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The Valuation of Multivariate Contingent Claims in Discrete Time Models

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ABSTRACT

There are several examples in the literature of contingent claims whose payoffs depend on the outcomes of two or more stochastic variables. Familiar cases of such claims include options on a portfolio of options, options whose exercise price is stochastic, and options to exchange one asset for another. This paper derives risk neutral valuation relationships (RNVRs) in a discrete time setting that facilitate the pricing of such complex contingent claims in two specific cases: joint lognormally distributed underlying variables and constant proportional risk aversion on the part of investors, and joint normally distributed underlying variables and constant absolute risk aversion preferences, respectively. This methodology is then applied to the valuation of several interesting complex contingent claims such as multiperiod bonds, multicurrency option bonds, and investment options.

THE EMPHASIS OF THE LITERATURE on options has mostly been on the valuation of claims whose payoffs depend on a single random variable such as the value of the underlying stock. However, there are several examples of contingent claims whose payoffs depend on the outcomes of two or more stochastic variables.[1] An example of such a claim is an option on a portfolio of options. For instance, if the equity of a firm is viewed as a call option, an option on a stock index, which is a portfolio of equity claims, can be viewed as an option on a portfolio of options. Similarly, in the case of a multiperiod bond, where there is some chance of default, the payoffs to the bondholder depend on the future cash flows of the firm and, in general, the bond would be a complex claim contingent on these cash flows with a joint probability distribution governing the stochastic behavior of the latter underlying variables. As will be discussed below, the equity claim, in this case, can be viewed as an option on a portfolio of options and the future cash flows of the firm.

Other examples of such multivariate contingent claims include an option whose exercise price is stochastic;[2] an option to exchange one asset for another;[3]

* Manchester Business School and New York University, respectively. We thank George Constantinides, Kose John, Patricia Smith, and especially David Nachman for helpful comments on an earlier draft of this paper. Subsequent to the writing of the second draft of this paper, we came across Stulz [26] and Sercu [21] who address a problem similar to that analyzed here.

[1] Stulz [26] deals with put and call options on the maximum or minimum of two risky assets. The problem dealt with here is more general since it deals with two or more assets. More importantly, the analysis here applies to contingent claims whose payoffs are *any* function of two or more stochastic variables. Indeed, some of the specific examples dealt with here, e.g., valuation of a multiperiod bond, call for functional relationships that go beyond options on the maximum or minimum of two stochastic variables.

[2] See Fischer [8] for a discussion and approach to the valuation of such a claim.

[3] Margrabe [14] analyzes such an option.

multicurrency option bonds, which give the holder the right to receive the principal (and coupon) payments in either of two currencies with a prespecified exchange rate between them; and an option to invest in one of two or more technologies. In all these cases, the payoffs to the holder of the claim depend on the outcomes of several stochastic variables through some prespecified functional relationship.

There are two possible approaches to the valuation of such complex contingent claims. The first is in the spirit of the Black-Scholes [3] and Merton [15] arbitrage analysis which consists of maintaining a riskless hedge which is self-financing (i.e., with zero net investment) by trading continuously in the underlying asset and the contingent claim. By extension, therefore, the valuation of the complex claims mentioned earlier would require trading in three or more securities, rather than two as was the case with the simple Black-Scholes call option pricing model and its extensions. The specific assets involved would be the contingent claim and assets representing each of the underlying stochastic variables. For instance, in the case of the multicurrency option bond whose payoffs depend on the cash flows of a firm as well as the exchange rate between the two currencies in which it is denominated, the hedge would involve positions in the multicurrency option bond, the equity of the firm, and the spot foreign exchange rate.[4] Once the hedge has been established, one can solve the partial differential equation subject to the boundary conditions as in Black-Scholes [3]. Alternatively, one can choose the simpler solution method which was originally suggested by Cox-Ross [6] to value call options. This involves identifying the variables relevant to the pricing of the claim and then noting that once the riskless hedge can be shown to be feasible, the valuation should be independent of investor attitudes towards risk. It then follows that the solution to the problem can be obtained with an arbitrary specification of investor preferences, in particular, universal risk neutrality under which the equilibrium rate of return on all assets is the riskless rate of interest. The contingent claim can then be priced as the present value of its expected terminal payoffs, discounted at the riskless rate of interest.

An alternative valuation method may be called for in many cases, since continuous trading in the relevant assets may not be possible or, in some cases, the assets may not be traded at all. As has been pointed out by Rubinstein [20] and Brennan [4], this approach calls for the restriction of investor preferences so as to eliminate the need for construction and maintenance of a riskless hedge. It should be mentioned in this context that, in some instances, the basic exogenous variables are cash flows rather than values, so that the stochastic

[4] This is similar in spirit to the approach in Stulz [26]. However, he derives the value of the multicurrency option bond in terms of, among other variables, the value of an option on the minimum of the value of the firm and the discounted value in the numeraire currency of the payments made in the foreign currency for which he provides a closed form solution. In this paper, we deal directly in terms of the exogenous variables, i.e., the value of the firm and the foreign exchange rate. This serves to illustrate a key distinction between the approach in Stulz and ours; he is concerned specifically with options on the minimum and maximum of two stochastic variables, while we deal with *any* contingent claim whose payoff is a function of two or more stochastic variables.

process generating the latter variables cannot be used directly. Since the cash flows are themselves not directly tradeable, the construction of a riskless hedge involving them is not possible in such cases. Further, if the basic exogenous variables are cash flows, it may be possible to incorporate various patterns of uncertainty resolution rather than relying on the assumption of a constant rate of uncertainty resolution that is characteristic of the continuous time approach. This flexibility may be advantageous in the study of many problems in corporate finance. Another advantage of the preference-based approach is that it permits the modelling of heterogenous probability assessments across investors which may be useful for purposes of empirical estimation. This is not feasible under the arbitrage approach since equilibrium does not exist in that case without homogeneous expectations.

It should be emphasized that, in both approaches to the valuation of contingent claims, the key element of the analysis is the derivation of a risk neutral valuation relationship (RNVR), i.e., a formula relating the value of the contingent claim to the value of the underlying assets and other exogenous parameters, compatible with arbitrary preference parameters, in particular with risk neutrality, under which all assets yield the same equilibrium expected rate of return. In the arbitrage approach, this is achieved by construction of a riskless hedge. In the discrete time case, preference restrictions achieve the same result.

The previous applications of the discrete time approach were by Merton [15], Rubinstein [20], and Brennan [4]. The first two papers deal with constant proportional risk aversion and joint lognormal distributions of the underlying asset and aggregate wealth. In an extension of this work, Brennan [4] shows first that constant proportional risk aversion on the part of the representative investor is both necessary and sufficient for a RNVR to hold for arbitrary bivariate lognormal distributions of the underlying asset and aggregate wealth. He also derives a similar result for the case of joint normally distributed returns, for which constant absolute risk aversion preferences for the representative investor are shown to be necessary and sufficient.

The purpose of this paper is to generalize the analysis of Brennan [4] to the case of complex contingent claims where the payoffs are dependent on two or more stochastic variables. In particular, we show that if the joint distribution of returns of the underlying stochastic variables and aggregate wealth is multivariate normal, then the necessary and sufficient condition for a RNVR to hold is that the representative investor exhibits constant absolute risk aversion. A similar result holds for multivariate lognormality of the stochastic variables and aggregate wealth with constant proportional risk aversion on the part of the investor. These RNVRs are then applied to several specific cases to value complex contingent claims using the appropriate boundary conditions.

