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Commodity Bonds and Consumption Risks

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ABSTRACT

This paper analyzes the economic role of commodity bonds by examining the nature of the demand for these securities. I show that while commodity bonds protect against relative price changes, they do so by introducing variability into the future real income stream. This variability limits the desirability of using commodity bonds to provide "price insurance" for future consumption. However, this variability may allow commodity bonds to hedge risks to consumption caused by stochastic changes in income. The analysis also suggests that it is this "income insurance" rather than "price insurance" that is important in hedging risks to future consumption.

IN 1863, THE Confederate States of America (CSA) took the unusual step of issuing bonds payable in bales of cotton. In 1980, The Sunshine Mining Company took the unusual step of issuing bonds payable in ounces of silver. Although the CSA commodity bond offering proved to be of an isolated (and rather short-lived) nature, recent developments suggest that commodity bond offerings like that of Sunshine Mining may become more commonplace. For example, both Mexico and the British Oil and Gas Corporation have issued petroleum bonds, and the Reagan administration has proposed issuing oil-indexed bonds to finance the strategic oil reserve. France has experimented with gold bonds, as have several private companies.[1]

Commodity-indexed debt also has appeared outside of organized bond markets. Shared Appreciation Mortgages, for example, represent a debt indexed to the price of housing. Even bank loans can be indexed to underlying commodities, as evidenced by the "oil-indexed" loans between Japanese banks and the Soviet Union.[2] Such developments suggest that debts linked to a wide variety of commodity indices are certainly feasible.

This paper analyzes the economic role of commodity bonds by considering the nature of the demand for these securities. Because commodity bonds can hedge

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[1] For discussion of the Reagan petroleum bond plan see "A New Plan to Finance Oil Reserve," *New York Times*, January 15, 1981; "Devious Device to Trim a Budget," *New York Times*, February 6, 1981; and "Building an Oil Reserve the Thrifty Way," *New York Times*, February 16, 1981. Britain's oil bonds are discussed in "Oil Bonds May Not Bring Riches," *World Business Weekly*, October 27, 1980. Gold bonds are featured in "Gold Indexed Bond is Costly for France," *New York Times*, June 30, 1980; "Hot Seller: Bonds Indexed to Oil and Gold," *Business Week*, August 3, 1979; and "Sale of Gold Bonds in Europe Planned," *New York Times*, January 28, 1981.

[2] Numerous articles on such indexed loans have appeared in the *Wall Street Journal*. See, for example, *WSJ*, February 15, 1979 (p. 33); June 11, 1979 (p. 18); August 15, 1979 (p. 29); April 7, 1980 (p. 7); April 8, 1980 (p. 35); and May 14, 1981 (p. 31).

changes in relative prices, they presumably provide a mechanism to hedge risks to future consumption. In the context of a dynamic programming model, I analyze the desirability and limitations of this hedging ability. I show that while commodity bonds protect against relative price changes they do so by introducing variability into the future real income stream. This variability limits the desirability and, hence, the potential importance of commodity bonds. In particular, I demonstrate that if bonds are priced fairly then a commodity bond will be demanded only if there is a minimum necessary consumption quantity or the bond's payoff negatively correlates with the individual's portfolio return. In either case, the commodity bond is valuable because it provides a form of insurance.

This paper can be viewed as belonging to the literature on financial market microstructure as well as to the growing literature examining the link between individuals' investment decisions and their consumption patterns. In this latter area (see Breeden [1]; Shiller [7]) the emphasis is on relating financial market behavior to underlying economic variables. The analysis here suggests that such an approach may also provide insight into financial market structure as well.

The paper is organized as follows. The next section examines the characteristics of commodity bonds and their relationship to existing financial assets. In Section II, a simple dynamic model is developed and the portfolio decision between commodity and conventional bonds is analyzed in a world of price and income uncertainty. The paper's conclusions are then presented in Section III.

I. The Characteristics of Commodity Bonds

A commodity bond is a financial security whose return is linked to the price of its underlying commodity. Unlike a conventional bond that pays a stated nominal interest rate and a stated nominal amount upon maturity, the commodity bond's payoff is a stated quantity of a particular commodity. For example, a $1000 face Silver Bond might be redeemable at maturity for 50 ounces of silver. The interim interest payments may or may not be likewise denominated in units of the particular commodity. Some commodity bonds also incorporate an option feature by allowing the holder to receive either the nominal face value or the designated commodity amount at maturity. Such option features have been analyzed by Schwartz [6], and will not be considered here.

