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On the Exclusion of Assets from Tests of the Mean Variance Efficiency of the Market Portfolio

SHMUEL KANDEL*

ABSTRACT

This paper presents an analysis of the testability of the mean variance efficiency of a market index when the returns on some components of the index itself are not perfectly observable. The results are basically not supportive of the notion that mean variance efficiency is testable on a subset of the assets. Bounding the market share of the missing asset and its expected return is not sufficient to produce a valid test. When the variance of the missing asset is bounded, and the amount of wealth that might be missing is small, it is possible, in principle, to reject correctly the mean variance efficiency of a market index.

IN HIS FAMOUS CRITIQUE of the asset pricing theory tests, Roll [7] concludes that the only testable hypothesis associated with the generalized two parameter asset pricing model of Black [1] is that the market portfolio is mean variance efficient. It has been emphasized by Ross [10] and Roll [7] that the mean variance efficiency of the market portfolio is mathematically equivalent to a positive linear relation between expected returns and the "betas" calculated against such a portfolio. Roll [7] deduces that the theory (CAPM) "... is not testable unless the exact composition of the market portfolio is known and used in the tests. This implies that the theory is not testable unless all individual assets are included in the sample." Errors of type one or two might occur when a market proxy is used instead of the true market portfolio even if the ex-ante true expected returns and the covariance-variance matrix of the returns on all the assets are used. These errors are the result of the choice of the portfolio, and as Roll puts it: "... even a small misspecification of the proxy's composition can lead to the wrong conclusion." (The market proxy might be mean variance efficient while the true market portfolio is not, and vice versa.) In principle, however, there are cases in which the above two conditions are not *both* necessary to *reject* the theory. One such exception, presented by Roll [7], is the case in which no totally positive portfolio is efficient. If any totally positive market proxy is used, a large sample would reject the null hypothesis unambiguously.

While the market portfolio identification problem constitutes a severe limitation to the testability of the theory, the question of the mean variance efficiency of a particular market index with respect to a subset of assets is a typical problem of statistical inference. The mean variance efficiency can be tested directly or by

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testing the linear security market line relation.[1] In the second type of tests, the number of assets for which the relation is tested may be much less than the number of assets that are included in the computation of the return on the market index. In a recent paper, Stambaugh [14] has investigated the sensitivity of inference about the linearity to changing the set of individual assets for which the linear relation is tested. He has found that the addition of just a few assets to the set of assets used to test the linear relation can produce changes in inference. The sensitivity of the tests to the number of market model equations is not surprising as (a) this is the nature of statistical inference, and (b) even if the tested market index is inefficient with respect to the set of all the assets included in it, it might still be efficient with respect to some subsets of assets. As Stambaugh [13] correctly says, this sensitivity exists whether or not one can identify the market portfolio. Stambaugh [14] also has addressed the empirical question whether alternative market indices produce similar inferences about mean variance efficiency. The tests conducted by him accept linearity and produce identical inferences across all market indices. He concluded that: "The impression ... is that inferences about the CAPM are not sensitive to altering the composition of the market index. ... It remains possible that alternative market portfolios can reverse inferences about the model. But the results of this sensitivity analysis almost surely indicate that such an occurrence is less likely than Roll's [7] arguments suggest." While the indices used in Stambaugh's tests *approximate* returns on portfolios of aggregate wealth and include a broad range of assets, it is clear that they are more similar to each other than to the true market portfolio. There are many other assets ("missing assets") whose returns are not perfectly observable every period, and are not included in the construction of these market indices. The question remains whether the lack of sensitivity of Stambaugh's tests to the choice of a particular market index constitutes evidence that these tests really test the theory.

The main goal of this paper is to analyze the testability of the mean variance efficiency of a market index, when the returns on some components of the index itself are not perfectly observable. This is the situation when there is a large sample of returns on many assets, the included assets, but not for all of them. The *relative* weights of the included assets in the market index may be known, and the total weight of the excluded assets might, or might not, be low. We will refer to the true portfolio of the missing assets as to the missing *asset*, and in general only partial information is available about its parameters. The question we address is what can be said, if anything, about the mean variance efficiency of the market index in these cases of partial observability. Our basic finding is that if there is no information about the missing asset then the hypothesis of the mean variance efficiency of the portfolio cannot be rejected. Or, to put the matter somewhat differently, the efficiency or inefficiency of a portfolio with respect to a subset of assets—by itself—tells us nothing about the efficiency of a larger