Section I of the paper presents the basic valuation model for any claim under the assumption that the law of one price holds. In the next section, the main results of the paper dealing with multivariate risk neutral valuation relationships (RNVRs) are derived. In Section III, we apply this basic approach to several interesting cases of complex contingent claims to illustrate the general applicability of the methodology.

I. A General Valuation Model under Aggregation

A general preference-based valuation model can be obtained if the law of one price holds under which two assets with identical payoffs in all future states are priced equally. The value of any claim dependent on $\underline{X}' = [X_1, X_2 \cdots X_n]$, the n underlying stochastic variables, can be written as

$$V = E[C(\underline{X})\theta] \tag{1}$$

where $C(\underline{X})$ specifies the functional relationship between the payoffs to the claim, C, and the n underlying stochastic variables, $X_1, X_2 \cdots X_n$, and θ is a random variable representing the pricing function for the random payoffs.[5] Equation (1) may be derived from the no-arbitrage condition implied by the law of one price (for example, using Farkas' Lemma as in Garman and Ohlson [9]) or from the first-order conditions of the representative investor, if the aggregation problem has been solved.[6] In the latter instance, the variable θ can be related to investors' preferences as follows.

In a single-period world, the representative investor maximizes his expected utility of end-of-period wealth[7,8], $E[U(w)]$, where

$$w = w_0 r + \underline{v}'[\underline{C}(\underline{X}) - \underline{P}] \tag{2}$$

$U(\cdot)$ is his utility for end-of-period wealth
w_0 is his initial wealth
r is the riskless rate of interest
$\underline{v}$ is his vector of demands for units of assets, $j = 1, 2, \cdots N$
$\underline{C}(\underline{X})$ is the vector of end-of-period payoffs associated with the contingent claims as a function of the underlying variables, $\underline{X}' = [X_1, X_2 \cdots X_n]$. In particular, these claims could be some linear combination of the underlying risks $\underline{X}$, for example, X_j, and
$\underline{P}$ is the vector of current prices of the claims $\underline{C}(\underline{X})$.

[5] This is a straightforward extension of Beja [1].

[6] The aggregation problem is solved if the equilibrium asset prices are independent of the distribution of wealth across investors. The conditions for aggregation are that either all investors have identical cautiousness and beliefs or all investors have exponential utility. See Rubinstein [19] and Brennan and Kraus [5] for details.

[7] The initial consumption decision is assumed here to have been taken already. Inclusion of this decision in the analysis would allow us to solve endogenously for the riskless rate of interest if aggregate consumption is specified. Since our concern here is to price the end-of-period payoffs associated with contingent claims in terms of observable variables including the riskless rate, we will take the latter as given exogenously.

[8] Strictly speaking, it is not necessary to assume a single-period model. In an analogous multiperiod framework, we may think of $U(c)$ as the direct utility of consumption, where c is the consumption at the end of the current period. Thus, we may allow for consumption at various points in time, both before and after the receipt of the cash flows from securities discussed below. This more general approach calls for referring to the direct utility of consumption $U(c)$ rather than the derived utility of wealth $U(w)$ discussed in the text of the paper. Also, the distributional assumptions discussed in the paper would then apply to c rather than w.

Assuming the usual regularity conditions, the requirement for a maximum is[9]

$$E[U'(w)\underline{C}(\underline{X})] = r\underline{P}E[U'(w)]$$

or

$$\underline{P} = r^{-1}\frac{E[U'(w)\underline{C}(\underline{X})]}{E[U'(w)]} \tag{3}$$

Since the economy is composed of several such representative investors, equilibrium prices can be obtained by aggregating their demands and clearing the market.

The current price of individual claims can be written in terms of their end-of-period payoffs as follows:

$$P = r^{-1}\frac{E[U'(w)\underline{C}(\underline{X})]}{E[U'(w)]}$$

$$= r^{-1}E[C(\underline{X})Z(\underline{X})]$$

where

$$Z(\underline{X}) = \frac{E[U'(w \mid \underline{X})]}{E[U'(w)]} \tag{4}$$

is the valuation function that can be defined in terms of the (relative) marginal utility of end-of-period wealth, conditional on the stochastic variables $\underline{X}$. In particular, the underlying risks $\underline{X}$ can themselves be priced by a particular application of this general formula,

$$\underline{V} = r^{-1}E[\underline{X}Z(\underline{X})] \tag{5}$$

where $\underline{V}$ is the vector of values of the underlying risks.

These formulae are simple extensions of the valuation relationship originally proposed by Beja [1], and used subsequently by Rubinstein [20] and Brennan [4], among others. In the following analysis, we will use the general multivariate valuation formula, (4), to study specific cases of risk neutral valuation relationships.

II. Multivariate Risk Neutral Valuation Relationships

We assume that in this economy, the payoffs of any asset and, in particular, the contingent claims being valued, are dependent, directly or indirectly on the vector of underlying stochastic variables, $\underline{X}$. From Equation (4), the value of the contingent claim is thus given by[10]

[9] The notation used here is shorthand for the more standard but cumbersome notation for the marginal utilities. The marginal utility of wealth $U'(w)$ should have as arguments both w and $\underline{X}$. Similarly, $E[U'(w)]$ and $E[U'(w \mid \underline{X})]$ are convenient, if somewhat inelegant representations of $E_{x,w}[U'(w;\underline{X})]$ and $E_w[U'(w;\underline{X}) \mid \underline{X}]$, respectively.

[10] This is the value of a particular contingent claim, say j, with the subscript dropped.

$$P = r^{-1}E[C(\underline{X})Z(\underline{X})]$$

$$= r^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} C(X_1, X_2 \cdots X_n)Z(X_1, X_2 \cdots X_n)$$

$$\cdot\phi(X_1, X_2 \cdots X_n)\, dX_1\, dX_2 \cdots dX_n \tag{6}$$

where $\phi(X_1, X_2 \cdots X_n)$ is the joint density function of $X_1, X_2 \cdots X_n$. It should be emphasized that, in addition to the payoff function of the contingent claim and the joint density function of the underlying stochastic variables, the above expression contains preference-specific information in the term $Z(\underline{X}) = Z(X_1, \cdots X_n)$.

If a RNVR exists, the valuation formula can be rewritten so that no information about preferences is required,

$$P = r^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} C(X_1 \cdots X_n)$$

$$\cdot\phi'(X_1, X_2 \cdots X_n)\, dX_1 \cdots dX_n \tag{7}$$

where $\phi'(X_1, \cdots X_n)$ is the modified joint density function involving a shift in the underlying parameters of the distribution.[11] The existence of a RNVR considerably simplifies the valuation of the contingent claim, since, under universal risk neutrality, the claim can be valued by discounting the expected value of its payoffs at the riskfree rate r as in Equation (7) above. We now turn to the derivation of two theorems which are generalizations of earlier results by Brennan [4].

A. The Multivariate Normal Distribution

THEOREM I. *A necessary and sufficient condition for a RNVR to hold under the assumption of an arbitrary multivariate normal distribution of the underlying stochastic variables and aggregate wealth is that the utility function of the representative investor exhibits constant absolute risk aversion.*

Proof of Sufficiency: If the utility function has constant absolute risk aversion, it is exponential and, in particular, the marginal utility can be written as

$$U'(w) = \exp(\alpha w) \tag{8}$$

[11] It should be noted that for sufficiency, the RNVRs derived below hold for every class of function C for which the integrals (6) and (7) are well-defined.