A commodity bond's quantity-denominated return structure is the fundamental difference between it and a conventional bond. With a conventional bond, the nominal return is known but the real return is not, whereas both the nominal and real monetary returns are unknown in the commodity bond. The uncertain real return of a commodity bond also differentiates it from an index bond; that is, a bond whose return depends on an aggregate price level index. Index bonds protect the holder from changes in the *absolute* price level; commodity bonds protect the holder from changes in *relative* price levels. Unless *all* prices in the economy change by exactly the same amount, these financial instruments will not be identical (see Fischer [2] for an analysis of index bonds).

The commodity bonds' quantity payoff suggests similarities between these bonds and forward contracts.[3] There are, however, some significant differences. One difference lies in their cash flows. The purchaser of a commodity bond pays the seller at the outset, while in a forward contract no such exchange occurs until the end. Further, commodity bonds typically pay interim coupon payments, but forward contracts pay off only at maturity. Finally, forward contracts are primarily short-term agreements; commodity bonds are designed as long-term instruments. These differences suggest interesting relationships between commodity bonds and existing financial securities.

To elaborate on this, consider a one-period world in which all contracts settle at the end of the period. In such a world, a short position in a commodity bond (that is, issuing a commodity bond) is equivalent to a short position in a forward contract for the commodity plus a short position in a money bond. Similarly, a long position in a commodity bond is equivalent to a portfolio consisting of a long bond and a long forward contract.

In a multiperiod setting with perfect capital markets, this same relationship holds if both the commodity bond and the money bond are pure discount securities. This relationship has been analyzed by Richard and Sundaresan [5].[4] Replicating a coupon-bearing commodity bond, however, is more complicated. In this case, a long position in a commodity bond is equivalent to a long position in a money bond and a long position in a portfolio of forward contracts selected to match the cash flows of the money bond.

These multiperiod equivalent portfolios are crucially dependent on using long-term forward contracts to replicate the commodity bond's quantity payoff. However, as mentioned earlier, such long-term contracts are not commonly available as investment vehicles.[5] One reason may be that while commodity bonds are liquid (i.e., may be traded without permission), forward contracts are not. In the forward contract both the long and the short sides have unfulfilled obligations. The short promises to deliver a certain quantity of the commodity; the long promises to pay a fixed price for it. By contrast in a commodity bond, the long meets all of its obligations when the bond is issued; only the issuer has an unfulfilled obligation.

[3] Commodity bonds also have traits in common with futures contracts because both securities involve making or taking delivery of a specific commodity at a prespecified date in the future. However, unlike a futures contract, a commodity bond need not trade on an organized exchange. Additionally, the cash flow timing is significantly different. Finally, most futures contracts are reversed prior to maturity and, hence, the commodity is never actually delivered. In a forward contract, and in a commodity bond, delivery will occur. For insightful discussion of futures markets see Telser [8] and Jarrow and Oldfield [3].

[4] Richard-Sundaresan [R-S] analyze this relationship using a significantly different approach. In R-S there is only one consumer and they solve for the equilibrium prices at which that individual is content to hold his particular given portfolio. In this paper, I analyze the portfolio decisions of a consumer (in a world of many consumers) who takes prices as given. Hence, R-S allow no trading and solve for equilibrium prices, while I take prices as given and solve for the equilibrium trading strategy. Footnote 15 describes how our results are related.

[5] Long-term forward contracts do exist in some industries. For example, a nuclear power plant might arrange for delivery of uranium via such a contract. In such circumstances, the integrity of each side is crucial, and only the largest, most established companies are likely to be involved. In spite of such selectivity, problems can still arise as evidenced by the recent Westinghouse case. See Joskow [4].

In the absence of long-term forward contracts, therefore, a commodity bond is a unique financial security. It is not clear however, that its unique properties are actually valuable. The next section investigates this by analyzing the demand for commodity bonds in a world of price and income uncertainty.

II. The Hedging Properties of Commodity Bonds

Why would anyone demand a commodity bond? Perhaps one reason is the protection that commodity bonds provide against changes in relative prices. Because conventional bonds pay a stated nominal return they expose the holder to price level risk. Although an index bond can mitigate this, the holder is still subject to the risk that relative prices may change. If an individual's consumption pattern is not always identical to that underlying the overall price level, a change in relative prices may make him worse off even if the absolute price level has remained the same or gone down. A commodity bond completely insures against these price risks; neither relative nor absolute price changes affect the bond's quantity payoff.