[1] Roll [8] reviews tests of mean variance efficiency and points out that the first approach is subject to computational and statistical problems and the second approach is faced with the main problem of errors in variables. Recently, Ross [11], Gibbons [3], and Kandel [5] developed Likelihood Ratio Tests, and Stambaugh [14] developed a Lagrangian Multiplier Test, which seem to avoid some of these difficulties.

portfolio. Furthermore, the same result holds if, in addition to the information about the relative weight of the missing asset, we have perfect or partial information about its expected return. There is, however, an empirically testable implication if perfect or partial information is available about the variance of the missing asset. If the variance is known or an upper bound for it is known, then, in principle, the mean variance of the market portfolio can be rejected. Rejection occurs when the portfolio of the included assets is sufficiently far from the efficient frontier in the mean variance space determined by these assets alone. If information is available about both the variance and the expected return, then the null hypothesis may be correctly rejected if the above distance is either too small or too large. This can be a basis for a future mean variance efficiency test in which a higher level of uncertainty about the market index is allowed.

I. The Model and Its Testability

Let R be a random vector of returns on n risky assets. Let E be the vector of expected returns, and let V be the variance-covariance matrix. To avoid degeneracies, we assume that expected returns are not all the same, and that the variance-covariance matrix is positive definite (p.d.).[2] We will use the same notation with an asterisk (E^*, V^*), when we refer to the parameters of the n + 1 assets: the included assets and the missing asset. A portfolio α is a vector which satisfies $\alpha' e = 1$ where $e' = (1, 1, 1, \cdots, 1)$. Short-selling of all the assets is allowed, and there is no risk-free asset in the market. The mean variance *efficient frontier* of all the portfolios which can be constructed from these assets is defined as the locus of portfolios that have the *smallest variance for a given expected return.* The efficient frontier is completely determined by the vector of expected returns and the variance-covariance matrix. We will distinguish between the efficient frontier in this sense of minimum variance and the stricter sense of efficiency that comes from maximizing return at each level of variance. In the mean variance space the efficient frontier corresponds to a parabola, and an efficient portfolio is a Mean Variance Efficient Portfolio (MVEP) if it corresponds to a point on the *positively sloped* segment of the parabola.[3]

Perfect Information

Let (E, V) be the parameters of the n returns on the included assets: the vector of expectations and the covariance-variance matrix, respectively.

Let μ, σ^2, and u be the parameters of the return on the missing asset: its expected return, variance, and the vector of covariances with the returns on the included assets, respectively.

Let $\eta = ((1 - \gamma)\alpha, \gamma)$, $\alpha \in R^n$, $\gamma \in R$, be a portfolio that assigns weight γ to the missing asset, and weight $1 - \gamma$ to a given portfolio α.

[2] In principle the variance-covariance matrix can be positive semidefinite. If this is the case, and if there are several zero-variance portfolios, then they should have the same expected return in order to avoid arbitrage opportunities (see Ross [10] or Roll [7]). For simplicity we ignore this case.

[3] A complete analysis of the efficient frontier is available in Merton [6] and Roll [7], and also in Sharpe [12], Black [1], and Fama [2].

Let (E^*, V^*) be the parameters of the returns on the $n + 1$ assets:

$$E^* = \begin{pmatrix} E \\ \mu \end{pmatrix} \qquad V^* = \begin{pmatrix} V & u \\ u' & \sigma^2 \end{pmatrix}.$$

Our first result is the workhorse for the subsequent analysis. It characterizes when a portfolio is mean variance efficient (MVEP).

LEMMA 1. *The matrix V^* is positive definite (and can, therefore, be a covariance matrix), and the vector $\eta = ((1 - \gamma)\alpha, \gamma)$ is mean variance efficient portfolio (MVEP) for (E^*, V^*) if and only if there exist scalars $x \geqq 0$ and y such that:*

$$(1 - \gamma)V\alpha + \gamma u = xE + ye \tag{1}$$

$$(1 - \gamma)u'\alpha + \gamma\sigma^2 = x\mu + y \tag{2}$$

$$u'V^{-1}u < \sigma^2 \tag{3}$$

Remark: When $x > 0$ the first two conditions are the well-known linear relation between the vector of expected returns and the vector of betas. Combining the set of n equations of condition 1 with the equation of condition 2 yields the following set of $n + 1$ equations:

$$V^*\eta = xE^* + y(e_{n+1}).$$

Let R^* be the random vector of returns on the $n + 1$ assets, and define $k_1 = y/x$ and $k_2 = 1/x$, then

$$E_i^* = k_1 + k_2 \operatorname{Cov}(R_i^*, \textstyle\sum_{j=1}^{n+1} \eta_j R_j^*) \qquad i = 1, 2, \cdots, n + 1.$$

When $x = 0$ the first two conditions imply that η is the global minimum variance portfolio of the $n + 1$ assets.

