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Short Sales Restrictions and Kinks on the Mean Variance Frontier

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ABSTRACT

With a short sales restriction, there may be switching points along the mean variance frontier corresponding to changes in the set of assets held. Traditional wisdom holds that each switching point corresponds to a kink, while Ross has claimed that kinks never occur. This paper shows that the truth lies between the two views, since the efficient frontier may or may not be kinked at a switching point. There is some indication that kinks are rare, since a kink corresponds to a portfolio in which all assets have the same expected return.

ACCORDING TO ROSS [3], the mean variance frontier will not have kinks (i.e., will be differentiable) even when there are short sale restrictions:

> Equation 10 also disproves a common fallacy in drawing the efficient set for risky assets with no short sales. At a switching point where the portfolio composition changes . . . there is no kink.

This paper gives a counterexample to Ross's statement and therefore shows that the conventional wisdom is correct, insofar as it predicts the existence of kinks on the frontier. We will see, however, that kinks occur only when the portfolio on the frontier consists of securities all having the same mean return, as happens if some security is on the frontier. In other words, there may or may not be a kink at a point on the frontier where the asset composition of the efficient portfolio changes.

Let E and V denote the mean vector and covariance matrix of asset returns (payoffs per dollar invested). Points on the mean variance frontier F with no short sales are given by the solutions for different values of expected return, μ, of the following problem.

Problem 1: Choose α to

$$\min \alpha V \alpha$$

subject to

$$\alpha e = 1, \qquad \alpha E = \mu, \qquad (\forall i)\alpha_i \geq 0.$$

In Problem 1, α_i is the fraction of wealth invested in security i, α is the vector of α_i's, and e is the vector of ones. In other words, portfolios on the frontier minimize variance subject to portfolio weights summing to one (the budget constraint), a given mean return, and no short sales (non-negative investment in each asset).

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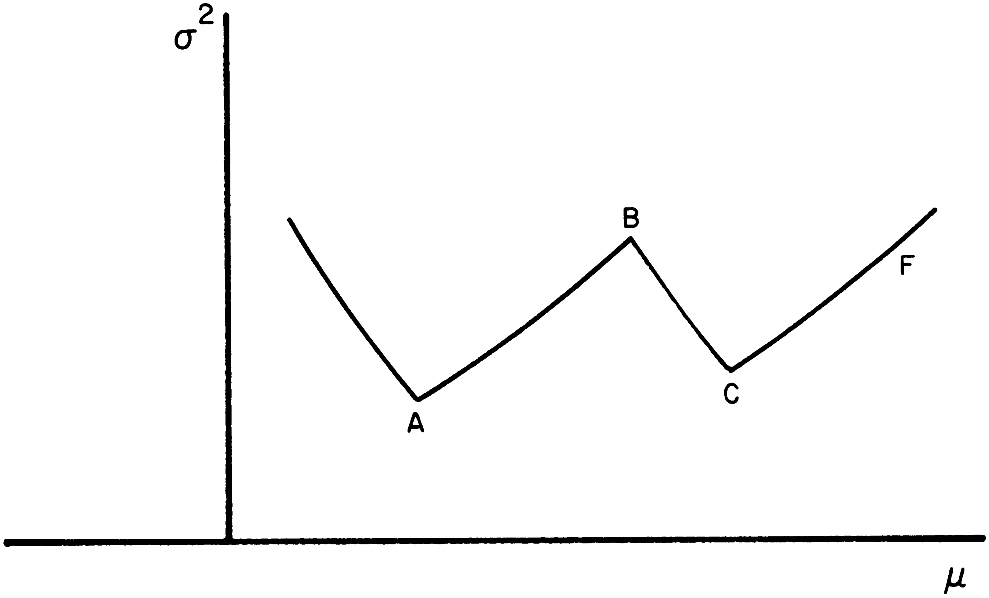

Figure 1. A mean variance frontier F with several kinks. This particular frontier cannot arise since kink B violates the convexity of the frontier.

We will always restrict μ to the interior of the interval $[\min_i E_i, \max_i E_i]$ for which Problem 1 is feasible.[1]

Our task is to study whether this frontier can have a kink, such as point A, B, or C, illustrated in Figure 1. Mathematically, a kink is a point at which the slope $d\sigma^2/d\mu$ of the frontier is discontinuous. It is easy to rule out upward pointing kinks, such as kink B in Figure 1, since the frontier is convex by the usual argument.[2]

What about downward pointing kinks, such as kinks A and C in Figure 1? The following example shows that downward pointing kinks may occur. This is the counterexample to Ross's statement.