The ability to hedge relative and absolute price level changes suggests that the demand for commodity bonds may be related to a desire to hedge risks to future consumption. A portfolio of commodity bonds can lock in a specific future consumption bundle that is invariant to actual future prices. In this section, I investigate the desirability of these hedging properties by examining the demand for commodity bonds in a simple two-period model.

In the model, an individual selects his portfolio in period one and consumes in period two. A portfolio may consist of either conventional bonds yielding an exogenously given certain interest rate r, or commodity bonds paying off in one unit of the underlying good.[6] Although numerous goods may exist in the economy, the model considers an individual who has demands for two goods, X and Y.[7] The price of good Y is constant across periods but the price of good X is not. Let the price of good X in period two, P, be described by a simple discrete distribution in which P equals k with probability q and l with probability $(1 - q)$. Without loss of generality, let $k > l$. Since the price of good Y is constant, all relative price risks are introduced by changes in the price of good X. For simplicity, I assume that commodity bonds exist only for good X.

Finally, I assume that an individual attempts to maximize his utility, where utility is given by the Stone-Geary function

$$U(X, Y) = (X - A)^{\alpha}(Y - B)^{\beta} \tag{1}$$

α, β, A, and B are parameters, and $\alpha + \beta$ is assumed less than or equal to one. The specification of a Stone-Geary utility function allows the inclusion of minimum quantity consumption. For example, if X is food, then an individual must consume A units to survive regardless of its price.

The individual faces a dynamic programming problem because his income in

[6] Throughout the analysis I am assuming that for both commodity and conventional bonds there is no default risk.

[7] Alternatively, good Y may be thought of as a composite of the other goods in the economy.

period two depends on his portfolio decisions in period one. The solution to such a problem involves working backwards by first solving for the optimal period two consumption. The period two demand functions can then be used to derive the indirect utility, or value, function. The optimal period one portfolio decisions can then be determined.

In period two, the individual selects X and Y to maximize his utility subject to his period two budget constraint. His problem is:

$$\underset{\{X,Y\}}{\text{MAX}}\ (X - A)^{\alpha}(Y - B)^{\beta}$$
$$\text{s.t.}\quad PX + P_y Y = I_2 \tag{2}$$

where I_2 is period two income, and P and P_Y are prices.[8] The first-order conditions can be solved for the demands

$$X^* = \frac{\alpha I_2 - \alpha B P_Y + \beta A P}{P(\alpha + \beta)}$$
$$Y^* = \frac{\beta I_2 - A\beta P + \alpha B P_Y}{P_Y(\alpha + \beta)} \tag{3}$$

The commodity demands are functions of income, prices, and parameters. Substituting these demand functions for X and Y into the utility function (1) yields the indirect utility, or value, function

$$V(P, P_Y, I_2) = U(X^*(\cdot), Y^*(\cdot))$$
$$= \left[\frac{\alpha(I_2 - BP_Y - AP)}{(\alpha + \beta)P}\right]^{\alpha}\left[\frac{\beta(I_2 - BP_Y - AP)}{(\alpha + \beta)P_Y}\right]^{\beta} \tag{4}$$

We can use this indirect utility function to solve the individual's period one portfolio problem.

In period one, the consumer selects conventional bonds, denoted M, and/or commodity bonds, denoted $\bar{X}$, to finance his period two consumption. The price of a conventional bond is one, and each conventional bond returns $(1 + r)$ in period two. A commodity bond has a price denoted c, and each commodity bond provides one unit of good X in period two. In period one, the period two price of X, P, is unknown and, hence, so is the value of period two income, I_2, because $I_2 = (1 + r)M + P\bar{X}$. Since the price of good Y is constant across periods, let P_y be normalized to one and, to simplify the calculations, let $B = 0$.[9] The period one problem is

$$\underset{\{M,\bar{X}\}}{\text{MAX}}\ E[V(\tilde{P}, \tilde{I}_2)]$$

[8] This problem only makes sense if $I_2 > PA + P_Y B$. This constraint on period two income will be imposed in Equation (5).

[9] With P_Y constant, a requirement to spend the amount $P_Y B$ on consumption of good Y in period two is analogous to simply subtracting that dollar amount from income. As a result, assuming $B = 0$ merely removes notational complexity without sacrificing any economic content. If B is assumed greater than zero, the main effect is to shift the lower bound of the feasible set out as it effectively reduces the income the individual has to spend.

$$\text{s.t.} \quad M + c\bar{X} = I_1;\ X \geq 0;\ M \geq 0$$

$$I_2 - AP \geq 0 \quad \text{for all } P. \tag{5}$$

The individual faces several constraints in choosing his optimal portfolio. First, I assume that short sales are not allowed and, hence, the individual must select non-negative quantities of money and bonds. The total size of his portfolio is then determined by the budget constraint. Additionally, because the individual must consume A units of good X in period 2 to survive, his income in period 2 must permit him to acquire A units, even under a "worst case" scenario. This constraint effectively requires the individual to have positive utility in all states of the world. In order to insure that the feasible set of bonds and commodity bonds is not empty, I assume that $I_1 > Ac$. Then $\bar{X} = I_1/c$ and $M = 0$ is a solution, but with any lower value for I_1, survival for period 2 cannot be guaranteed.