Proof: The matrix V is a positive definite matrix, so V^* is positive definite if and only if its determinant is positive. Since[4] $|V^*| = |V|(\sigma^2 - u'V^{-1}u)$, condition 3 of Lemma 1 is necessary and sufficient for V^* to be positive definite. The vector $\eta = ((1 - \gamma)\alpha, \gamma)$ is MVEP for (E^*, V^*) if and only if every other portfolio has either a lower expected return or a higher variance. So $\eta = ((1 - \gamma)\alpha, \gamma)$ is MVEP for (E^*, V^*) if and only if it is the solution of the following convex programming problem for some m:

$$\min_{p \in R^{n+1}} p'(V^*)p, \qquad p'e = 1, \qquad p'E^* \geqq m.$$

Conditions 1 and 2 of Lemma 1 are first-order necessary conditions of this problem. Since $\eta = ((1 - \gamma)\alpha, \gamma)$ is a portfolio by definition, and as this is a convex programming problem, these conditions are also sufficient. (For a further discussion of the mathematics of the efficient set see Roll [7].) Q.E.D.

In the remainder of this paper we will assume that E and V are the known parameters of the included assets, and that $\eta = ((1 - \gamma(\alpha, \gamma)$ is a given portfolio, $0 < \gamma < 1$. We assume that perfect information is not available about the parameters of the missing asset—otherwise it would be "included"—and in each

[4] For a proof see Johnston [4].

case the parameters are assumed to belong to a "feasible set" determined by the available information. We will explore the implications of increasing information about these parameters. The first case assumes that we have no information about the missing asset.

Case 1: No Information

THEOREM 1. *There exist $\mu \in R$, $\sigma^2 \in R_+$, and $u \in R^n$ such that if $E^* = \begin{pmatrix} E \\ \mu \end{pmatrix}$ and $V^* = \begin{pmatrix} V & u \\ u' & \sigma^2 \end{pmatrix}$ then V^* is p.d. and $\eta = ((1-\gamma)\alpha, \gamma)$ is MVEP for (E^*, V^*).*

Proof: The following procedure shows how μ, σ^2, and u can be chosen in such a way that the three conditions of Lemma 1 are satisfied.

1. Let $x > 0$ and y be any two scalars.
2. Consider condition 1 as a set of linear equations in the unknown vector u and solve for u.
3. Let σ^2 be any scalar such that $\sigma^2 > u' V^{-1} u$ (condition 3).
4. Consider condition 2 as an equation in one unknown, μ, and solve for μ. Q.E.D.[5]

Remark: Theorem 1 implies that even if the market portfolio's weights are known and a test of its mean variance efficiency is conducted, the null hypothesis cannot be rejected unless all the individual assets are included in a large sample, or assumptions are made on the parameters of the missing assets. To put the same point another way, any portfolio can be made to be a part of an efficient portfolio for some choice of the missing parameters. Unfortunately, this same negative conclusion is reached when we assume that partial or even perfect information is available only about the expected return of the missing asset.

Case 2: Information about the Expected Return

THEOREM 2. *Assume that μ is known, and let $E^* = \begin{pmatrix} E \\ \mu \end{pmatrix}$. There exist $\sigma^2 \in R_+$ and $u \in R^n$ such that $V^* = \begin{pmatrix} V & u \\ u' & \sigma^2 \end{pmatrix}$ is p.d. and the portfolio $\eta = ((1-\gamma)\alpha, \gamma)$ is MVEP for (E^*, V^*).*

Proof: The following procedure shows how σ^2 and u can be chosen, given μ, in such a way that the three conditions of Lemma 1 are satisfied.

1. Let $x = 0$ and $y = 1/(2L)$ where $L \equiv e' V^{-1} e$.
2. Consider condition 1 as a set of equations in the unknown vector u and solve for u,

$$u = [(\tfrac{1}{2}L)e - (1-\gamma)V\alpha]/\gamma.$$

[5] Another way to prove Case 1 is to take an arbitrary μ and apply Case 2.