In his analysis, Brennan [4] modifies the distribution from ϕ to ϕ', the risk neutral density function, so as to shift the location parameter by the ratio of the mean of the distribution of the rate of return on the underlying asset to the riskless rate of interest. While this happens to be correct in this case, this does not follow from the definition of a RNVR. All that is required, in general, is that the modification be such that the resultant density function involves only "objective" information and no preference parameters. A linear shift happens to be appropriate given the linearity property of normally distributed random variables and the linear adjustment for the certainty equivalent in the CAPM given in (15).

In this paper, we do not address the more difficult problem of risk neutral shifts that involve changes to the character of the underlying distribution. In other words, we impose the restriction that $\phi(\cdot)$ and $\phi'(\cdot)$ belong to the same family of distributions, e.g., normal.

where α is negative, indicating risk aversion. Thus, the conditional relative marginal utility function can be written as

$$Z(\underline{X}) = \frac{E[U'(w \mid \underline{X})]}{E[U'(w)]}$$

$$= \frac{E[\exp(\alpha w \mid \underline{X})]}{E[\exp(\alpha w)]} \tag{9}$$

after substituting in Equation (4).

Since the joint distribution of $(w, X_1, X_2 \cdots X_n)$ is multivariate normal, the conditional distribution of $(w \mid X_1, X_2 \cdots X_n)$ is normal.[12] Specifically, if the joint distribution of $(X_1, X_2 \cdots X_n)$ is multivariate normal with

$$\phi(X_1, X_2 \cdots X_n) = N(\underline{\mu}, \Sigma) \tag{10}$$

where $\underline{\mu}$ is the vector of means of the $\underline{X}$ and Σ is the variance-covariance matrix, the conditional distribution of $(w \mid \underline{X})$ can be written as

$$\phi(w \mid \underline{X}) = N[\mu_w + \Sigma_{wx}\Sigma^{-1}(\underline{X} - \underline{\mu}),\quad \sigma_w^2 - \Sigma_{wx}\Sigma^{-1}\Sigma_{xw}] \tag{11}$$

where μ_w and σ_w^2 are the unconditional mean and variance of w and Σ_{wx} is the vector of covariances between w and $\underline{X}$, $[\sigma_{w1}, \sigma_{w2} \cdots \sigma_{wn}]$. Substituting for $\phi(w \mid \underline{X})$ and $\phi(w)$ and evaluating Equation (9) yields

$$Z(X_1, \cdots X_n) = \exp\left[\alpha\Sigma_{wx}\Sigma^{-1}(\underline{X} - \underline{\mu}) - \frac{\alpha^2}{2}\Sigma_{wx}\Sigma^{-1}\Sigma_{xw}\right] \tag{12}$$

Using the above equation in the expression for value, (6), with

$$\phi(X_1, \cdots X_n) = \frac{1}{(2\pi)^{n/2}\mid\Sigma\mid^{1/2}}\exp\left[-\frac{1}{2}(\underline{X} - \underline{\mu})'\Sigma^{-1}(\underline{X} - \underline{\mu})\right]$$

where $\mid\Sigma\mid$ is the determinant of Σ, yields after substitution and simplification,

$$P = r^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{C(\underline{X})}{(2\pi)^{n/2}\mid\Sigma\mid^{1/2}}\exp\left[-\frac{1}{2}\{\underline{X} - (\underline{\mu} + \alpha\Sigma_{wx})\}'\Sigma^{-1}\{\underline{X} - (\underline{\mu} + \alpha\Sigma_{wx})\}\right]dX_1 \cdots dX_n \tag{13}$$

This pricing relationship can be used to value any asset whose payoffs depend on the outcomes of the stochastic variables $X_1, \cdots X_n$. In particular, the risks $\underline{X} = (X_1, X_2 \cdots X_n)$ can themselves be priced using (6) and (13) above as in Equation (5), so that

[12] It is assumed implicitly that either the contingent claims $C(\underline{X})$ are in zero net supply (i.e., like put and call options bought and written between investors) or that the cash flow stream $\underline{X}$ is split up according to predetermined rules to form two or more contingent claims (i.e., similar to debt and equity in a no-tax frictionless economy).

$$\underline{V} = r^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{\underline{X}}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2}\{\underline{X} - (\underline{\mu} + \alpha \Sigma_{wx})\}' \Sigma^{-1} \{\underline{X} - (\underline{\mu} + \alpha \Sigma_{wx})\}\right] dX_1 \cdots dX_n \tag{14}$$

where the integral is the vector of the means of the multivariate normal distribution with a shift of $\alpha \Sigma_{wx}$ so that

$$\underline{V} = r^{-1}[\underline{\mu} + \alpha \Sigma_{wx}] \tag{15}$$

where α is a (negative) parameter of risk aversion.

It should be noted that (15) is the standard mean-variance capital asset pricing model of Sharpe [22], Lintner [13], and Mossin [17], where the prices of the stochastic variables $\underline{X}$ are the discounted values of the certainty equivalents of end-of-period cash flows. The certainty equivalents, in turn, are the expected cash flows less the price of risk times the measure of risk which is the vector of covariances with aggregate wealth.

Substituting (15) back in (13) results in

$$P = r^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{C(\underline{X})}{(2\pi)^{n/2} |\Sigma|^{1/2}} \cdot \exp\left[-\frac{1}{2}\{\underline{X} - r\underline{V}\}' \Sigma^{-1} \{\underline{X} - r\underline{V}\}\right] dX_1 \cdots dX_n \tag{16}$$

which is a RNVR, since no information regarding preferences is required for its evaluation.

For the proof of necessity, see the Appendix.

B. The Multivariate Lognormal Distribution

An analog of the above result can be derived when the joint distribution of wealth and the underlying stochastic variables is multivariate lognormal.

THEOREM II. *A necessary and sufficient condition for a RNVR to hold under the assumption of an arbitrary multivariate lognormal distribution of the underlying stochastic variables and aggregate wealth is that the utility function of the representative investor exhibits constant proportional risk aversion.*

Proof of Sufficiency: If the utility function exhibits constant proportional risk aversion, the marginal utility function is of the form

$$U'(w) = w^{\gamma} \tag{17}$$

From (4), the conditional marginal utility function can be written as

$$Z(\underline{X}) = \frac{E[U'(w \mid \underline{X})]}{E[U'(w)]} = \frac{E[w^{\gamma} \mid \underline{X}]}{E[w^{\gamma}]} \tag{18}$$

Since the joint distribution of $(w, X_1 \cdots X_n)$ is multivariate lognormal, the conditional and marginal distributions of $\underline{X}$, w, and w^{γ} are lognormal, with

$$\phi(X_1, X_2 \cdots X_n) = L(\underline{\mu}, \Sigma) \tag{19}$$

where $\underline{\mu}$, and Σ are the mean vector and variance-covariance matrix of the underlying multivariate normal distribution. Hence,

$$\phi(w \mid \underline{X}) = L[\mu_w + \Sigma_{wx}\Sigma^{-1}(\underline{\ln X} - \underline{\mu}), \quad \sigma_w^2 - \Sigma_{wx}\Sigma^{-1}\Sigma_{xw}] \tag{20}$$

where μ_w, $\sigma_w{}^2$, and Σ_{wx} refer to the parameters of the underlying multivariate normal distribution of $\underline{\ln X}' = [\ln X_1, \ln X_2 \cdots \ln X_n]$.