The quantity of commodity bonds the individual demands depends upon the price, c, which in turn depends upon the supply of commodity bonds. To determine this supply, consider first a world in which all bonds are supplied by risk neutral firms. In such a world, a firm would be willing to supply a commodity bond if the price were given by

$$c = \frac{E(P)}{(1 + r)} \tag{6}$$

If firms are risk neutral, any price other than (8) would result in an arbitrage opportunity. At a higher price, firms would try to sell an infinite amount; at a lower price firms would go infinitely short.[10] Thus, I assume that the commodity bond is priced risk neutrally.

Figure 1 depicts the set of all feasible choices for the commodity bond, $\bar{X}$. The upper bound arises from the individual's overall wealth level.[11] The lower bound incorporates the worst case scenario ($P = k$) into the individual's minimum consumption requirement. Provided that income is high enough ($I_1(1 + r) > kA$), the only restriction (other than the period 1 budget) on feasible $\bar{X}$ is the non-negativity required by the short sale restriction. If income is less than the "worst

[10] This suggests that the supply curve of commodity bonds is a horizontal line at a price equal to c. The equilibrium quantity of commodity bonds is then found by summing the demand curves across all individuals, and finding its intersection with the supply curve.

[11] The constraints $M + c\bar{X} = I_1$ and $I_2 - AP \geq 0$ for $P = l$ yield

$$\bar{X} \leq \frac{(1 + r)I_1 - Al}{E(P) - l}$$

But $M + c\bar{X} = I_1$ and $\bar{X}$, $M \geq 0$ yield

$$\bar{X} \leq \frac{I_1(1 + r)}{E(P)}$$

When $(1 + r)I_1 \geq AE(P)$,

$$\frac{I_1(1 + r)}{E(P)} \leq \frac{(1 + r)I_1 - AL}{E(P) - l}$$

Hence, the budget constraint and non-negativity constraints determine the upper bound of the feasible set.

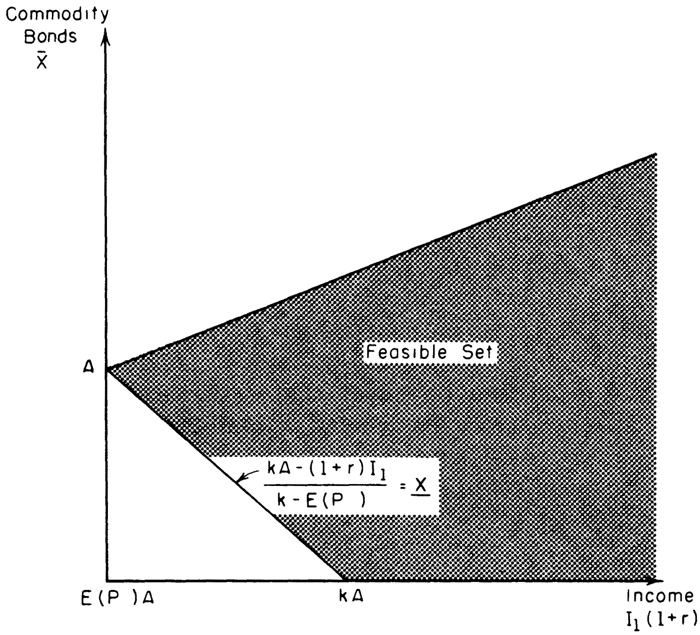

Figure 1. The Feasible Set for Commodity Bonds

case" outcome, then some commodity bonds must be held to guarantee survival. Defining $\underline{X}$ to be this minimum level, it is easy to show that

$$\underline{X} = \frac{kA - (1 + r)I_1}{k - E(P)} \tag{7}$$

The period one problem can be solved for the optimal demand for commodity bonds, $\bar{X}^*$, which, if interior to the feasible set of Figure 1, is given by

$$\bar{X}^* = \frac{I_1(1 + r)(1 - H) - A(l - Hk)}{E(P)(1 - H) + kH - l} \tag{8}$$

where

$$H \equiv \left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}$$

Equation (8) is the demand curve for the commodity bond. The demand for commodity bonds, if interior, is then

$$\bar{X}^* = \frac{I_1(1 + r)\left[1 - \left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right] - Ak\left[\frac{l}{k} - \left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right]}{E(P)\left[1 - \left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right] - k\left[\frac{l}{k} - \left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right]} \tag{9}$$