3. Consider condition 2 as an equation in one unknown, σ^2, and solve for σ^2,

$$\sigma^2 = \frac{1}{\gamma^2}\left\{(1-\gamma)^2(\alpha' V\alpha) + \frac{4\gamma - 2}{4L}\right\}.$$

Since

$$u' V^{-1} u = \frac{1}{\gamma^2}\left\{(1-\gamma)^2(\alpha' V\alpha) + \frac{4\gamma - 3}{4L}\right\},$$

this implies that $\sigma^2 > u' V^{-1} u$, which is condition 3 of Lemma 1.[6] Q.E.D.

The first positive result occurs when we assume some information about the variance of the missing asset. Now we will be able to reject efficiency in some cases.

Case 3: Information about the Variance

THEOREM 3. *Assume that σ^2 is known. There exist $\mu \in R$ and $u \in R^n$ such that $V^* = \begin{pmatrix} V & u \\ u' & \sigma^2 \end{pmatrix}$ is p.d. and η is MVEP for (E^*, V^*), $\left(E^* = \begin{pmatrix} E \\ \mu \end{pmatrix}\right)$, only if $\sigma^2 > \Delta$, where*

$$\Delta \equiv (\alpha' V\alpha - \alpha^{*\prime} V\alpha^*)(1-\gamma)^2/\gamma^2,$$

and α^* is the portfolio on the efficient frontier with respect to (E, V) which has the same expected return as α. (This case is illustrated in Figure 1.)

Proof: Define $m \equiv \alpha' E$, $L \equiv e' V^{-1} e$, $M \equiv e' V^{-1} E$, $N \equiv E' V^{-1} E$, and $D = NL - M^2$. We know that[7]

$$\alpha^* = (1/D)[(Lm - M)V^{-1}E + (N - Mm)V^{-1}e],$$

and we obtain that $\alpha^{*\prime} V\alpha^* = a' V\alpha^*$ since $\alpha' e = \alpha^{*\prime} e = 1$ and $\alpha' E = \alpha^{*\prime} E = m$. To simplify the mathematical expressions in the proof we introduce a change of variables: r and p instead of x and y. For any scalars x and y there exist scalars r and p such that:[8]

$$x = (1/D)[(1-\gamma)(Lm - M) + \gamma(Lr - Mp)]$$
$$y = (1/D)[(1-\gamma)(N - Mm) + \gamma(Np - Mr)]$$

This together with condition 1 of Lemma 1 imply that there exist μ and u such that $\eta = ((1-\gamma)\alpha, \gamma)$ is MVEP for (E^*, V^*) *only if* there exist scalars r and p such that:

$$u = [(1-\gamma)/\gamma]V(\alpha^* - \alpha) + [((Lr - Mp)/D)E + ((Np - Mr)/D)]e. \quad (1')$$

[6] In this proof, we show that μ and u can be chosen to make η the global minimum variance portfolio. In fact, it is a simple extension of the proof to show that we can take $x > 0$ sufficiently small and then the portfolio η will lie on the strictly positive part of the frontier.

[7] For a proof see Merton [6] or Roll [7].

[8] For any scalars x and y this is a system of two equations in the two unknowns r and p. Note that $NL - M^2 > 0$ since V is p.d. (Cauchy-Scwartz inequality).