Substituting for $\phi(w \mid \underline{X})$ and $\phi(w)$ and evaluating Equation (18) yields

$$Z(X_1, \cdots X_n) =$$

$$\exp\left[-\gamma\Sigma_{wx}\Sigma^{-1}\ \underline{\mu} - \frac{\gamma^2}{2}\ \Sigma_{wx}\Sigma^{-1}\Sigma_{xw} + \gamma\Sigma_{wx}\Sigma^{-1}\ \underline{\ln X}\right] \tag{21}$$

Using the above relation in the expression for value, (6), with

$$\phi(X_1, \cdots X_n) = \frac{1}{(2\pi)^{1/2}\,|\,\Sigma\,|^{1/2}\ \Pi_1^n X_j} \exp\left[-\frac{1}{2}(\underline{\ln X} - \underline{\mu})'\Sigma^{-1}(\underline{\ln X} - \underline{\mu})\right]$$

leads after completing the square and simplifying to

$$P = r^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \frac{C(\underline{X})}{(2\pi)^{n/2}\,|\,\Sigma\,|^{1/2}\Pi_1^n X_j}\exp\left[-\frac{1}{2}\{\underline{\ln X}\right.$$

$$\left. - (\underline{\mu} + \gamma\Sigma_{wx})\}'\Sigma^{-1}\{\underline{\ln X} - (\underline{\mu} + \gamma\Sigma_{wx})\}\right] dX_1 \cdots dX_n \tag{22}$$

The above formula can be used to value any asset whose payoffs depend on the stochastic variables, $X_1, \cdots X_n$, and, in particular, the risks $\underline{X}' = [X_1, X_2 \cdots X_n]$ themselves, as in Equation (5). This implies that

$$\underline{V} = r^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \frac{\underline{X}}{(2\pi)^{n/2}\,|\,\Sigma\,|^{1/2}\Pi_1^n X_j}\exp\left[-\frac{1}{2}\{\underline{\ln X}\right.$$

$$\left. + (\underline{\mu} + \gamma\Sigma_{wx})\}'\Sigma^{-1}\{\underline{\ln X} - (\underline{\mu} + \gamma\Sigma_{wx})\}\right] dX \cdots dX_n \tag{23}$$

where the integral is the vector of means of the multivariate lognormal distribution shifted by $\gamma\Sigma_{wx}$, so that

$$\underline{V} = r^{-1}[\underline{\exp L}] \tag{24}$$

where

$$\underline{\exp L} = \begin{bmatrix} \exp(\mu_1 + \sigma_1^2/2 + \gamma\sigma_{w1}) \\ \exp(\mu_2 + \sigma_2^2/2 + \gamma\sigma_{w2}) \\ \cdot \\ \cdot \\ \cdot \\ \exp(\mu_n + \sigma_n^2/2 + \gamma\sigma_{wn}) \end{bmatrix}$$

Equation (24) is the market equilibrium valuation relationship with lognormally distributed cash flows, which is analogous to the mean-variance capital asset pricing model of Equation (15).

Substituting (24) back in (22) and simplifying, yields

$$P = r^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{C(\underline{X})}{(2\pi)^{n/2} |\Sigma|^{1/2} \Pi_1^n X_j} \exp\left[-\frac{1}{2} \left\{ \underline{\ln X} - \ln r\underline{V} + \frac{1}{2} \underline{S} \right\}' \Sigma^{-1} \left\{ \underline{\ln X} - \ln r\underline{V} + \frac{1}{2} \underline{S} \right\} \right] dX_1 \cdots dX_n \quad (25)$$

where $S' = [\sigma_1^2, \sigma_2^2 \cdots \sigma_n^2]$, which is a RNVR since the relationship is independent of preference parameters.

For a proof of necessity see the Appendix.[13]

III. Applications of Multivariate Contingent Claims Pricing Theory

The general results of the previous section can be applied to the valuation of any complex contingent claim whose payoffs depend on two or more stochastic variables. In what follows, we illustrate the use of this methodology in certain interesting valuation problems.

A. An Option on a Mutual Fund

Consider the case of a call option on a mutual fund of equities, for example, the capital shares of a dual fund or an option on a stock index. In a single-period context, it is well known that the equities in the mutual-fund can themselves be viewed as simple call options, i.e.,

$$S_j = V\{\mathrm{Max}[(X_j - Y_j), 0]\} \quad (26)$$

where S_j is the value of stock j and is given by the value of the maximum of the end-of-period cash flows of the firm, X_j, less the promised payments to bondholders, Y_j, or zero. The notation $V\{\ \}$ indicates "the value of $\{\ \}$." Now, a portfolio of such securities can be valued as

$$M = \textstyle\sum_{j=1}^n w_j S_j = \sum_{j=1}^n w_j V\{\mathrm{Max}[(X_j - Y_j), 0]\} \quad (27)$$

where M is the value of the portfolio and w_j is the proportion invested in asset j. A call option on such a portfolio can be valued as

$$C = V\{\mathrm{Max}[\textstyle\sum_{j=1}^n w_j V[\mathrm{Max}\{(X_j - Y_j), 0\}] - K, 0]\} \quad (28)$$

where C is the value of the call option, M is the value of the portfolio, and K is the exercise price of the option.

Suppose the end-of-period joint distribution of firm values is multivariate

[13] This is the shift necessary to derive a valuation model with lognormally distributed variables. In that sense, it is a specific RNVR and the comments in footnote 11 apply here as well.

lognormal. With constant proportional risk aversion on the part of the representative investor, we can write the value of each equity claim as a RNVR from Theorem II.

$$S_j = r^{-1} \int_{Y_j}^{\infty} (X_j - Y_j)\phi'(X_j)\, dX_j \tag{29}$$

where

$$\phi'(X_j) = \frac{1}{\sqrt{2\pi}\sigma_j X_j} \exp\left[-\frac{1}{2}\left\{\frac{\ln X_j - rV_j}{\sigma_j}\right\}^2\right]$$

and V_j is the value of the end-of-period cash flow of the firm. Equation (29) above leads to the Black-Scholes formula as shown in Rubinstein [20].

Substituting equity values from (29) in (28), the call option on the mutual fund can be valued. For example, if there are two stocks 1 and 2 in the portfolio, a call with exercise price K on the portfolio has a value

$$C = r^{-1} \int_{Y_1}^{\infty} \int_{Y_2}^{\infty} \mathrm{Max}[\{w_1(X_1 - Y_1) + w_2(X_2 - Y_2) - K\}, 0]\phi'(X_1, X_2)\, dX_2\, dX_1 \tag{30}$$

where

$$\phi'(X_1, X_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)X_1X_2} \exp\left\{-\frac{1}{2(1-\rho^2)}A\right\}$$

and

$$A = [\{(\ln X_1 - A_1)/\sigma_1\}^2 - 2\rho\{(\ln X_1 - A_1)/\sigma_1\}\{(\ln X_2 - A_2)/\sigma_2\} + \{(\ln X_2 - A_2)/\sigma_2\}^2]$$

$$A_j = \ln r + \ln V_j - \sigma_j^2/2, \qquad j = 1, 2$$

B. Corporate Bonds

A bond specifies an obligation to make a series of payments $Y_1, Y_2 \cdots Y_n$ at time $t = 1, 2 \cdots n$, whose security is represented by the cash flows generated by the firm $X_1, X_2 \cdots X_n$. The value of such a bond in the one-period (univariate) case, where all the X_t's apart from the terminal cash flow X_n are zero, has been derived by Black-Cox [2] and Geske [10, 11] extending previous work on discount bonds by Merton [16]. In the general case where X_t, $t < n$, are nonzero and can be paid out, subject to bond covenants, in dividends, the valuation problem involves call options on a multivariate stochastic process and can be solved using the results derived in Section II. We will illustrate the general solution with a two-period example.