It is useful to determine when this demand is, in fact, interior. Substracting the lower bound $\underline{X}$ from $\bar{X}^*$ yields

$$\bar{X}^* - \underline{X} = \frac{(k-l)[(1+r)I_1 - AE(P)]}{\left[E(P)\left[1-\left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right] - k\left[\frac{l}{k} - \left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right]\right](k - E(P))} \tag{10}$$

As the denominator and numerator are both positive, the demand for commodity bonds is generally interior. However, the extent to which an individual demands bonds over the required amount depends upon his attitude toward risk. These risk preferences are captured by the $\alpha + \beta$ terms. As $\alpha + \beta$ goes to one, the denominator goes to infinity and $\bar{X}^* - \underline{X}$ goes to zero. The limiting case of $\alpha + \beta = 1$ corresponds to assuming risk neutrality. A risk neutral individual, therefore, will hold only enough commodity bonds to guarantee survival.

A risk averse individual will demand commodity bonds above this minimum level. The less risk averse the individual (that is, the closer is $\alpha + \beta$ to one) the fewer bonds he will hold. Since his demand is interior, his optimal choice can be further characterized by analyzing the commodity bond demand curve (Equation (9)). In particular, the effect of income on bond demand is given by

$$\frac{\partial \bar{X}^*}{\partial I_1} = \frac{(1+r)\left[1-\left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right]}{E(P)\left[1-\left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right] - k\left[\frac{l}{k} - \left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}\right]} < 0 \tag{11}$$

As income increases, the individual decreases his demand for commodity bonds. If his income is high enough, even the most risk averse individual will demand zero commodity bonds. Indeed, in the absence of short sale restrictions, a wealthy risk averse individual would even issue a commodity bond. Equations (10) and (11) also dictate that no matter how risk averse the individual, his demand for commodity bonds is never greater than A. This yields an intriguing result: if no minimum consumption quantity existed, there would be no demand for the commodity bond at all. Figure 2 illustrates these characteristics of commodity bond demand.

The ability to hedge changes in relative prices of consumption goods, therefore, appears not as valuable a property as might be expected. To understand why this is so, it is useful to consider the properties of individuals' consumption demands and how commodity bonds relate to their consumption decisions. Provided the demand curve for a particular good is downward sloping, a utility maximizing individual's demand for that good is already partially hedged with respect to price changes. That is, an increase in the price of good X results in the demand for X falling. Hence, the quantity demanded of any consumption good will change with shifts in relative prices. If an absolute minimum consumption quantity is always demanded, however, this hedging property will be absent, and the hedging ability of the commodity bond may become valuable. The commodity bond can protect the individual from being unable to purchase the needed consumption quantity. The commodity bond, therefore, provides a form of "price insurance."

If there is no minimum consumption quantity (or if we consider an individual whose income is always sufficient to purchase such a minimum quantity), however, the "insurance" provided by the commodity bond comes at a very high

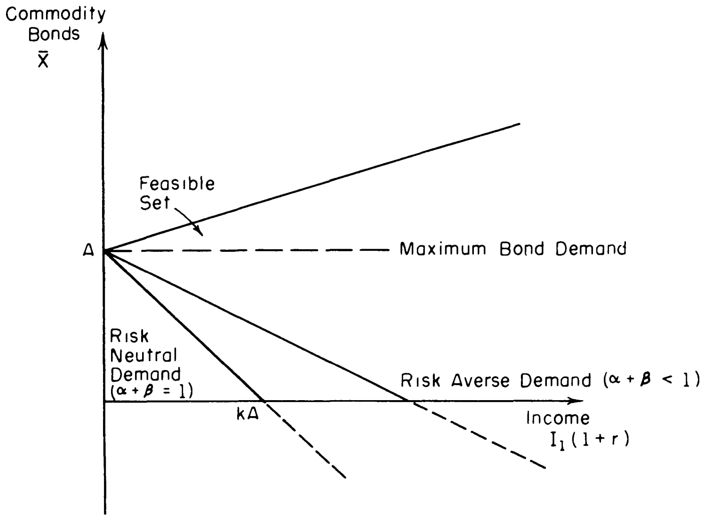

Figure 2. The Demand for Commodity Bonds

price.[12] The reason is that the commodity bond affects an individual's period two income. While the commodity bond provides the same quantity of the good across states, the real *value* of the quantity to the individual depends on both the commodity's price and the individual's own price index. As relative prices change, the optimal quantity the individual demands also changes, and this, in turn, affects his individual price index. This index change distorts the real payoff of the commodity bond and, as can be demonstrated, the loss incurred by holding commodity bonds when prices are low is greater than the gain from such a position when prices are high. As a result, the individual may prefer to self-insure these price risks rather than purchase the commodity.

