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On the Jensen Measure and Marginal Improvements in Portfolio Performance: A Note

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ABSTRACT

The marginal performance contribution made by new assets in a portfolio is identified. The maximum change in a portfolio's Sharpe performance from the addition of new assets is a simple function of a generalized Jensen index and the unexplained covariances from a multivariate market model. Deviations from a higher dimension market line may be used to rank the desirability of asset additions to an existing portfolio. Statistical tests for the equality of the performance contributions by new assets is possible.

THE PORTFOLIO PERFORMANCE MEASURES of Treynor [11] and Jensen [12] are based on the systematic or beta measure of risk. When originally developed, it was advocated that these measures of performance were useful for comparing and ranking performances of various portfolios. After many years of use in this area, it was demonstrated by Roll [7] and Ross [8] that it is theoretically inappropriate to use these performance measures for this purpose, since differences in portfolios' performance indicate only inefficiency in the index used for the computation of beta. Dybvig and Ross [1] subsequently proved that a portfolio can be improved by adding (or shorting) a small investment in another asset if the asset lies above (or below) the security market line. A potentially important use of the market line was therefore preserved.

The purpose of this note is the development of a procedure which identifies the exact performance contribution made by one or more assets when they enter the investor's optimal portfolio of assets. It is demonstrated that the change in squared Sharpe [10] performance is a simple function of a generalized Jensen index and the unexplained covariances from a multivariate market model. New assets may be ranked according to the magnitude of their performance contribution. The improvements in a portfolio are easily computable from the same linear regression employed to calculate an asset's Jensen measure. Thus, the appeal of the security market line as a useful financial tool may be preserved.

The first section briefly establishes the efficient set algebra required to develop the marginal contribution made by new assets in a portfolio. Section II presents the performance contribution dependent on the number of new assets and whether partial or full optimal adjustment of the portfolio weights are permitted. Section III concludes the note.

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I. Efficient Set Mathematics

The Markowitz [4] efficient set of risky asset portfolios is comprised of the portfolios that minimize portfolio variance for a given mean excess return μ_p (i.e., in excess of the riskless rate of interest or zero beta rate) subject to the constraint that investment proportions sum to one. In the presence of a riskless asset, such as treasury bills, the Sharpe [9] efficient set of risky and riskless assets becomes the set of linear combinations of the riskless asset and a unique risky asset portfolio m.

To develop the expression for the slope of the efficient frontier in excess mean return-standard deviation space, consider a population of N assets with mean excess return vector $\boldsymbol{\mu}_{(N\times 1)}$ and nonsingular covariance matrix $\boldsymbol{\Sigma}_{(N\times N)}$. Merton [5, p. 1866] and Roll [7, Appendix] have shown that the proportions of risky assets in the tangency portfolio are given by the $(N \times 1)$ vector

$$\mathbf{X}_m = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/(\mathbf{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}), \tag{1}$$

where $\mathbf{e}_{(N\times 1)}$ is the unit vector, and primes indicate the transpose operation. The mean excess return and standard deviation of the optimal tangency portfolio are therefore

$$\mu_m = \boldsymbol{\mu}'\mathbf{X}_m = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/(\mathbf{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) = a/b, \tag{2}$$

and

$$\sigma_m = (\mathbf{X}_m'\boldsymbol{\Sigma}\mathbf{X}_m)^{1/2} = (\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{1/2}/(\mathbf{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) = \frac{\sqrt{a}}{b}, \tag{3}$$

where

$$a = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \tag{4}$$

and

$$b = \mathbf{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

are called efficient set constants. The Sharpe performance of the optimal risky asset portfolio is also the slope of the tangent line which is given by

$$\mathrm{Sh}_m = \mu_m/\sigma_m = (a/b)(b/\sqrt{a}) = \sqrt{a}. \tag{5}$$

Jobson and Korkie [3] have termed $a = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ or $\sqrt{a}$ the potential performance of the N asset set, since the latter represents the maximum attainable Sharpe performance from the N assets when the assets are held in the optimal proportions given by (1). The Equations (2) through (5) are illustrated in Figure 1.