Example 1: Let

$$V = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{vmatrix} \quad \text{and} \quad E = \begin{vmatrix} 1 \\ 3 \\ 4 \end{vmatrix}.$$

Then asset 1 is riskless and assets 2 and 3 are perfectly correlated. The efficient frontier with no short sales is shown in Figure 2a. The kink occurs at $\mu = 3$, where the efficient portfolio contains only asset 2. At all other interior points on

[1] Whenever the problem is feasible it has a solution, since the objective function is continuous and the constraint set is compact.

[2] Take any portfolios α and β on the frontier. Then for $\gamma \in (0, 1)$, $\gamma\alpha + (1 - \gamma)\beta$ does not involve short sales but by Cauchy-Schwartz has smaller variance than $\gamma\alpha V\alpha + (1 - \gamma)\beta V\beta$. See, for example, Elton et al. [1].

the frontier, the efficient portfolio consists of a combination of asset 2 and one of the other two assets.

An easy way to see that the frontier is kinked is by looking at mean standard deviation space (Figure 2b), which is a smooth "kink-preserving" transform of

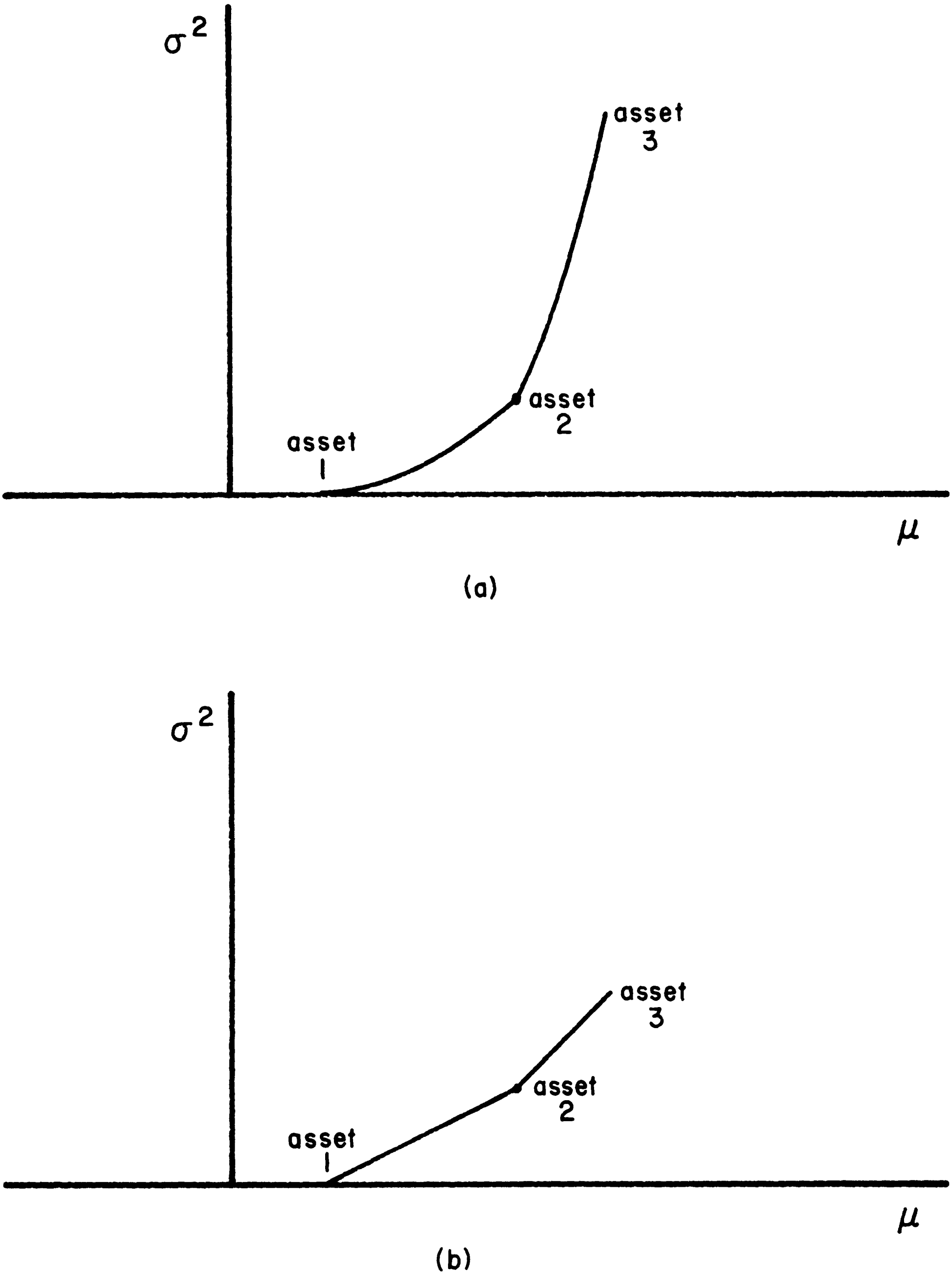

Figure 2. A kinked frontier (Example 1) in (a) mean variance space and in (b) mean standard deviation space.

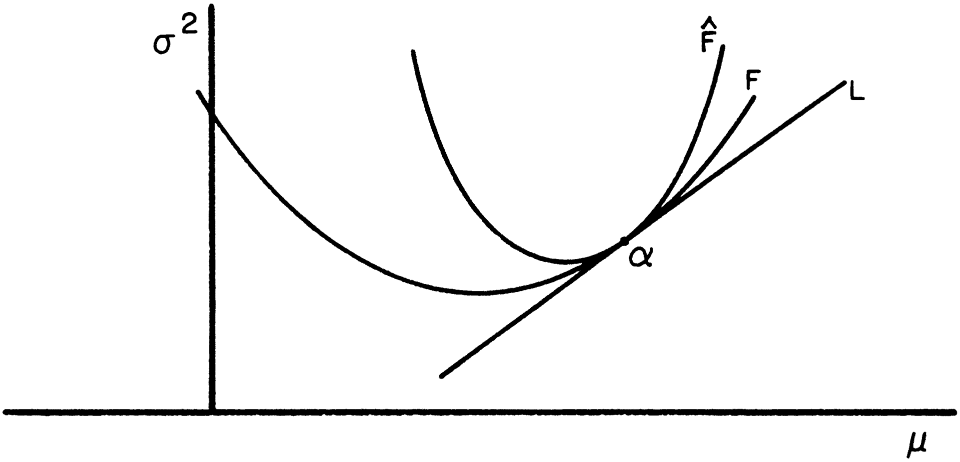

Figure 3. If the frontier $\hat{F}$ for assets in α (allowing short sales) is smooth, the frontier F which does not allow short sales is locally squeezed between $\hat{F}$ and any support L (guaranteed by convexity) of F at α. Therefore, F cannot be kinked at α unless $\hat{F}$ is degenerate.

mean variance space (for $\sigma^2 \neq 0$). Since asset 1 is riskless and assets 2 and 3 are perfectly correlated, linear combinations are straight lines in this space, and the frontier is clearly kinked. Q.E.D.