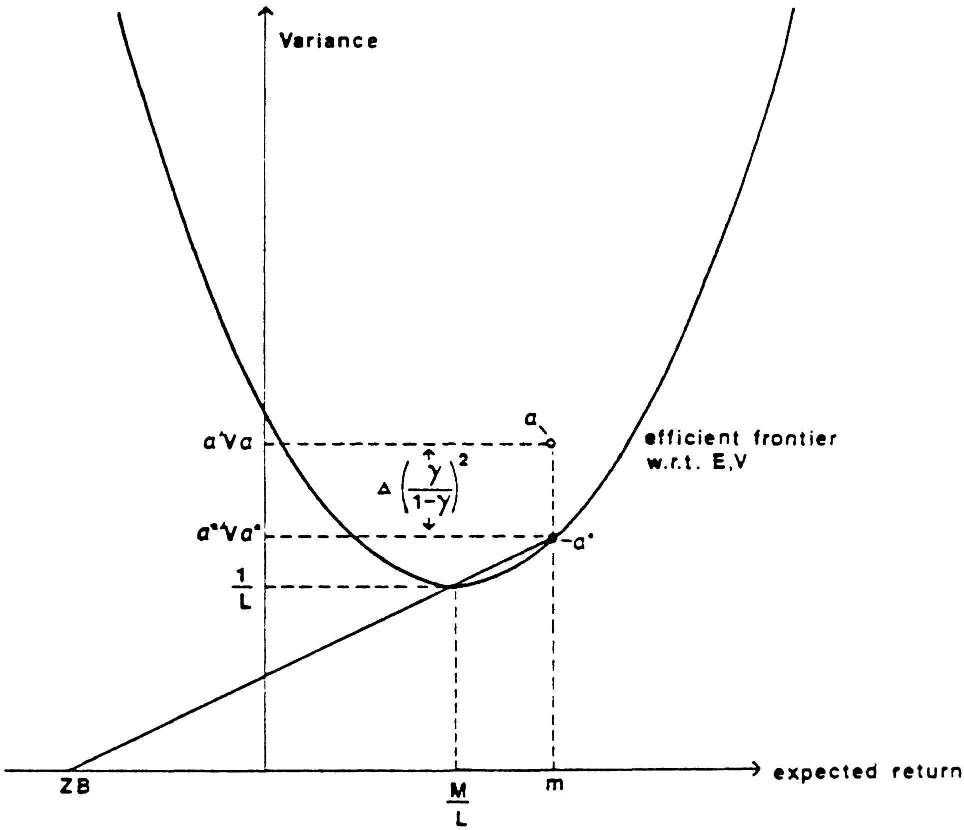

Figure 1.

Note that $e'V^{-1}u = p$ and $E'V^{-1}u = r$ since $\alpha'E = \alpha^{*\prime}E = m$ and $\alpha'e = \alpha^{*\prime}e = 1$. It follows that

$$\Delta = [(\alpha^* - \alpha)'V(\alpha^* - \alpha)](1 - \gamma)^2/\gamma^2,$$

and

$$\begin{aligned} u'V^{-1}u &= \Delta + (1/D)(Lr^2 - 2Mpr + Np^2) \\ &= \Delta + (1/D)[(re - pE)'V^{-1}(re - pE)]. \end{aligned}$$

The necessary condition 3 of Lemma 1 can now be stated as:

$$\Delta + (1/D)[(re - pE)'V^{-1}(re - pE)] < \sigma^2. \qquad (3')$$

Since V is positive definite and $D > 0$, we conclude that $\sigma^2 > \Delta$ is a necessary condition for η to be MVEP. Q.E.D.

Remarks:

1. Suppose that the feasible set of values of the variance, determined by the available information, is bounded. If the upper bound $\bar{\sigma}^2$ (or the true σ^2 in case of perfect information) is less than Δ then the index is not mean variance

efficient. For $0 < \gamma < 1$, Δ is a decreasing function of γ, so the same result holds if the true total weight of the missing assets is replaced by an upper bound of it, $\bar{\gamma}$. In summary, a sufficient condition for rejecting the null hypothesis is:

$$\bar{\sigma}^2 < \frac{(1-\bar{\gamma})^2}{\bar{\gamma}^2}(\alpha' V\alpha - \alpha^{*\prime} V\alpha^*)$$

2. The condition of Theorem 3 becomes sufficient, as well as necessary, when α^* is MVEP (i.e., positively efficient). In this case we can take $r = p = 0$ and get that $x > 0$. In the proof, we have shown that condition 3 of Lemma 1 is satisfied, and u and μ can be solved from conditions 1 and 2 of Lemma 1.

3. We see that even if σ^2 is known, it will be difficult to reject the efficiency of the "big" portfolio if the market share of the missing assets is large (close to 1) or if the subset portfolio, α, is close to being efficient.

4. An interesting question, which is beyond the scope of this analysis, is how to apply the result in this case to the empirical situation in which both E and V are to be estimated using a finite sample. One way is to employ the maximum likelihood technique and to have the condition of Theorem 3 as a constraint in the maximization of the likelihood function under the null hypothesis. (For Likelihood Ratio Tests of mean variance efficiency see Ross [11], Gibbons [3], and Kandel [5].) Another way is to use the sample parameters of the included assets, $\hat{E}$ and $\hat{V}$, instead of the true parameters, and to try to infer whether $(\alpha' \hat{V}\alpha - \hat{\alpha}^{*\prime} \hat{V}\hat{\alpha}^*)$ is significantly greater than $\bar{\sigma}^2 \frac{\bar{\gamma}^2}{(1-\bar{\gamma})^2}$ (note that both $\hat{V}$ and $\hat{\alpha}^*$ are random variables). In either case, the sampling distribution is very complex, and combining the analysis with the statistical inference does not seem to be tractable.

Finally, we consider the case where we have some joint information about both the mean and the variance of the missing asset.