This point may be illustrated by considering the individual's period two indirect utility function given by Equation (4). If there is no minimum consumption quantity (i.e., $A = B = 0$), Equation (4) can be rewritten as

$$V(P, I_2) = \Omega \frac{[I_2]^{\alpha+\beta}}{P^{\alpha} P_Y^{\beta}} \tag{12}$$

where Ω is the constant term. Normalizing P_Y to one and substituting the definition of I_2 yields

$$V(P, I_2) = \Omega \frac{[M(1 + r) + P\bar{X}]^{\alpha+\beta}}{P^{\alpha}} \tag{13}$$

[12] Another way to view this is to consider the individual's price and income elasticities of demand for good X. If there are no minimum consumption quantities (i.e., $A = B = 0$) then the income elasticity, η_I, and the price elasticity, η_P, both equal one. Hence, if prices rise, for example, the amount of good X the individual demands falls so that the total expenditure remains the same. The commodity bond is thus redundant and will not be demanded. If A and B are greater than zero, then η_I and η_P are both less than one, and demand is more inelastic. In this case, the hedging ability of the commodity bond is useful because the individual will not fully offset price and income changes with changes in commodity demand.

The numerator is period two income; the denominator is the individual's price index.

As $\alpha + \beta$ approaches one, the V function becomes linear in income but is convex in prices. Hence, although the individual would take a "fair gamble" on income, he would not accept such a gamble if its payoffs positively correlated with his individual price index. The commodity bond involves such a gamble. When prices are high, the bond's payoff rises but so does the individual's price index. Conversely, when prices are low the bond's payoff and the price index are also low. Because prices enter Equation (13) nonlinearly, however, the gains and losses to real income when prices change are not symmetric. The gain to real income when prices are high is more than offset by the loss to real income when prices are low.[13] The "price insurance" provided by the commodity bond thus comes at the expense of greater real wealth variability.

This wealth variability problem can be mitigated if the commodity bond's price reflected the real purchasing power effect the bond had on the individual. Specifically, the bond would have to sell at a price that compensated the individual for the increased risks it imposed. To determine this price, consider the individual's indifference curves between commodity bonds and conventional bonds. For a given income level, the price at which the individual would hold a specific quantity of commodity bonds is determined by the tangency of the budget constraint with his indifference curve. This is given by

$$C^* = \frac{E[P]}{(1+r)} + \frac{\operatorname{cov}(MU_{\bar{I}_2}/\Omega(\alpha+\beta), P)}{(1+r)E[MU_{\bar{I}_2}/\Omega(\alpha+\beta)]} \tag{14}$$

where MU_{I_2} is the marginal utility of adjusted income, $\bar{I}_2$, and $\bar{I}_2$ is defined as $(1+r)M + P(\bar{X} - A)$. This price may be larger or smaller than the risk neutral price depending upon where the individual's demand is evaluated. For example, if (14) is evaluated at $\bar{X} = A$, the required price simplifies to

$$C^* = \frac{E[P]}{(1+r)} + \frac{\operatorname{cov}(P, P^{-\alpha})}{(1+r)E[P^{-\alpha}]} \tag{15}$$

At this price the commodity bond is "indexed" to the consumer's real price index.[14] Since this is less than the risk neutral price, presumably only a risk averse firm would issue such a bond. The difference between the risk neutral price in Equation (6) and the indexed price in Equation (15) represents the

[13] This property can be illustrated by a simple example. Suppose there are two equally likely states of the world in which the price of good X will be either 10 or 5. Let the individual's price index be given by P^{α}, and let $\alpha = 0.4$. The real payoff to a commodity bond is $10/10^{0.4} = 3.98$ in state one and $5/5^{0.4} = 2.63$ in state two. Alternatively, suppose the individual can purchase a conventional bond yielding 7.5 in either state (since 7.5 is the expected value of the state prices, the risk neutral prices of both bonds in period one should be the same). The real payoff to the conventional bond is $7.5/10^{0.4} = 2.98$ in state one and $7.5/5^{0.4} = 3.94$ in state two. Note that while the commodity bond is worth 1.0 more than the conventional bond in state one, it is worth 1.32 less in state two. Hence the losses when prices are low are greater than the gain when prices are high.