The degeneracy of the covariance matrix is not essential for the example. It is shown in the Appendix that the frontier is still kinked at $\mu = 3$ if a small i.i.d. noise term is added to each asset, making V nonsingular. Intuitively, all that is needed to get a kink is that asset 2 be sufficiently correlated with asset 1 and/or asset 3, so that an improvement for an agent holding asset 2 would involve *shorting* assets 1 and 3.

Ross based his claim that the frontier is differentiable (unkinked) on the *fact* that the slope of the frontier is one of the Lagrange multipliers of Problem 1 and the incorrect *intuition* that this multiplier should be continuous in the parameter μ. An intuition I find more helpful is related to Figure 2 of Roll [2] and Figure 3 of this paper.[3] For any α on F, consider the frontier $\hat{F}$ formed by allowing an agent to buy and sell short only those assets held in positive quantities of α. Then the portfolio α is on $\hat{F}$, and therefore $\hat{F}$ can be generated by linear combinations of α and some other portfolio on $\hat{F}$. Near α, these linear combinations contain positive weights in assets held in α, and by definition of $\hat{F}$, zero weight in all other assets. Therefore, near α, there are portfolios on $\hat{F}$ which are feasible for Problem 1, so that $\hat{F}$ is locally dominated by F.[4] Unless $\hat{F}$ is degenerate, F is sandwiched between two tangent differentiable functions, namely $\hat{F}$ and the linear support to α ensured by the convexity of F (discussed above and in footnote 2). This implies that F is differentiable at α. We have just about proven the following theorem.