Case 4: Information about μ and σ^2

THEOREM 4. *Assume that μ and σ^2 are known, and let $E^{*\prime} = (E', \mu)$. There exists $u \in R^n$ such that $V^* = \begin{pmatrix} V & u \\ u' & \sigma^2 \end{pmatrix}$ is p.d., and $\eta = ((1-\gamma)a, \gamma)$ is MVEP for (E^*, V^*) only if*

$$\underline{\sigma}^2(\gamma, \mu) < \sigma^2 < \bar{\sigma}^2(\gamma, \mu)$$

where

$$\underline{\sigma}^2(\gamma, \mu) \equiv \Delta + \frac{a - \sqrt{bc}}{2\gamma^2} = \Delta + \frac{a - \sqrt{a^2 - d^2}}{2\gamma^2},$$

$$\bar{\sigma}^2(\gamma, \mu) \equiv \Delta + \frac{a + \sqrt{bc}}{2\gamma^2} = \Delta + \frac{a + \sqrt{a^2 - d^2}}{2\gamma^2},$$

$$a \equiv (1-\gamma)^2(\alpha^* V\alpha^*) + \gamma^2(\alpha_\mu V\alpha_\mu),$$

$$b \equiv ((1-\gamma)\alpha^* + \gamma\alpha_\mu)' V((1-\gamma)\alpha^* + \gamma\alpha_\mu) = a + d,$$

$$c \equiv ((1-\gamma)\alpha^* - \gamma\alpha_\mu)' V((1-\gamma)\alpha^* - \gamma\alpha_\mu) = a - d,$$

$$d \equiv 2\gamma(1-\gamma)\alpha^{*\prime} V\alpha_\mu,$$

and α_μ is the portfolio on the efficient frontier with respect to (E, V) whose expected return is μ:

$$\alpha_\mu = \frac{L\mu - M}{D} V^{-1}E + \frac{N - M\mu}{D} V^{-1}e.$$

Proof: Let

$$S_{\mu,\sigma^2} = \left[(E^*, V^*) \mid E^* = \begin{pmatrix} E \\ \mu \end{pmatrix}, \quad V^* = \begin{pmatrix} V & u \\ u & \sigma^2 \end{pmatrix}, u \in R^n, \quad V \text{ is p.d.} \right].$$

In a similar way to the proof of the previous case, it can be shown that $\eta = ((1-\gamma)\alpha, \gamma)$ is MVEP for some $(E^*, V^*) \in S_{\mu,\sigma^2}$ only if there exist scalars p and r such that:

$$u = ((1-\gamma)/\gamma)V(\alpha^* - \alpha) + ((Lr - Mp)/D)E + ((Np - Mr)/D)e \tag{1$'$}$$

$$\gamma Dk = \xi_0 - r\xi_1 - p\xi_2 \tag{2$'$}$$

$$(Lr^2 - 2Mpr + Np^2)/D < k \tag{3$'$}$$

where

$$\xi_0 \equiv [\mu(Lm - M) + (N - Mm)](1-\gamma)$$

$$\xi_1 \equiv (1-\gamma)(Lm - M) - \gamma(L\mu - M)$$

$$\xi_2 \equiv (1-\gamma)(N - Mm) - \gamma(N - M\mu)$$

$$k \equiv \sigma^2 - \Delta$$

In order to see for which pairs of (μ, σ^2) there exist r and p such that (2′) and (3′) are satisfied, we consider the following minimization problem:

$$\min_{r,p}(Lr^2 - 2Mpr + Np^2)/D$$

subject to

$$\xi_2 p + \xi_1 r = \xi_0 - \gamma Dk.$$

Clearly, the two conditions are satisfied if and only if the optimal value of this problem is less than k. The optimal value (for given ξ_0, ξ_1, ξ_2, and k) is:

$$(\xi_0 - \gamma Dk)^2/(cD^2)$$

where

$$cD^2 = (N\xi_1^2 + 2M\xi_1\xi_2 + L\xi_2^2)$$
$$= [((1-\gamma)\alpha^* - \gamma\alpha_\mu)' V((1-\gamma)\alpha^* - \gamma\alpha_\mu)]D^2.$$

The necessary conditions are satisfied only if

$$k^2\gamma^2 - ka + d^2/4\gamma^2 < 0,$$

where

$$d \equiv 2\gamma(1-\gamma)(\alpha^* V\alpha_\mu) = 2\gamma\xi_0/D,$$

$$a \equiv [(1-\gamma)^2(a^* V\alpha^*) + \gamma^2(\alpha_\mu V\alpha_\mu)] = c + d = b - d,$$

and this is satisfied if and only if

$$(a - \sqrt{a^2 - d^2})/(2\gamma^2) < k < (a + \sqrt{a^2 - d^2})/(2\gamma^2).$$

Recall that $k = \sigma^2 - \Delta$ and get the condition of Case 4. Q.E.D.