Since we are interested in valuing the claim on Y_1 and Y_2 together, the resultant valuation is independent of questions of seniority between them. Thus, in the general case, the obligation Y_2 can be modelled as a claim which has first claim over the cash flow X_2. The residual equity value of the claim against X_2 as of time 1, S_{12}, together with the cash flow X_1, therefore, forms the security for the claim Y_1. Thus, as in the single-period case, the equity value at the end of the first period is a simple call option whose value at time 1 is

$$S_{12} = V_1[\text{Max}(X_2 - Y_2, 0)] \tag{31}$$

where V_1 indicates the value at time 1 of the residual cash flows at time 2. The value of the equity at time 0 can then be modelled as a call option against the sum of X_1 and S_{12}. Specifically,[14]

$$S_0 = V_0[\text{Max}(S_{12} + X_1 - Y_2, 0)] \tag{32}$$

where S_0 is the equity value of cash flows X_1 and X_2 at time 0. The value of the bonds or any other multiperiod contingent claim can be determined in a similar manner, with appropriate boundary conditions.

In general, it is necessary to specify the stochastic processes governing S_{12} and X_1 in order to value the equity at time 0. One possibility is to assume multivariate joint normality of the cash flows and constant absolute risk aversion investors as in Stapleton and Subrahmanyam [23]. In this case, it can be shown that viewed from time 0, the joint distribution of the value of the second-period cash flow X_2 as of time 1 (conditional on the information set at time 1), V_{12}, the cash flow, X_1, and the aggregate wealth at time 1, W_1, is multivariate normal. A similar argument can be made for the cash flow X_2 and aggregate wealth at the end of period 2 viewed from time 1, W_2. Further, the derived utility of wealth functions at time 1 and 2 can be shown to have constant absolute risk aversion, with suitably modified parameters. In this case, with multivariate normality and constant absolute risk aversion at each stage, we can write the value of the equity or, for that matter, any contingent claim as a RNVR from Theorem I.

Given the state of the world at time 1, ψ_1, and, in particular, X_1, we can determine both S_{12} and V_{12}, since the latter is simply S_{12} with $Y_2 = 0$, i.e., the total value of the cash flow X_2, conditional on the information set ψ_1. Hence, S_{12} can itself be expressed as a function of the total value V_{12} and the value of the

[14] This hinges on the accuracy of the definition of bankruptcy used in Geske [10, 11] and in Stapleton and Subrahmanyam [24], under which the firm is in default if it is unable to issue the new equity whose proceeds are necessary to make required payments to bondholders. This implicitly assumes that the firm is unable to reverse earlier investment decisions and sell existing assets for cash. In turn, this would be consistent with a stringent debt covenant whereby long term bondholders (claimants on Y_2) have priority over the long term cash flow. If the covenants are any looser or if the seniority between the bonds is not at issue, it is possible for the firm to liquidate at least a part of the assets producing the cash flow X_2 before declaring bankruptcy. If the resale value of the assets at time 1 is q times the market value V_{12}, then

$$S_0 = V_0[\text{Max } q \cdot V_{12} + X_1 - Y_1, 0]$$

Of course, this reduces the probability that the bondholders Y_2 will be paid and hence would be appropriately reflected in their value. Since our concern in this paper is with the general principles underlying the valuation of complex contingent claims, we do not go into the details of the effects of various types of covenants on equity valuation.

equity at time 0 can be written as a RNVR so that[15]

$$S_0 = r_1^{-1} \int_{-\infty}^{+\infty} \int_{\bar{V}_{12}}^{+\infty} (S_{12} + X_1 - Y_1)\phi'(V_{12}, X_1)\, dV_{12}\, dX_1 \tag{33}$$

where $\bar{V}_{12}$ is the breakeven value of the firm at which the equity value, $S_{12} = (Y_1 - X_1)$, is just sufficient to meet the shortfall in the payment to the bondholder after utilizing the cash flow from the current period, and $\phi'(\cdot)$ is the risk neutral density computed as in Equation (16). Specifically, if the joint density function of X_1 and X_2 is normal, the risk neutral valuation relationship now becomes

$$S_0 = r_1^{-1} \int_{-\infty}^{+\infty} \left[\int_{\bar{V}_{12}}^{+\infty} (S_{12} + X_1 - Y_1)\phi'(V_{12} \mid X_1)\, dV_{12} \mid X_1 \right] \phi'(X_1)\, dX_1 \tag{34}$$

where $\phi'(X_1)$ is the marginal risk neutral density of X_1 and $\phi'(V_{12} \mid X_1)$ is the conditional risk neutral density of V_{12} given X_1.

By applying Theorem I to this case we can write,

$$\phi'(X_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp - \left[\frac{1}{2\sigma_1^2}(X_1 - r_1 V_{01}) \right] \tag{35}$$

where V_{01} is the value of cash flow X_1 at time 1, and

$$\phi'(V_{12} \mid X_1) = \frac{1}{\sqrt{2\pi}\sigma(V_{12} \mid X_1)} \exp - \left[\frac{1}{2\sigma^2(V_{12} \mid X_1)} \{(V_{12} \mid X_1) - r_1 V_{02}\} \right] \tag{36}$$

where $\sigma(V_{12} \mid X_1)$ is the standard deviation of V_{12} conditional on X_1, and V_{02} is the period 0 value of V_{12} (which in turn is the value of X_2 at time 1).

Given the value of the equity in Equation (34), it is possible to derive the value of the risky debt on a no-tax frictionless world via the Modigliani-Miller theorem, given the value of the cash flows of the firm. From Stapleton and Subrahmanyam

[15] Note that since X_1 and X_2 are joint normally distributed by assumption, S_{12}, which is a call option on X_2, is the value of a truncated normal distribution. Hence the joint distribution of X_1 and S_{12} cannot be written down analytically. However, since the relationship between S_{12} and V_{12} is one-to-one, we can deal with the joint distribution of X_1 and V_{12} which, in this case, is joint normal. This allows the valuation of S_0 to be in accordance with the RNVR of Theorem I. This illustrates an important feature of the approach derived here. So long as the underlying stochastic variables are joint normally or joint lognormally distributed, one can value contingent claims on derived assets which are not normally or lognormally distributed. This is because the claims can in turn be redefined as (nonlinear) contingent claims on the underlying stochastic variables, allowing us to use the RNVR result of Theorem I.

This is in contrast to the continuous time approach where the riskless hedge involving such nonlinear contingent claims would be complex. For example, in Stulz [26, p. 181], he analyzes the problem of valuing secured debt, where the payoffs to the security are a function of two stochastic variables. In his case, however, the probability distributions of these two variables are complex functions of the underlying lognormal distribution of the value of the firm's assets, and hence he is unable to write down the value of the option. However, using the approach of this paper, it is possible to value the option as a RNVR, since it is sufficient that the underlying distribution be lognormal; even though the payoffs of the option are dependent on nonlinear functions of this variable, the functional relationships can be handled through the limits of the integration.