THEOREM 1: *If a risky portfolio α on the frontier F (not allowing short sales) has*

[3] This same intuition is discussed by Solnik [4].

[4] Of course, $\hat{F}$ may dominate F far from α. If all assets are held in α, then $\hat{F}$ and F are locally the same.

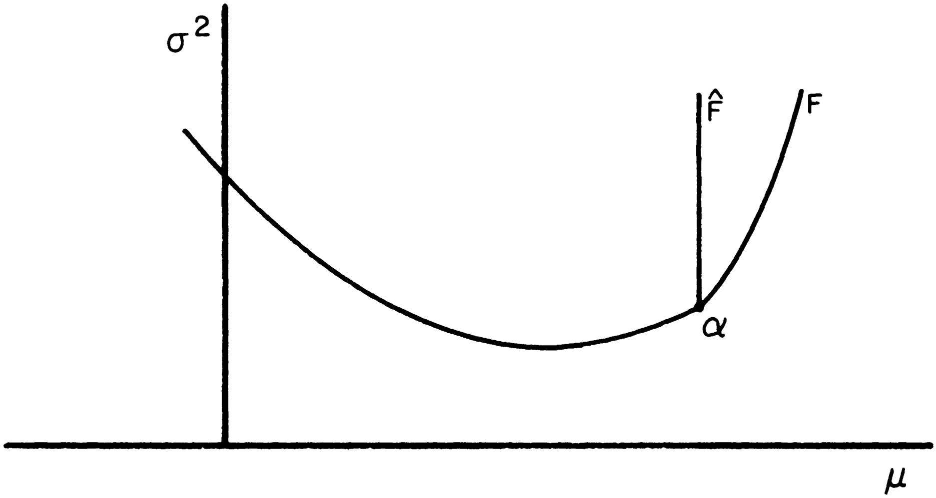

Figure 4. If there is a kink, then every asset held at the kink has the same expected return. Consequently, $\hat{F}$ is either a single point (as in Example 1) or a vertical ray as shown here.

positive weight in any pair of assets with differing expected returns, then there is no kink in F at α. (Equivalently, if there is a kink at the risky portfolio α, then α is a portfolio of securities of equal expected return.)

Proof: If α has positive weight in two securities with unequal expected return, then the frontier $\hat{F}$ defined above includes points at all expected returns and is differentiable at α (since α is risky).[5] Applying the argument above, we can conclude that F is differentiable at α, i.e., has no kink at α. Q.E.D.

COROLLARY: *If all securities have different expected returns, then a kink on the frontier can occur only if a single security is held at that point.*

Proof: Immediate, given Theorem 1. Q.E.D.

Figure 4 illustrates why the geometric argument illustrated in Figure 3 no longer holds for the portfolio α held at a kink; since all assets held at α have the same expected return, the set of portfolios formed from assets in α is either a single point or a vertical ray, and consequently the frontier $\hat{F}$ is a single point.

The converse to Theorem 1 and its Corollary is false. In other words, it is possible to have a smooth frontier even if a single security is on the frontier or if a portfolio of assets with equal means is on the frontier. This is shown (by counterexample) in the Appendix.

[5] See the Appendix of Roll [2] for a collection of results about the unconstrained mean variance frontier. In mean standard deviation space, the frontier is typically a hyperbola which is everywhere differentiable. In one limiting case, there is a riskless asset, and the frontier is a line reflected at the riskless asset. In this case, the frontier is differentiable except at the riskless asset. In an uninteresting degenerate case, corresponding to the existence of arbitrage, the frontier is a horizontal line, which is of course differentiable. In the other degenerate case, in which all assets have the same mean, the frontier is a point, which is ruled out by the condition that two of the assets held in α have differing means. For us, therefore, an unconstrained mean standard deviation frontier is differentiable at every risky α. Also, there is no kink on an unconstrained frontier in *mean variance* space, even when there is a riskless portfolio.