The mathematics of the efficient set are such that, if the riskless rate exceeds the mean return on the global minimum variance portfolio (or its excess mean return b/c is negative), then the tangency portfolio is on the lower boundary of the efficient set hyperbola. In this case, the excess mean return μ_m and Sharpe measure Sh_m are negative. That is,

$$\mu_m,\ \mathrm{Sh}_m \begin{cases} > 0, & \text{if } b/c > 0 \\ < 0, & \text{if } b/c < 0, \end{cases}$$

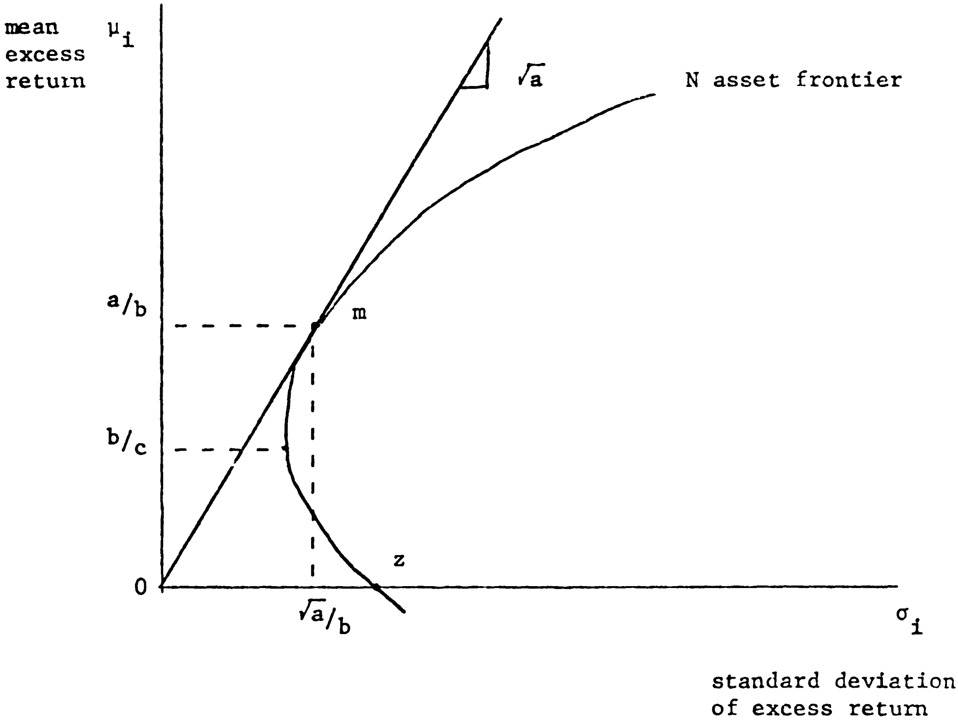

Figure 1. Parameters of the Efficient Set Illustrating the Mean Excess Return a/b, the Standard Deviation $\sqrt{a}/b$, and the Sharpe Performance $\sqrt{a}$ of the Tangency Portfolio m (with Zero Beta Portfolio z) When the Global Minimum Variance Portfolio's Mean Excess Return b/c Is Positive

where $c = \mathbf{e}'\mathbf{\Sigma}^{-1}\mathbf{e}$. When quantifying the contribution that additional assets make to an efficient portfolio, it is necessary to impose the efficiency condition of positive excess mean returns on the global minimum variance portfolio and therefore on the tangency portfolio.

II. The Performance Contribution of Additional Assets

The contribution of additional assets to a portfolio of assets may be determined by examining the difference in the potential performance Δa before and after the assets are added to the optimal portfolio. That is,

$$\Delta a = a - a_1 = \boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}_1'\mathbf{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1, \tag{6}$$

where there are $N = N_1 + N_2$ assets after the addition and N_1 assets before the addition of N_2 new assets and the partitioning of the N assets gives the required mean vectors and covariance matrices

$$\boldsymbol{\mu} = \begin{array}{cc} (1) & \\ \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} & \begin{array}{c} (N_1) \\ (N_2) \end{array} \end{array} \quad \text{and} \quad \mathbf{\Sigma} = \begin{array}{cc} \begin{array}{cc} (N_1) & (N_2) \end{array} & \\ \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix} & \begin{array}{c} (N_1). \\ (N_2) \end{array} \end{array}$$

In order for the increase in performance given by (6) to make economic sense, there must exist positive excess mean returns on the global minimum variance portfolios of the N_1 and N asset sets. That is, $b/c = \mathbf{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/(\mathbf{e}'\boldsymbol{\Sigma}^{-1}\mathbf{e}) > 0$ and $b_1/c_1 = \mathbf{e}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1/(\mathbf{e}_1'\boldsymbol{\Sigma}_{11}^{-1}e_1) > 0$ or alternatively the excess means, μ_m and μ_{m1} on the respective tangency portfolios, must be positive.