[23], given the assumptions of joint normality and CARA preferences, the values of the cash flows X_1 and X_2, respectively, can be written as

$$V_{01} = r_1^{-1}[\mu_1 - \lambda_1\{\mathrm{Cov}(X_1, X_1^M + X_2^M r_2^{-1})\}]$$

and

$$V_{02} = r_1^{-1}[\mu_2 r_2^{-1} - \lambda_1\{\mathrm{Cov}(X_2 r_2^{-1}, X_1^M + X_2^M r_2^{-1})\}] \tag{37}$$

where X_1^M and X_2^M are the cash flows from the market portfolio at time 1 and 2, respectively. The value of the debt obligations is, therefore, $V_{01} + V_{02} - S_0$. Given the promised payments Y_1 and Y_2, the promised yield to maturity can be computed by the usual methods.

C. The Multicurrency Option Bond

Another interesting application of this methodology is the valuation of multicurrency option bonds.[16] These bonds, which are traded in the Euro-markets, have payoffs denominated in one of two or more currencies at the option of the bondholder. Thus, in a pure discount version of such an instrument, for example, a firm may undertake to pay either \$1000 or £ 500 at the maturity of the bond. If, at that time, the exchange rate is greater than £ 1 = \$2, it pays for the bondholder to choose the sterling basis for the payoffs. Of course, if the bonds are subject to default risk as well, the available cash flows of the firm impose an additional constraint. Thus, the payoffs to the bondholder can be written as

$$\mathrm{Min}[\mathrm{Max}(B, BX_1/e), X_2] \tag{38}$$

where X_1 and X_2 are the exchange rate between the two currencies and the cash flow of the firm, respectively, at the maturity of the bond, B is the promised payment in the base currency, and e is the exchange rate specified in the contract.

In the absence of default risk, Feiger and Jacquillat [7] show this instrument to be a combination of a straight bond denominated in the base currency and an option contract on the foreign exchange rate. The introduction of default risk considerably complicates the valuation problem, since the payoffs to the bondholder are dependent on states of the world defined by the outcomes of the two stochastic variables X_1 and X_2.

If the cash flow of the firm and the foreign exchange rate are joint lognormally distributed, the value of the bond can be written as a RNVR using Theorem II.

$$P = \int_0^{+\infty}\int_0^{+\infty} \mathrm{Min}[\mathrm{Max}(B, BX_1/e), X_2]\phi'(X_1, X_2)\, dX_1\, dX_2 \tag{39}$$

where $\phi'(X_1, X_2)$ is the risk neutral density function involving (X_1, X_2). This

[16] As mentioned in footnote 4, this problem has been analyzed independently by Stulz [26]. The distinction between the approach here and in Stulz's paper is that while we derive the valuation of such a contingent claim directly, he writes it in terms of a riskless bond, a simple put option and a call option, the minimum of the value of the firm, and the discounted value in the numeraire currency of the payments made in the foreign currency option is chosen. Both approaches are equally valid.

can be rewritten after substituting the boundary conditions into the limits of integration as

$$P = \int_0^{+\infty} \int_0^{+\infty} \mathrm{Min}[\mathrm{Max}(B, BX_1/e), X_2]\phi'(X_1, X_2)\, dX_1\, dX_2 \tag{39}$$

$$+ \int_{BX_1/e}^{+\infty} \int_e^{+\infty} \frac{BX_1}{e} \phi'(X_1, X_2)\, dX_1\, dX_2$$

$$+ \int_B^{BX_1/e} \int_e^{+\infty} X_2\phi'(X_1, X_2)\, dX_1\, dX_2 \tag{40}$$

If the distribution of X_1 and X_2 is joint lognormal, $\phi'(X_1, X_2)$ can be written as in Equation (30) except that σ_1 and σ_2 are the standard deviations of the normal variates $\ln X_1$ and $\ln X_2$, respectively, and ρ is the correlation between them.

A case similar in principle to the multicurrency option bond was discussed in Stapleton and Subrahmanyam [25]. Consider the case of a bond issued by a government.[17] In a world of inflation, the payoffs on the bond are

$$Y = \min(B, X\pi) \tag{41}$$

where B is the promised obligation in nominal terms, X is the part of output available for payment of the loan (after meeting consumption and investment needs), and π is the (relative) price level. In real terms, the payoffs are

$$Y' = \pi^{-1}\min(B, X\pi) = \min(B\pi^{-1}, X) \tag{42}$$

This can be modelled as writing a covered call option with an uncertain excercise price as in Fischer [8] if the price level and output are joint lognormal, with constant proportional risk aversion on the part of the representative investor.

The price of the bond is

$$P = r^{-1} \int_0^{\infty} \left\{ \int_0^{B\Pi^{-1}} X + \int_{B\Pi^{-1}}^{\infty} BP^{-1} \right\} \phi'(X, \pi)\, dX\, d\pi \tag{43}$$

where $\phi'(X, \pi)$ is the risk neutral joint density function of the two stochastic variables from Theorem II, as in Equation (30) above.

D. The Investment Option

A firm investing in new capital equipment frequently obtains the option to use the equipment to produce two or more commodities. Similarly, additional investment in research allows the firm to produce a new product using the most

[17] Stapleton and Subrahmanyam [25] focuss on special cases of such bonds and analyze the determinants of inflation uncertainty. Most of these cases reduce to complex claims of the type analyzed here.

economic of two or more technologies.[18] In both cases, the firm owns a complex option whose payoffs can be written as

$$Y = \text{Max}[(X_1 - Y_1), (X_2 - Y_2), 0] \tag{44}$$

where X_1 and X_2 are the revenues from the alternatives 1 and 2, which can be obtained after additional expenditures Y_1 and Y_2, respectively.

The value of this investment option can be computed using the RNVR principle. If X_1 and X_2 are joint normally distributed, the value can be written as

$$\begin{aligned} P = &\int_{-\infty}^{Y_2} \int_{Y_1}^{\infty} (X_1 - Y_1)\, \phi'(X_1, X_2)\, dX_1\, dX_2 \\ &+ \int_{Y_2}^{+\infty} \int_{-\infty}^{Y_1} (X_2 - Y_2)\phi'(X_1, X_2)\, dX_1\, dX_2 \\ &+ \int_{Y_2}^{+\infty} \int_{Y_1+(X_2-Y_2)}^{+\infty} (X_1 - Y_1)\phi'(X_1, X_2)\, dX_1\, dX_2 \\ &+ \int_{Y_2+(X_1-Y_1)}^{+\infty} \int_{Y_1}^{+\infty} (X_2 - Y_2)\phi'(X_1, X_2)\, dX_1\, dX_2 \end{aligned} \tag{45}$$

where $\phi'(X_1, X_2)$ is the risk neutral density function for the joint normal case and where σ_1 and σ_2 are the standard deviations of the cash flows X_1 and X_2 and ρ is the correlation between them. The values V_1 and V_2, the values of the underlying stochastic variables, can be obtained from the mean-variance capital asset pricing model as in Equation (15).