Appendix

The Kuhn-Tucker conditions for Problem 1 are given by:

$$V\alpha = \gamma e + \lambda E + \delta$$
$$\delta_i = 0 \qquad \text{if } \alpha_i > 0$$
$$\delta_i \geq 0 \qquad \text{all } i.$$

For Example 1 modified by adding i.i.d. error terms to each asset, E and V are given by

$$V = \begin{vmatrix} \epsilon & 0 & 0 \\ 0 & 1+\epsilon & 2 \\ 0 & 2 & 4+\epsilon \end{vmatrix} \quad \text{and} \quad E = \begin{vmatrix} 1 \\ 3 \\ 4 \end{vmatrix},$$

where $\epsilon \geq 0$. (For $\epsilon = 0$, we are analyzing Example 1.) We can use the first-order conditions to show that for ϵ small (specifically $0 \leq \epsilon \leq 1/3$), the mean variance frontier with no short sales consists of all convex combinations of assets 1 and 2 and all convex combinations of assets 2 and 3.

If $\alpha = (1 - \alpha_2, \alpha_2, 0)$, then let $\gamma = -\alpha_2/2 + 3\epsilon/2 - 2\alpha_2\epsilon$, let $\lambda = \alpha_2/2 - \epsilon/2 + \alpha_2\epsilon$, and let $\delta = (0, 0, \alpha_2/2 + \epsilon/2 - 2\alpha_2\epsilon)$. This satisfies complementary slackness ($\delta_3 \geq 0$), since we are taking $\epsilon \leq 1/3$. For $\alpha = (0, \alpha_2, 1 - \alpha_2)$, let $\gamma = -4 + 2\alpha_2 - 3\epsilon + 7\alpha_2\epsilon$, let $\lambda = 2 - \alpha_2 + \epsilon - 2\alpha_2\epsilon$, and let $\delta = (2 - \alpha_2 + 2\epsilon - 5\alpha_2\epsilon, 0, 0)$. This satisfies complementary slackness ($\delta_1 \geq 0$), again since $\epsilon \leq 1/3$. This exhausts all $\mu \in [\min_i E_i, \max_i E_i] = [1, 4]$, so we have found the whole frontier.

To see that F is kinked at $\mu = 3$ where $\alpha = (0, 1, 0)$, note that $d\sigma^2/d\mu = \lambda$ since σ^2 is the value of problem 1 and λ is the multiplier on the "μ" constraint. Therefore, $\lim_{\mu\uparrow 3}(d\sigma^2/d\mu) = \lim_{\alpha_2\to 1}(\alpha_2/2 - \epsilon/2 + \alpha_2\epsilon) = 1/2 + \epsilon/2$, and $\lim_{\mu\downarrow 3}(d\sigma^2/d\mu) = \lim_{\alpha_2\to 1}(2 - \alpha_2 + \epsilon - 2\alpha_2\epsilon) = 1 - \epsilon$. For $0 \leq \epsilon < 1/3$, $1/2 + \epsilon/2 < 1 - \epsilon$, and there is a kink at $\mu = 3$. For $\epsilon = 1/3$, $1/2 + \epsilon/2 = 1 - \epsilon$, and there is no kink, although $\alpha = (0, 1, 0)$ is an efficient portfolio on the interior of the frontier. This shows that the converse of Theorem 1 is false, as noted in the text. Since V is positive definite in this example, it follows that $\alpha = (0, 1, 0)$ is the unique portfolio on the frontier at $\mu = 3$, so this counterexample to the converse of Theorem 1 and its Corollary does not rely on non-uniqueness of efficient portfolio at α.

REFERENCES

1. E. Elton, M. Gruber, and M. Padberg. "Simple Criteria for Portfolio Selection." *Journal of Finance* 31 (December 1976), 1341–48.
2. R. Roll. "A Critique of the Asset Pricing Theory's Tests." *Journal of Financial Economics* 4 (May 1977), 129–76.
3. S. Ross. "The Capital Asset Pricing Model (CAPM), Short-sale Restrictions and Related Issues." *Journal of Business* 32 (March 1977), 177–83.
4. B. Solnik. "Properties of the Efficient Frontier with Short-sales Restriction." Working paper. Chambre de Commerce et d'Industrie de Paris, 1981.