Remarks:

1. Similar to Case 3, we can use this result to correctly reject the mean variance efficiency of a portfolio when there is partial or perfect information about the expected return and the variance of the portfolio of the excluded assets.
2. Notice that $\underline{\sigma}^2(\gamma, \mu)$ and $\bar{\sigma}^2(\gamma, \mu)$ are well-defined for every γ and μ since b, c, and γ are non-negative. $b > 0$ for every γ and μ. $c = 0$ only when $\mu = m$ and $\gamma = 0.5$. Therefore, $(\underline{\sigma}^2(\gamma, \mu), \bar{\sigma}^2(\gamma, \mu))$ is a well-defined open interval for every γ and μ, except for $\mu = m$ and $\gamma = 0.5$.
3. We get that $\underline{\sigma}^2(\gamma, \mu) \geqq \Delta$ since $a \geqq \sqrt{bc}$. Equality holds if and only if $\alpha^* V\alpha_\mu = 0$, which means that α_μ is the "zero-beta" portfolio to α^* and μ is the "zero-beta" return to α^*. (Note that both α^* and α_μ are on the efficient frontier w.r.t. (E, V).[9])
4. The necessary condition of Case 4 is necessary and *sufficient* for $\eta = ((1-\gamma)\alpha, \gamma)$ to be *on the efficient frontier* for some $(E^*, V^*) \in S_{\mu,\sigma^2}$. Recall that in the introduction we made a distinction between a portfolio that is *on the efficient frontier* and a MVEP, which is a positively efficient portfolio.
5. The portfolio $\eta = ((1-\gamma)\alpha, \gamma)$ is MVEP for some $(E^*, V^*) \in S_{\mu,\sigma^2}$ if and only if there exist scalars r and p which satisfy conditions (1′), (2′), (3′) and an additional condition (4′):

$$x = (1/D)[(1-\gamma)(Lm - M) + \gamma(Lr - Mp)] \geqq 0 \tag{4'}$$

If the optimal solution (r^*, p^*) of the minimization problem in the proof of Case 4 satisfies this constraint, then the condition of Case 4 is sufficient, as well as necessary, for η to be MVEP for some $(E^*, V^*) \in S_{\mu,\sigma^2}$. Otherwise, we have to solve the set of linear equations,

$$\gamma Dk = \xi_0 - r\xi_1 - p\xi_2, \tag{2'}$$

and

$$(1/D)[(1-\gamma)(Lm - M) + \gamma(Lr - Mp)] = 0, \tag{4'}$$

[9] The zero-beta *return* to a MVEP α is θ if and only if θ is the expected return on a portfolio orthogonal to α. The zero-beta *portfolio* to α is the portfolio on the efficient frontier whose expected return is θ. These terms are well-defined only when α is on the efficient frontier. A discussion of orthogonal portfolios and the definitions of the zero-beta return and portfolio are available in Roll [8] and Kandel [5].