Since $\boldsymbol{\Sigma}$ is assumed to be nonsingular, Δa may be expanded by first employing the expression for the inverse of partitioned matrices (see for example Morrison [6, p. 68]). That is,

$$\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1 + [\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1]'[\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}]^{-1}[\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1].$$

Substitution in (6) gives

$$\Delta a = [\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1]'[\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}]^{-1}[\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1]. \tag{7}$$

Examination of (7) indicates that the vector $\boldsymbol{\alpha} = [\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1]$ is the $(N_2 \times 1)$ vector of intercept terms in the multivariate regression of the excess returns from the N_2 new assets on the excess returns from the original N_1 assets.[1] Thus, α_i an element of $\boldsymbol{\alpha}$, can be thought of as the generalized (to more than one independent variable) Jensen measure of asset i in the regression

$$r_{it} = \alpha_i + b_{1i}r_{1t} + b_{2i}r_{2t} + \cdots + b_{N_1i}r_{N_1t} + \epsilon_{it},$$
$$i = N_1 + 1,\ N_1 + 2,\ \cdots,\ N, \tag{8}$$

where r_{it} is the excess return from asset i in period t and b_{ji}, $j = 1, 2, \cdots, N_1$, is the coefficient of asset j's excess return in regression i. Similarly, examination of (7) indicates that the matrix

$$\boldsymbol{\Sigma}_{22.1} = [\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}]$$

is the $(N_2 \times N_2)$ covariance matrix of error terms from the same multivariate regression of the excess returns from N_2 assets on the excess returns of the N_1 assets.[2] Thus, the change in the optimal squared Sharpe performance due to the N_2 new assets is the quadratic form

$$\Delta a = \boldsymbol{\alpha}'\boldsymbol{\Sigma}_{22.1}^{-1}\boldsymbol{\alpha} = \textstyle\sum_{i=N_1+1}^{N} \sum_{j=N_1+1}^{N} \alpha_i\alpha_j\sigma_{22.1}^{ij}, \tag{9}$$

where $\sigma_{22.1}^{ij}$ is an element of the inverse of $\boldsymbol{\Sigma}_{22.1}$.

The marginal contribution of the N_2 assets can be couched in an arbitrage framework. Consider a portfolio p (with investment weights X_i summing to one), constructed from $N = N_1 + N_2$ assets, which has zero weights on the N_2 assets and which is efficient in the mean-standard deviation space of the N_1 assets. An arbitrage portfolio, relative to p, is a marginal portfolio represented by the marginal changes in weights ΔX_i, $i = 1, 2, \cdots, N$ which must sum to zero over the N assets. If no arbitrage opportunity exists, the marginal performance

[1] $\boldsymbol{\alpha}$ is also the mean vector for the N_2 assets after controlling for the N_1 assets and is often written $\boldsymbol{\mu}_{2.1}$.

[2] $\boldsymbol{\Sigma}_{22.1}$ is also termed the partial covariance matrix of the N_2 assets after controlling for the N_1 assets. An element of $\boldsymbol{\Sigma}_{22.1}$ is denoted by $\sigma_{ij.1}$, $i, j = 1, 2, \cdots, N_2$. If only one ($N_1 = 1$) independent variable exists (for instance, the returns on a portfolio p), then an element of the partial covariance matrix controlling for p is denoted $\sigma_{ij.p}$, $i, j = 1, 2, \cdots, N_2$.

increase Δa resulting from the marginal investments ΔX_i must be zero. Equivalently, for a given risk level, σ_p, the arbitrage mean return $\Delta\mu_p$ must be zero to preserve $\Delta a = 0$. The portfolio p must therefore be efficient in the mean-standard deviation space of all N assets. The mean returns on N_1 assets are sufficient to span the mean return space of all N assets and therefore the mean returns on all N assets can be written as exact linear combinations of the N_1 assets' mean excess returns as in arbitrage pricing theory, or as exact linear functions of the mean excess return on p as in capital asset pricing theory.

A. Special Cases

The foregoing analysis derived the general expression for the contribution of N_2 assets to the squared Sharpe performance of N_1 assets when all assets are combined optimally in their respective portfolios. When restrictions on the numbers of assets N_1 and N_2 exist, more familiar results arise.