E. Other Complex Contingent Claims

In all the above cases, the complex option is a nontrivial extension of simple contingent claim valuation such as call option pricing. This is because the value of the option in all these cases is of the form

$$P = r^{-1} \int_{f_1(X_1)}^{\infty} \int_{f_2(X_2)}^{\infty} C(X_1, X_2)\phi'(X_1, X_2)\, dX_1\, dX_2 \tag{46}$$

where $C(X_1, X_2)$ is the functional relationship between the payoffs on the claim and X_1 and X_2 and underlying stochastic variables, and $\phi'(X_1, X_2)$ is the risk neutral density function. The r^{-1} will appear in the formula when X_1 and X_2 refer to end-of-period stochastic variables. The complication arises because the limits of integration may involve functions $f_1(\cdot)$ and $f_2(\cdot)$ of the variables X_1 and X_2 themselves. In a few cases, however, this difficulty does not arise and the complex claim can be simplified and rewritten with a change of variables as a

[18] Myers [18] views such investment opportunities as call options and shows that levered firms that own such options have an incentive to underinvest in some future states of the world. A variant of this case, which they refer to as "capital convertibility" is analyzed in Grier and Tennenbein [12], using the assumption of universal risk neutrality.

simple contingent claim of the Black-Scholes type. We will illustrate the simpler case using two examples.

The first case analyzed previously by Fischer [8] involves an option with an uncertain exercise price, where the payoffs to the holder of a European call and put option, respectively, can be written as

$$Y = \text{Max}[S - X, 0] \qquad \text{and} \qquad \text{Max}[X - S, 0] \tag{47}$$

where both the stock price S and the exercise price X at the maturity of the option are stochastic. We will use the European call to illustrate our method in what follows. If S and X are joint lognormally distributed and investors' preferences exhibit constant proportional risk aversion, we can write the value as a RNVR from Theorem II.

$$P = r^{-1} \int_0^{+\infty} \int_0^{+\infty} \text{Max}[(S - X), 0]\phi'(S, X)\, dX\, dS \tag{48}$$

where $\phi'(S, X)$ is the risk neutral density function for the bivariate case. Equation (48) can be rewritten after redefining the limit, defining $Y = S - X + X_0$, where X_0 is the exercise price at time 0 (today), and transforming the variables as

$$P = r^{-1} \int_{X_0}^{\infty} (Y - X_0)\phi'(Y)\, dY \tag{49}$$

where Y is a lognormal variate with the risk neutral density function $\phi'(Y)$.[19]

From Theorem II,

$$\phi'(Y) = \frac{1}{\sigma Y\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left\{\ln Y - \ln rV + \frac{\sigma^2}{2}\right\}^2\right] \tag{50}$$

where σ^2, the variance of the underlying normal distribution, is defined by $\sigma^2 = \sigma_S^2 - 2\rho\sigma_S\sigma_X + \sigma_X^2$. Here σ_S^2 is the variance of the end-of-period stock price, σ_X^2 is the variance of the end-of-period exercise price, ρ is the correlation between them and V is the current value of the claim Y.

Changing the random variable Y to the standardized normal variate z and simplifying, yields

$$P = V \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{2} + \sigma z\right)\exp\left(-\frac{1}{2}z^2\right) dz$$

$$- rX_0 \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dt \tag{51}$$

[19] Strictly speaking, if S and X are joint lognormal, Y cannot be lognormal. However, if S and X follow a joint Weiner process in continuous time, then S, X, and Y will be joint lognormal in discrete time if continuous hedging is possible in the Black-Scholes-Merton sense. The analysis here may be regarded as an approximation or the risk neutral valuation of end-of-period payoffs generated by continuous joint Wiener processes with continuous hedging, using the Cox-Ross [6] argument. However, if the approximation is regarded as inappropriate, the integration can be carried out over the joint distribution of S and X by numerical methods.

where

$$d_2 = \frac{\ln V - \ln X_0 + \ln r - \frac{\sigma^2}{2}}{\sigma}$$

Rewriting the above in terms of cumulative normal densities leads to[20]

$$P = VN(d_1) - r^{-1}S_0N(d_2) \tag{52}$$

where d_2 is defined as above and

$$d_1 = d_2 + \sigma$$

The formula in Equation (52) is almost identical to the standard Black-Scholes value for the call option except that V now stands for the value of the stock less the current value of a position that perfectly hedges against changes in the exercise price from its current level X_0. Also, as explained earlier, σ^2 is now defined to incorporate uncertainty from both the stock price and the exercise price at maturity.

Another instance of a complex claim that can be reduced to the standard case is the case of an option to exchange one asset for another, with the exercise price held constant, as analyzed by Margrabe [14]. Since he views this type of option as an increment over the option to buy one instrument, there is no time dimension and hence no riskless rate of return enters the formula for value.

The payoffs to such an option are

$$\text{Max}[(S_1 - S_2), 0] \tag{53}$$

where S_1 and S_2 are the respective stock prices of the two underlying assets.

Using steps essentially similar to the previous case, we can derive the value of such an option as[21]

$$P = V_1N(d_1) - V_2N(d_2) \tag{54}$$

where

$$d_1 = \frac{\ln(V_1/V_2) + \frac{\sigma^2}{2}}{\sigma}$$

$$d_2 = d_1 - \sigma$$

and V_1 and X_2 are the respective stock prices and $\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$ where σ_1 and σ_2 are the respective standard deviations (of the underlying normal

[20] This result is at variance with the analysis in Fischer [8], since his formula involves returns other than the riskless rate of interest. This is not in line with the literature on option pricing since, if a perfect hedge can be set up, a RNVR exists and no rates of return other than the riskless rate should enter the expression for value.

[21] Once again, as in the previous case, this can be viewed as an approximation. See footnote 19 above.

distributions) and ρ is the correlation coefficient between them. This formula is the discrete time analog of Equation (7) in Margrabe [14].

IV. Conclusion

We have used a generalization of earlier results in Brennan [4] to show how complex contingent claims whose payoffs depend on two more more stochastic variables can be priced using risk neutral valuation relationships. Two important cases that yield RNVRs are multivariate normality with constant absolute risk aversion and multivariate lognormality with constant proportional risk aversion. In both cases, the result may be interpreted as arising from the fact that for these combinations of preferences and return distributions, the risk borne by investors by not hedging continuously can be priced by shifts in the means of the underlying distributions. Thus, they are concerned only with terminal payoffs and, since the valuation is risk neutral, discount them at the riskless rate of interest.

The derivation of the RNVR with restrictions on preferences in a discrete time setting is similar in spirit to showing that a riskless hedge can be constructed and maintained in a continuous time framework; in both cases, the end-of-period expected payoffs can be discounted at the riskless rate of interest under the assumption of universal risk neutrality. However, in the case of many multivariate contingent claims, the derivation of the riskless hedge in the continuous time context may be quite complex. By contrast, in the discrete time framework used here, once a RNVR has been derived, any claim contingent on the set of multivariate stochastic variables can be directly valued by the methods developed here.

Appendix

Proof of Necessity for the Multivariate Normal Distribution

If a RNVR exists and takes the form of a mean shift of the underlying distribution from $\underline{\mu}$ to $r\underline{V}$, the expression for the value of the contingent claim is

$$P = r^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{C(\underline{X})}{(2\pi)^{n/2} |\Sigma|^{1/2}} Z(X) \cdot \exp\left[-\frac{1}{2} \{\underline{X} - \underline{\mu}\}' \Sigma^{-1} \{X - \underline{\mu}\}\right] dX_1 \cdots dX_n$$

$$= r^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{C(\underline{X})}{(2\pi)^{n/2} |\Sigma|^{1/2}} \cdot \exp\left[-\frac{1}{2} \{\underline{X} - r\underline{V}\}' \Sigma^{-1} \{\underline{X} - r\underline{V}\}\right] dX_1 \cdots dX_n \quad \text{(A1)}$$

The first equation above is an application of the general valuation relationship (6) for multivariate joint normality of aggregate wealth and the risks $\underline{X}$. The second part of the relationship is the RNVR for this case, which does not involve any parameters of risk aversion.

