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Option Pricing When the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note

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IN THEIR CLASSIC PAPER, Black and Scholes [2] present two derivations of the European call option pricing formula. The first, which we will refer to as the "replication derivation," relies on the fact that the returns from the option can be replicated by holding a continuously revised position in the underlying asset and bonds. This implies that if an option is not priced according to the Black-Scholes formula, there is a sure profit to be made by some combination of either short or long sales of the option and underlying asset. This argument is seemingly independent of considerations relating to capital market equilibrium. The second derivation, which we will call the "equilibrium derivation," evolves from the requirement that the option earn an expected rate of return commensurate with the risk involved in holding the option as an asset. This note considers the pricing of a call option on an asset which earns an expected rate of return lower than that necessary to induce investors to hold it. We show in this case that even if the replication strategy is feasible, it is inefficient. A consequence of this is that the correct option pricing formula is different than the Black-Scholes formula. Unlike the Black-Scholes formula, our formula requires as an input the difference between the expected rate of return on the underlying asset and that expected rate of return which would be required to induce holding of the asset.

Other papers have made a similar point. Constantinides [3] and Cox et al. [4] have argued that in equilibrium the prices of traded contingent claims must satisfy a partial differential equation which is identical to the Black-Scholes partial differential equation only when the underlying asset pays no dividends and earns a rate of return sufficient to induce holding of the asset. More recently, Ingersoll [5] has noted that commodities often pay a nonpecuniary service yield, the value of which must be taken into account in deriving the option price.

None of these authors, however, explain why the Black-Scholes formula is sometimes inappropriate. The purpose of this note is to demonstrate why the Black-Scholes formula needs modification even if replication is feasible. We provide an intuitive derivation of the correct formula and discuss why divergences of the equilibrium price from the Black-Scholes formula do not result in arbitrage opportunities.

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I. The Replication Derivation

Consider a tradeable asset[1] with price P, following the Ito process

$$\frac{dP}{P} = \alpha_p\, dt + \sigma_p\, dz_p \tag{1}$$

Suppose that the equilibrium expected rate of return necessary to compensate investors for bearing the risk of holding this asset is $\alpha > \alpha_p$.

The formula for the price of a call option written on this asset will be correct if two conditions are met:

i) At maturity, the formula has the value

$$\text{Max}[0, P_T - X] \tag{2}$$

where X is the exercise price and T is the maturity date.

ii) The formula implies that the option earns a rate of return consistent with capital market equilibrium.

Merton [6], in his derivation of the Black-Scholes formula, shows that an investor can replicate the payoff of a call option on the asset by holding a continuously adjusted, self-financing portfolio containing the asset and a position in risk-free bonds. This suggests that, to prevent arbitrage, the price of the call should equal the cost of the replicating portfolio. As Merton's derivation demonstrates, the cost of the replicating portfolio is precisely the value given by the Black-Scholes formula:

$$\begin{aligned} V(P_0, X, s) &= P_0 N(d_1) - Xe^{-rs}N(d_2) \\ d_1 &= [\ln(P_0/X) + (r + \sigma_p^2/2)s]/\sigma_p\sqrt{s} \\ d_2 &= d_1 - \sigma_p\sqrt{s} \end{aligned} \tag{3}$$

where s is the time to maturity, r is the risk-free rate of interest, and $N(\cdot)$ is the standard normal cumulative density function.

The option price (3) satisfies condition (i), in that it has the correct payoff at maturity. It fails to satisfy condition (ii), however. This is easily seen for a particular case by setting $X = 0$ in (3). Then $V(P, 0, s) = P$, and

$$\frac{1}{dt} E\left[\frac{dV}{V}\right] = \frac{1}{dt} E\left[\frac{dP}{P}\right] = \alpha_p < \alpha.$$

The replicating portfolio in this special case is trivial; it involves holding one unit of the underlying asset. Therefore, the replicating portfolio—and thus the option—earns a below-equilibrium rate of return.

[1] For our purposes an underlying asset is any financial asset or commodity that can be bought, sold, and held. Examples are stocks, corn, and pencils. We do not include a commodity such as electricity, which cannot be held, solely because there is no question of literally setting up a replicating portfolio using it. An option on such a commodity would obviously be priced according to our formula (6) below. See Constantinides [3] for a discussion of this latter point.

It is also easy to see that in general (for any exercise price) the option will earn too low a rate of return. Since (3) is also the cost of a portfolio of bonds and the underlying asset, we can write

$$V(P, X, s) = \omega_1(P, s)P + \omega_2(P, s)B(s) \tag{4}$$

where $B(s)$ is the price of a one-dollar face value pure discount bond maturing in s periods, $\omega_1 > 0$ is the quantity of the underlying asset held, and $\omega_2 < 0$ is the quantity of bonds held in the replicating portfolio.

Obviously bonds earn an equilibrium rate of return; it follows that V earns an equilibrium rate of return only if P earns an equilibrium rate of return. Since by assumption, P earns a below-equilibrium rate of return and is held in positive quantity in the replicating portfolio, V must also earn a below-equilibrium rate of return.