[14] Note that the "indexed" price is different for every individual since it depends on the individual's consumption pattern. Only if the individual's consumption pattern is the same as that underlying the overall price level will the bond be equivalent to an index bond in the Fischer sense (see [2]).

"premium" the firm must pay the individual for bearing the risk. In effect, the individual would be insuring the firm against adverse price changes.[15]

To this point the analysis has delineated the role of commodity bonds in hedging risks to future consumption caused by changes in relative prices. A further risk to consumption may arise from changes in income. If an individual's future income stream is uncertain, the payoff structure of the commodity bond may be valuable. Income uncertainty can be incorporated by allowing income in period two, I_2, to include a random endowment $\tilde{w}$. This random term can be thought of as an exogenously given lump sum, or as a random return to an existing portfolio. The period two budget constraint thus becomes

$$P_Y Y + PX = I_2 = M(1 + r) + P\bar{X} + \tilde{w} \tag{16}$$

Introducing a second random variable into the analysis greatly increases the complexity of the individual's decision problem. For tractability, I consider the special case where $\tilde{w}$ and $\tilde{P}$ are either perfectly negatively correlated or perfectly positively correlated. The random endowment term can then be described by a discrete distribution with $\tilde{w}$ equal to w^1 with probability q and w^2 with probability $(1 - q)$. Let $\underset{\sim}{w}$ denote the minimum value for $\tilde{w}$. The feasibility constraint on the minimum $\bar{X}$ required to guarantee survival in the worst case scenario ($P = k$, $w = \underset{\sim}{w}$) becomes

$$\hat{\underset{\sim}{X}} = \frac{Ak - (1 + r)I_1 - \underset{\sim}{w}}{k - E(P)} \tag{17}$$

As before, the individual solves the period two problem for his commodity demands X^* and Y^*, substitutes these into his indirect utility function $V(\cdot)$, and then solves the period one portfolio problem. The resulting commodity bond demand is

$$\bar{X}^* = \frac{I_1(1 + r)(1 - H) + w^2 - w^1 H - A(l - Hk)}{E(P)(1 - H) + kH - l} \tag{18}$$

where the H term is defined in Equation (8). Subtracting the boundary condition (17) from this commodity bond demand yields

$$\bar{X}^* - \hat{\underset{\sim}{X}} = \frac{\begin{array}{c}\{(k - l)[(1 + r)I_1 + AE(P)]\} + w^2 k - w^2 E(P) \\ - w^1 kH + w^1 E(P)H + \underset{\sim}{w}E(P) - \underset{\sim}{w}E(P)H - \underset{\sim}{w}l + \underset{\sim}{w}kH\end{array}}{\left[E(P)(1 - H) - k\left(\frac{l}{k} - H\right)\right](k - E(P))} \tag{19}$$

Equation (19) gives the quantity of commodity bonds the individual demands

[15] This point can be illustrated by considering the first-order conditions of the general portfolio problem. It can be shown that the individual sets $\mathrm{cov}(V', P)/E[V'] = (1 + r)c - E[P]$, where primes denote derivatives. This equation is very similar to Equation (40) in Richard-Sundaresan. In their model, since the individual's portfolio is given (and hence so is the covariance), the price c, must adjust to offset this covariance. Because of this, they argue that normal backwardation or contango can arise if the security is not a good consumption hedge. In my model, if c is set risk neutrally then the individual's optimal strategy is to select his portfolio to set the covariance to zero. The amount of the asset needed to do so depends on its "insurance" or hedging ability.

above the necessary survival amount. Suppose that the random variables $\tilde{w}$ and $\tilde{P}$ are negatively correlated; that is, when prices are low, wealth is high, and conversely. In this case, $\underline{w} = w^1$ and (19) reduces to

$$\bar{X}^* - \hat{\underline{X}} = \frac{\{(k-l)[(1+r)I_1 - AE[P]]\} + w^2(k - E[P]) + w^1(E[P] - l)}{\left[E(P)(1-H) - k\left(\frac{l}{k} - H\right)\right](k - E(P))} \tag{20}$$

The bracketed term $\{\cdot\}$ in the numerator is identical to the earlier consumption-based demand in Equation (10); the denominator is also the same as in (10). Hence Equation (20) contains the consumption-based demand found earlier plus an investment demand. This investment component simplifies to $(k - l)(qw^1 + (1 - q)w^2)$, which is positive if and only if the mean of the random endowment is positive. Thus, a random endowment negatively correlated with P affects commodity bond demand only through its mean and the effect of the mean is the same as a change in income, $(1 + r)I_1$. If the return on the commodity bonds negatively correlates with a random income stream with positive mean, therefore, the individual will demand more commodity bonds.