A necessary condition for the RNVR to exist is that[22]

$$Z(\underline{X})\exp[-\tfrac{1}{2}\{\underline{X} - \underline{\mu}\}'\Sigma^{-1}\{\underline{X} - \underline{\mu}\}] = \exp[-\tfrac{1}{2}\{\underline{X} - r\underline{V}\}'\Sigma^{-1}\{\underline{X} - r\underline{V}\}] \quad \text{(A2)}$$

or

$$Z(\underline{X}) = \exp[\underline{X}'\Sigma^{-1}(\underline{\mu} - r\underline{V})]\exp -\tfrac{1}{2}[\underline{\mu}'\Sigma^{-1}\underline{\mu} - r^2\underline{V}'\Sigma^{-1}\underline{V}] \quad \text{(A3)}$$

The exponential valuation relationship is consistent with a conditional relative marginal utility of wealth function of the form $\exp(\underline{\beta}'\underline{X})$, where β' is a row vector involving only technological variables.

By definition, from Equation (4), $Z(\underline{X})$ depends on the conditional distribution of w given X. From (11) above, this distribution is normal with conditional mean $\mu_w + \Sigma_{wx}\Sigma^{-1}(\underline{X} - \underline{\mu})$ and conditional variance $\sigma_w^2 - \Sigma_{wx}\Sigma^{-1}\Sigma_{xw}$. If Equation (A2) holds for arbitrary normal distributions of w and X, it should hold for the particular case when w and $\underline{X}' = [X_1, X_2 \cdots X_n]$ are perfectly correlated with each other, so that the conditional variance of w is zero. Here, all pairwise correlation coefficients between the X's are unity and the correlation matrix is singular. Hence, the vectors $\underline{X}$, $\underline{\mu}$, and $\underline{V}$ can be replaced by scalars x, μ, and V, the vector Σ_{wx} by the scalar σ_{wx} and the matrix Σ by σ^2. Hence, from Equations (4) and (A3) we can write

$$\begin{aligned} Z(\underline{X}) &= Z(x) = \exp[x(\mu - rV)/\sigma^2]\exp[(\mu^2 - r^2V^2/\sigma^2)] \\ &= U'[\mu_w + \frac{\sigma_w}{\sigma}(x - \mu)]/E[U'(w)] \end{aligned} \quad \text{(A4)}$$

It is easy to see that $U'(\cdot)$ is an exponential function by equating the argument in $U'(\cdot)$ to say y, eliminating x and substituting in $Z(\underline{X})$. It then follows that $U(\cdot)$ is an exponential function and hence exhibits constant absolute risk aversion.

Proof of Necessity for the Multivariate Lognormal Distribution

If a RNVR exists and takes the form of a mean shift in the multivariate lognormal distribution from $\underline{\exp}(\underline{\mu} + \tfrac{1}{2}\underline{S})$ to $\underline{\exp}\, r\underline{V}$, (from $\underline{\mu}[r\underline{V} - \tfrac{1}{2}\underline{S}]$ for the underlying multivariate normal distribution), the expression for the value of the contingent claim is given by

$$P = r^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{C(\underline{X})}{(2\pi)^{n/2}|\Sigma|^{1/2}\Pi_1^n X_j} Z(\underline{X})$$

$$\cdot\exp\left[-\frac{1}{2}\{\underline{\ln X} - \underline{\mu}\}'\Sigma^{-1}\{\underline{\ln X} - \underline{\mu}\}\right] dX_1 \cdots dX_n$$

[22] For (A2) to follow from such a RNVR, the class $\mathscr{L}$ of functions C used to define the RNVR, for which (A1) must hold, must be sufficiently rich to imply that the measures in (A1) are equal, which is equivalent to (A2) if there is a version of Z that is continuous in $\underline{X}$. But equality of the measures in (A1) is necessary for (A2) and is equivalent to (A1) holding for all functions C for which the integrals in (A1) are defined. A similar comment applies to Equations (A5) and (A6) below.

$$= r^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{C(\underline{X})}{(2\pi)^{n/2}|\Sigma|^{1/2}\Pi_1^n X_j} \exp\left[-\frac{1}{2}\left\{\underline{\ln X} - \left(r\underline{V} - \frac{1}{2}\underline{S}\right)\right\}'\Sigma^{-1}\left\{\underline{\ln X} - \left(r\underline{V} - \frac{1}{2}\underline{S}\right)\right\}\right] dX_1 \cdots dX_n \tag{A5}$$

The first relation above follows from the general multivariate valuation Equation (6) for the case of multivariate lognormality. The second step is the RNVR for this case and hence does not involve any parameters of risk aversion.

Thus, for a RNVR to exist, it is necessary that

$$Z(\underline{X})\exp[-\tfrac{1}{2}\{\underline{\ln X} - \underline{\mu}\}'\Sigma^{-1}\{\underline{\ln X} - \underline{\mu}\}] = \exp[-\tfrac{1}{2}\{\underline{\ln X} - (r\underline{V} - \tfrac{1}{2}\underline{S})\}'\Sigma^{-1}\{\underline{\ln X} - (r\underline{V} - \tfrac{1}{2}S)\}] \tag{A6}$$

or

$$Z(X) = \exp[\underline{\ln X}'\Sigma^{-1}\{\underline{\mu} - (r\underline{V} - \tfrac{1}{2}\underline{S})\}] \cdot \exp -\tfrac{1}{2}[\underline{\mu}'\Sigma^{-1}\underline{\mu} - (r\underline{V} - \tfrac{1}{2}\underline{S})'\Sigma^{-1}(r\underline{V} - \tfrac{1}{2}S)] \tag{A7}$$

Thus, the valuation relationship implies a conditional relative marginal utility function of the form

$$Z(\underline{X}) = a \exp[\underline{B}' \ \underline{\ln X}] \tag{A8}$$

where a is a scalar and $\underline{B}'$ is a row vector of constants involving technological variables.

By definition, from Equation (4), $Z(\underline{X})$ depends on $(w|\underline{X})$. This conditional distribution is lognormal with the parameters specified in (20). If (A8) holds for arbitrary multivariate distributions of w and X, it should hold, in particular, when the ln w and $\underline{\ln X}$ are perfectly positively correlated, so that the conditional variance of the underlying normal distribution (i.e., of ln w) is zero. Since the variables in the vector $\underline{\ln X}$ are perfectly correlated with each other, the variance-covariance matrix is singular. The vector $\underline{\ln X}$, $\underline{\mu}$, and $\underline{V}$ can be replaced by scalars ln X, μ, and v, respectively, the vector Σ_{wx} by the scalar σ_{wx} and the matrix Σ by σ^2. Further, since the expectation and derivative operators are transitive, we can write

$$E[U'(w \mid \underline{X})] = U'[E(w \mid X)] \tag{A9}$$

Combining (4) and (A8) we can write

$$Z(\underline{X}) = Z(x) = a \exp[\underline{B}' \ln x] = U'[\exp\{\mu + \sigma_w/\sigma(\ln x - \mu)\}]/E[U'(w)] \tag{A10}$$

By equating the argument in $U'(\cdot)$ to say y, eliminating ln X and substituting in the expression for $Z(X)$, we can show that $U'(\cdot)$ is a power function. It follows that $U(\cdot)$ is also a power function and hence exhibits constant proportional risk aversion.

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