The idea that the price of a call option should equal the Black-Scholes price stems from the notion that the price of a contingent claim should equal the cost of a strategy that replicates the returns of that claim. We have just seen, however, that if the Black-Scholes price were the price of the option, the option would earn a subnormal rate of return. This is clearly not an equilibrium situation. We would expect an excess of writers to buyers at this price, which would drive down the call price.[2]

II. The Equilibrium Derivation

The argument in the previous section is quite general and makes it clear that there is no hope of finding the equilibrium option price by a replication strategy. Because the underlying asset earns a below-normal rate of return, any portfolio which consists of bonds and a long position in the underlying asset will also earn a subnormal rate of return. The correct option price must therefore be derived by other means.

One alternative is to follow Constantinides [3] and impose the condition that the option earn a rate of return commensurate with its risk. This requires specifying a particular model of capital market equilibrium. We adopt a slightly more general version of this argument, which does not require specifying a particular equilibrium model.

As discussed, the correct option price must satisfy (i) and (ii). Suppose that we repeat the Merton derivation, only we replace P by P^*, where

$$P_t^* = P_t e^{-(\alpha - \alpha_p)(T-t)} \tag{5}$$

P^*, unlike P, earns an equilibrium rate of return.[3] Replacing P by P^* in the

[2] Recall that options are a zero net supply asset. Thus, in the context of a pricing model such as the CAPM, the equilibrium rate of return is that which induces zero aggregate demand. If the rate of return on the option is too low, there will be negative aggregate demand.

[3] Using Ito's Lemma, it can be shown that P_t^* follows the Ito process

$$\frac{dP^*}{P^*} = \alpha\, dt + \sigma_p\, dz_p$$

Thus, P^* has the same risk as P, but earns the expected rate of return α.

Merton derivation yields (3), only with P replaced by P^*:[4]

$$W(P_0, X, s; \delta) = P_0 e^{-\delta s} N(d_1^*) - X e^{-rs} N(d_2^*)$$
$$d_1^* = [\ln(P_0/X) + (r - \delta + \sigma_p^2/2)s]/\sigma_p \sqrt{s}$$
$$d_2^* = d_1^* - \sigma_p \sqrt{s} \tag{6}$$

where $\delta = \alpha - \alpha_p$. Because $P_T = P_T^*$, it is clear that (6) satisfies (i). Further, P^* earns an equilibrium rate of return, so that from (4), the option also earns an equilibrium rate of return. Thus condition (ii) is also satisfied. Therefore, the option pricing formula (6) is the equilibrium price of the call. We emphasize that even though there may be no asset with price P^*, this argument identifies the option value which satisfies condition (ii).[5]

To verify this argument for a special case, suppose again that $X = 0$. Then $W(P, X, s; \delta) = Pe^{-\delta s} = P^*$. The risk of the option is then the same as P_t^*, and it appreciates at the same expected rate α. Note that when the rate of return shortfall δ equals zero, then (6) reduces to (3).[6]

It is clear from comparing (3) and (6) that $W(P_0, X, s; 0) = V(P_0, X, s)$. Also, it is easy to show that $\frac{\partial W}{\partial \delta} < 0$. Therefore $W(P_0, X, s; \delta) < V(P_0, X, s)$ if $\delta > 0$, which shows that the equilibrium option price will be lower than (3). The replication strategy is too costly because it involves holding some of the asset, which by assumption earns a subnormal rate of return.

Equation (6) is also the formula for the value of a European call option on a stock which pays dividends at the proportional rate δ. Formally, the rate of return shortfall, δ, is like a dividend, in that it is a portion of the total required rate of return on the underlying asset which does not accrue to the option-holder and thus affects the option price. The question of whether this shortfall actually accrues to the holder of the underlying asset (as it does for stock dividends) is irrelevant for pricing the option.

III. Absence of Arbitrage Opportunities

We have argued that the correct price of a European call option on an asset which earns a below-equilibrium rate of return will be less than the Black-Scholes price V, if $\delta > 0$. It is commonly supposed, however, that divergences from V will result in arbitrage profits. Because W is less than V, the arbitrage would involve short-selling the underlying asset and buying the call. Lending the asset for purposes of a short-sale, however, implies a willingness by some investor to hold

[4] It should be noted that P^* can be interpreted as the price that would be paid today for time T delivery of the underlying asset. Thus, $F = P^* e^{r(T-t)}$ is the forward price for the commodity, and (6) can be seen to be equivalent to Black's [1] formula for the price of a commodity option, with Fe^{-rs} substituted for P^*. This makes it clear that if there is a futures market for the underlying asset, then the replication and equilibrium prices are the same when the futures contract is used in the replication.

[5] It should be clear that only one equilibrium price can exist. Otherwise arbitrage opportunities exist.

[6] This is why Black and Scholes are able to use both arbitrage and equilibrium derivations to arrive at their formula. For a nondividend-paying stock, $\delta = 0$.

the asset. Because the asset earns too low an expected rate of return, no one would lend the asset for short-sale. Therefore, the arbitrage cannot be undertaken.

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