Consider the case where only one ($N_2 = 1$) new asset i is added to an optimal portfolio of N_1 assets and all $N = N_1 + N_2$ assets' proportions are readjusted to the new optimal portfolio. Then the performance contribution of the new asset is from (9) equal to the square of the adjusted Jensen measure

$$\Delta a_i = (\alpha_i^2/\sigma_{ii.1}),$$

where α_i and $\sigma_{ii.1}$ are the intercept (generalized Jensen measure) and partial (error) variance from the multiple regression of the new asset excess returns on the excess returns from the original N_1 assets (Equation (8)). Consider also the special case where the original N_1 asset portfolio is treated as a single asset p and a new portfolio is formed from a linear combination of the original portfolio p and a new asset i. The performance contribution of the new asset is the square of the adjusted Jensen measure

$$\Delta a_i = \alpha_i^2/\sigma_{ii.p},$$

where α_i and $\sigma_{ii.p}$ are the intercept and error term variance from the simple market model regression of the asset's excess returns on the excess returns from the original portfolio given by

$$r_{it} = \alpha_i + b_i r_{pt} + \epsilon_{it}, \tag{10}$$

If only one asset addition to the portfolio is permitted, then the best asset addition is the one that achieves the largest adjusted Jensen measure $(\alpha_i^2/\sigma_{ii.p})$. Thus, the candidate assets may be ranked according to the magnitude of this performance measure. This gives some legitimacy to the popular practice of computing both the Sharpe and the Jensen measures over a number of mutual funds, providing the Jensen measures are adjusted by $\sigma_{ii.p}$. Suppose there are n funds and a market index p to be evaluated with excess returns available on all $n + 1$ assets for T periods. The ranking of the funds by the Sharpe measure indicates the best to worst performance over the sample period assuming that only one of the n funds is purchased. On the other hand, fund ranking by the adjusted Jensen measure $\alpha_i^2/\sigma_{ii.p}$, $i = 1, 2, \cdots, n$, which is obtained from (10), indicates the funds making the best to worst additions to the index portfolio p assuming only one fund could be purchased and held with p in the same period.

Thus, it is reasonable to rank order funds by both the Sharpe and adjusted Jensen measures since the funds may be held by themselves or with other assets. However, the definition of the reference portfolio p is flexible and obviously would depend on the assets held by those investors who also purchase a particular mutual fund.[3]

B. Significance Tests

The marginal contribution of a set of N_2 new assets to the potential performance of an original set of N_1 assets given by

$$\Delta a = (\alpha' \Sigma_{22.1}^{-1} \alpha),$$

is similar in form to the squared Sharpe (potential) performance of the N_2 assets considered alone given by

$$a_2 = (\mu_2' \Sigma_{22}^{-1} \mu_2).$$

Since α and $\Sigma_{22.1}$ can be thought of as the mean μ_2 and covariance matrix Σ_{22} after removal of the effects of the N_1 assets, the marginal contribution $\Delta a = (\alpha' \Sigma_{22.1}^{-1} \alpha)$ can be thought of as the potential performance of the N_2 assets with the effect of the N_1 assets removed. $\sqrt{\Delta a}$ is therefore the slope of the tangency in corrected excess mean return-standard deviation space.

Jobson and Korkie [3] have developed test statistics for testing the hypothesis that two sets of assets have equivalent tangency slopes (potential performance). Thus, the test may be used to test the hypothesis that $\Delta a = 0$. This test procedure may also be applied to test the hypothesis that, for a given set of N_1 assets, the additional contribution Δa^* made by a new set of N_2^* assets is equivalent to the additional contribution Δa^{**} made by an alternative set of N_2^{**} assets which includes the N_2^* assets. In other words, the hypothesis may be restated as, the potential performance of the $(N_1 + N_2^*)$ assets is equivalent to the potential performance of the larger set of $(N_1 + N_2^{**})$ assets.

III. Conclusion

This note has shown that the improvement in the squared Sharpe performance forthcoming from the addition of new assets to a portfolio, with optimal adjustment in asset weights, is a simple function of the generalized Jensen measures adjusted by the partial covariance matrix. The vector of Jensen measures and the partial covariance matrix are obtained from the multivariate regression of the excess returns from the new assets on the excess returns from the original assets in the portfolio. A potentially important use has been made of deviations (as measured by the adjusted Jensen measure) from a hyperplane with dimension equal to or larger than the one-dimensional security market line. A statistical test of the hypothesis of equivalent performance contributions of sets and subsets

[3] When the excess return regression's independent variable(s) are not the available market index (or market assets) it is misleading to label the deviations as being from a (higher dimension) market line. Some other nomenclature like the N_1-dimensional security reference line might be preferable.

of new assets is possible. Our results may be of use in determining whether one investor class outperforms another class, in analyzing expanded market indexes, and in examining the related issue of market efficiency.

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