_0) = H$.

[8] If $D > T$, $D = H$ implies that the issuer plans to refinance the debt from maturity to the end of the holding (borrowing) period H.

REFERENCES

1. G. O. Bierwag. "Immunization, Duration and the Term Structure of Interest Rates." *Journal of Financial and Quantitative Analysis* 12 (December 1977), 725–42.

2. ——— and C. Khang. "An Immunization Strategy is a Minimax Strategy." *Journal of Finance* 34 (May 1979), 389–99.
3. L. Fisher and R. L. Weil. "Coping with the Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies." *Journal of Business* 44 (October 1979), 408–31.
4. H. G. Fong and O. A. Vasicek. "A Risk-Minimizing Strategy for Multiple Liability Immunization." Unpublished paper. (February 1982).
5. C. A. Hessel and L. Huffman. "The Effect of Taxation on Immunization Rules and Duration Estimation." *Journal of Finance* 36 (December 1981), 1127–42.
6. J. R. Hicks. *Value and Capital*, 2nd Edition. Oxford: The Clarendon Press, 1946.
7. C. Khang. "Bond Immunization When Short-Term Rates Fluctuate More than Long-Term Rates." *Journal of Financial and Quantitative Analysis* 14 (December 1979), 1085–90.
8. W. Kolb and G. D. Gay. "Immunizing Bond Portfolios with Interest Rate Futures." *Financial Management* 11 (Summer 1982), 81–89.
9. F. R. Macaulay. *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the U.S. Since 1856.* New York: National Bureau of Economic Research, 1938.
10. F. M. Redington. "Review of the Principles of Life-office Valuations." *Journal of the Institute of Actuaries* 78, 3 (1952), 286–315.
11. P. A. Samuelson. "The Effect of Interest Rate Increases on the Banking System." *American Economic Review* 35 (March 1945), 16–27.