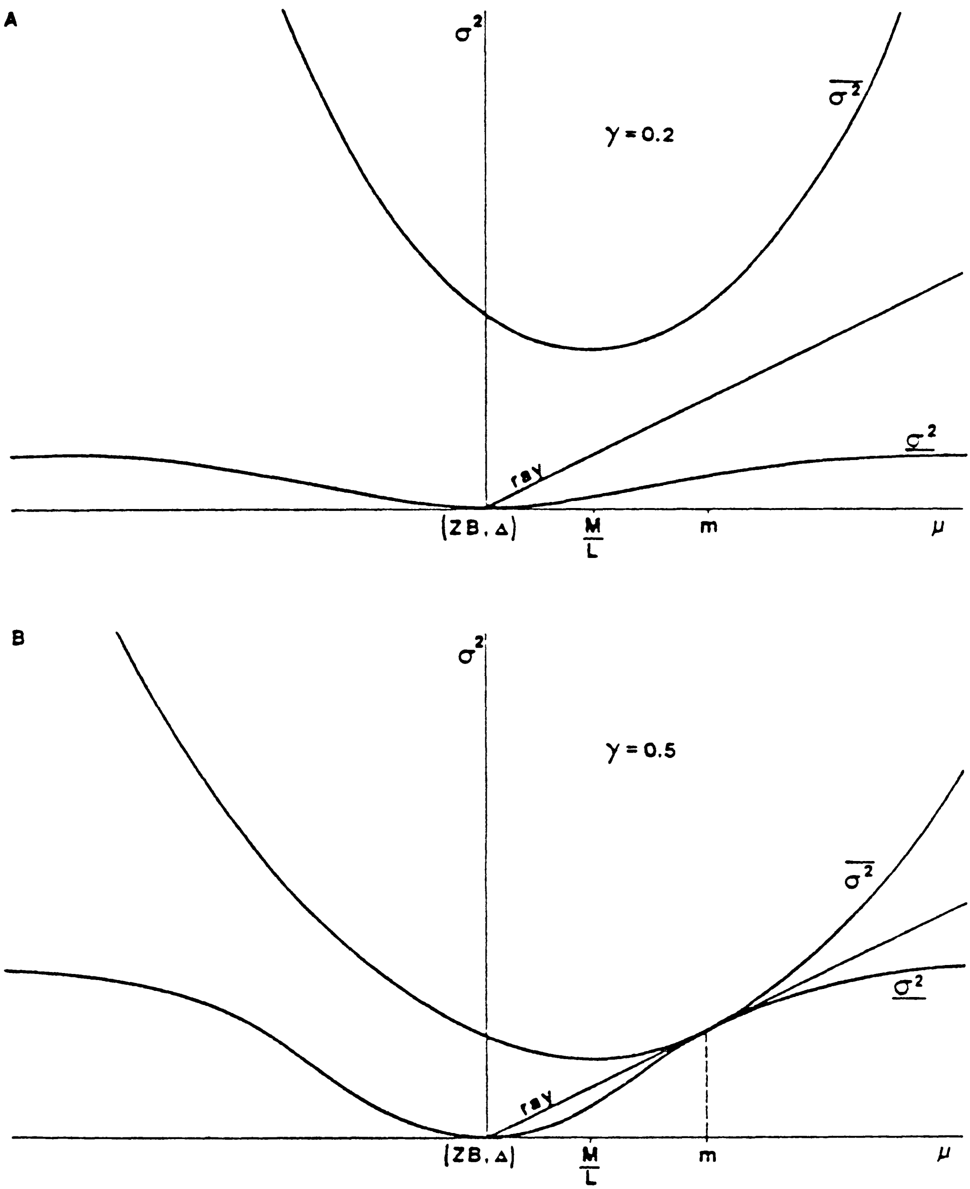

Figure 2.

substitute the solution (r, p) into $(3')$, and find which pairs of (μ, σ^2) are consistent with it.

6. Consider the following ray in the mean variance space:

$$\sigma^2(\mu) = \Delta + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{N - Mm}{D}\right) + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{Lm - M}{D}\right)\mu$$

where μ is greater (less) than the zero-beta return to α^* if m is greater (less) than the expected return on the global minimum variance portfolio. ($\mu > (N - Mm)/(M - Lm)$ if $m > M/L$, and $\mu < (N - Mm)/(M - Lm)$ if $m < M/L$.) It can be easily shown that each point on this ray satisfies the condition of Case 4. Since the set of values of the variance on this ray is not bounded from above and its infimum is Δ, $\eta = ((1 - \gamma)\alpha, \gamma)$ is on the efficient frontier for some $(E^*, V^*) \in S_{\mu,\sigma^2}$ if and only if $\sigma^2 > \Delta$. More than that, it can be shown that if $m > M/L$ then $\eta = ((1 - \gamma)\alpha, \gamma)$ is MVEP for some $(E^*, V^*) \in S_{\mu,\sigma^2}$, where (μ, σ^2) is any point on this ray. In Figure 2, we present some examples of the graphs of $\underline{\sigma}^2$, $\bar{\sigma}^2$, and the ray.

II. Conclusions

Our results are basically not supportive of the notion that mean variance efficiency is testable on a subset of the assets. Bounding the expected return on the missing asset alone is not sufficient to produce a valid test. Adding bounds on the amount of wealth that might be missing is of little use. However, when the variance of the missing asset, or an upper bound for it, is known, it is possible, in principle, to reject correctly the mean variance efficiency of the market portfolio. In this case, the theory can be rejected when the market share of the missing assets is relatively small, and an index portfolio of a subset of assets, with the same relative weights as in the market portfolio, is shown to be sufficiently inefficient on this subset. It is interesting that our results highlight the importance of the variance of the missing asset. However, the requirements that the variance be bounded and that the market share of the missing asset be small, constitute a severe limitation on using a subset of assets to test the theory.

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