Alternatively, suppose that the random variables $\tilde{w}$ and $\tilde{P}$ are positively correlated (prices high, income high, etc.). Then $\underline{w} = w^2$ and it can be demonstrated that

$$\bar{X}^* - \hat{\underline{X}} =$$

$$\frac{\{(k-l)[(1+r)I_1 - AE(P)]\} + w^2(k-l) - (w^1 - w^2)\left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)}(k - E(P))}{\left[E(P)(1-H) + k\left(\frac{l}{k} - H\right)\right](k - E(P))} \tag{21}$$

With w^2 now the minimum, the positive investment term $(w^2(k - l))$ in Equation (21) is less than the investment terms in Equation (20). Since the final term in (21) is clearly negative, the individual decreases his demand for commodity bonds. How much he decreases his demand depends upon his risk preferences. As $\alpha + \beta$ goes to one, $\bar{X}^* - \hat{\underline{X}}$ converges to $\dfrac{w^2 - w^1}{k - E(P)} < 0$. Thus, with positive correlation and $\alpha + \beta$ close to one the optimal $\bar{X}$ is $\hat{\underline{X}}$.[16] This result contrasts with the earlier result that when there is no income uncertainty the optimal $\bar{X}$ is above $\underline{X}$ for all $\alpha + \beta < 1$. The difference results because, when the commodity bond's payoff positively correlates with his income stream, the individual would

[16] Note that $\bar{X}^* - \hat{\underline{X}} \leq 0$ if

$$H = \left(\frac{l}{k}\right)^{\alpha/(\alpha+\beta-1)} \geq \frac{(k-l)[(1+r)I_1 - AE(P)] + w^2(k-l)}{(w^1 - w^2)(k - E(P))}$$

But $\lim_{\alpha+\beta\to1} H = \infty$, so for large enough $\alpha + \beta$, $\bar{X}^* - \hat{\underline{X}} \leq 0$.

prefer to avoid such bonds as they exacerbate the income uncertainty he already faces.

Equations (20) and (21) reinforce our earlier result that price variability *per se* is not the overriding factor in the decision to hold commodity bonds. Rather, it is the overall wealth variability and its correlation with the bond's return that influences demand. If wealth variability is large, the demand for commodity bonds will increase if the correlation between wealth and bonds is negative; demand will decrease if the correlation is positive. Hence, commodity bonds are desirable to the extent that they offset changes in stochastic income. Since changes in real wealth affect consumption, commodity bonds can hedge risks to future consumption by "smoothing out" this income stream. The demand for commodity bonds is thus similar to the demand for any other investment vehicle; its value depends upon the role it plays in a diversified portfolio.

III. Conclusion

The analysis suggests that the unique properties of commodity bonds are desirable only under certain conditions. If bonds are supplied by risks neutral firms, a commodity bond will only be demanded if there is a minimum consumption level or if the bond's payoff helps smooth out wealth variability. In either of these cases, the commodity bond provides a form of "insurance" and, hence, is valuable in hedging risks to future consumption.

This insurance aspect of the commodity bond provides some interesting insights into both the economic role of commodity bonds and individual's hedging behavior. As with any insurance contract, a commodity bond is only valuable if the protection it provides outweighs its cost or premium. In the case of hedging risks to consumption introduced by changes in relative prices, this protection will be valuable when the individual must purchase a particular good and may not always have sufficient income to do so. If the individual can substitute other goods, however, he has no need for commodity-specific insurance. Perhaps more important, if the individual's income is high enough, he would be better off self-insuring (or even, if possible, providing such insurance to others). This suggests that to protect future consumption what is needed is not "price insurance" but rather "income insurance." To the extent that commodity bonds can provide such insurance, they will have a role to play in the financial markets.

This analysis of the demand for commodity bonds has delineated the benefits and limitations of this unique security. There remain, however, many important questions to be answered. In particular, an analysis of the supply of commodity bonds might yield insights into the nature of firms' financing decisions. Although such an analysis is beyond the scope of this paper, the analysis here does suggest the conditions under which such issuances could be successful.

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