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An International Study of Tax Effects on Government Bonds

ROBERT H. LITZENBERGER and JACQUES ROLFO*

ABSTRACT

It is shown that coupon bonds alone are not sufficient to span time-dated claims on ordinary income, capital gains, and non-taxable wealth. In an incomplete bond market where the pure dated claims are not spanned by existing bonds, marginal rates of substitution between present consumption and pure dated claims on ordinary income, capital gains income, and non-taxable wealth, respectively, can differ across bondholders. However, the relative pricing of coupon bonds in each of these countries is shown to be consistent with the tax status of the major (non-tax-exempt) holders of government debt.

IN A COMPLETE BOND MARKET, pure claims on the interest, the capital gains, and the principal repayment components of cash flows can be obtained by holding portfolios of existing bonds. In such a capital market, prices of existing bonds imply for each date a unique set of an income tax rate, a capital gains tax rate, and a marginal rate of substitution between present and future wealth that would be required for an investor to optimally hold positive amounts of all existing bonds. An investor who would optimally hold all existing bonds is denoted as a “representative investor.”

In a capital market consisting solely of coupon bonds, the three classes of pure claims cannot be spanned by existing bonds. In such a market, there does *not* exist for each date a unique set of tax rates and marginal rate of substitution for a comparable “representative investor.” However, if there are two bonds maturing at each cash flow date (i.e., dates of principal and/or coupon payments), conditional on a prespecified capital gains tax rate, it is possible to derive for each date a tax rate on interest income and a marginal rate of substitution. Thus, existing bond prices imply a locus of interest income and capital gains tax rates consistent with “representative investors” holding positive amounts of all existing bonds. Rational bond pricing would require this locus to include the interest income and capital gains tax rates of the major holders of diversified bond portfolios.

This study examines the impact of taxes on the relative pricing of non-callable, default-free coupon bonds in four major bond markets: the Federal Republic of Germany, Japan, the United Kingdom, and the United States. The analysis builds on a number of related theoretical and empirical studies. On a theoretical level, Robichek and Niebuhr [11] show that the tax treatment of capital gains

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can have a substantial effect on the yield curve. Using U.S. data, McCallum [7] and Pye [10] examine the effect of preferential capital gains tax rates on the pricing of government debt. In the context of a static model of term structure, McCulloch [8] provides estimates of the tax rate implied by the pricing of U.S. government debt; he finds that the introduction of taxes substantially improves the explained variation in his term structure model. Using a limited number of bond issues, Van Horne [18] shows that the implied tax rates varied inversely with yield levels for individuals and for corporate investors suggesting the existence of tax-related clientele effects.

The introduction of taxes into an econometric model of the term structure improves the model's explanatory power more in the case of the United States and the United Kingdom than for Japan and Germany due to the greater interest-rate volatility of the former, and the resulting wider range of coupons available on outstanding bonds.¹ Over the period covered by our study, from January 1973 to March 1980, the relative pricing of default-free coupon bonds in the four markets studied is consistent with the tax status applicable to major holders of government bonds.²

The empirical procedures used in the present study are similar to McCulloch's; the discount function is approximated by a cubic spline, i.e., a succession of cubic polynomials with continuous first and second derivatives at the connection ("knot") points. However, a number of modifications have been made to this approach. In particular, the different tax treatment of capital gains in each of the four countries under study as well as a number of more technical changes have been accounted for.

McCulloch interprets his estimate of the implicit tax rate as a weighted average tax rate of market participants. Schaefer [12, 13, 14] argues that, without restrictions on short sales, an equilibrium with divergent tax rates does not exist. He finds that with short sales restrictions on coupon bonds the implicit tax rate is bounded, not determined, by individual tax rates. In contrast to Schaefer's study, the present study does not impose short-selling restrictions on coupon bonds; however, non-negativity restrictions on holdings of pure claims are imposed.

The present study discusses the interpretation of the implied tax rates in complete and incomplete bond markets. The locus of econometrically derived interest income and capital gains tax rates in a given country encompasses points that closely approximate the set of tax rates applicable to some of the major holders of government bonds.³

Section I analyzes the valuation of fixed-income securities in a complete bond market. Sections II and III discuss the identification of implied tax rates in complete and incomplete bond markets, respectively. Sections IV and V, respec-

¹ The percentage improvement is based on the comparison between the "average" minimum standard error of estimates when the income tax rate is optimally derived for a prespecification of the capital gains tax rate of zero and the average standard error of estimates for the no-tax case.

² A major class of private holders of government debt is defined as a category of holders that held on average during the period 10% or more of aggregate private holdings.

³ That is, the combination of income and capital gains tax rates which minimizes the standard error of estimates between observed and estimated bond prices.

tively, discuss the estimation procedure and present a detailed example. The results are presented in Section VI, then summarized and interpreted in Section VII.

I. Valuation of Fixed-Income Securities in a Complete Capital Market

A coupon bond selling at a discount may be viewed as a complex security consisting of a bundle of pure dated claims on interest income (for every coupon date), and of pure dated claims on capital gains income and on non-taxable wealth (for the maturity date). The j th coupon bond has a constant interest income payment C_j at each coupon payment date over its life. (Coupons are paid semi-annually in the United States and in the United Kingdom, and annually in Germany and Japan.) At maturity, the j th coupon bond gives rise, when selling at a discount, to a capital gain payment of $(100 - P_j)$, and to a non-taxable repayment of P_j . In the United States, when a bond is selling at a premium (i.e., above par), the investor has the option, which he usually finds advantageous to exercise, of amortizing the premium against ordinary income over the life of the bond. Thus, at each coupon date, there is an interest income payment of $[C_j - (P_j - 100)/n]$ (where n denotes the number of coupon payments during the remaining life of the bond), and a non-taxable payment of $(P_j - 100)/n$. In addition, there is a non-taxable repayment of 100 at maturity.

In a complete bond market, the pure dated claims on interest income, on capital gains income, and on non-taxable wealth would be spanned by existing securities. Portfolios of existing securities could be constructed with respective payoffs of one unit of interest income, one unit of capital gains income, and one unit of non-taxable wealth at a given date. Arbitrage would assure identical prices of feasible portfolios of existing securities providing identical streams of time-dated claims on interest income, capital gains income, and non-taxable wealth. Thus, the price of a coupon bond could be expressed in terms of the prices of pure dated claims on interest income, capital gains income, and non-taxable wealth. Ignoring existing unrealized gains or losses (the valuation model takes the perspective of the buyer), the market value of a coupon bond selling at a discount in a complete bond market is expressed as:^{4,5}

$$P_j = C_j \sum_{t=T_1}^{T_n} d_i(t) + (100 - P_j) d_g(T_n) + P_j d(T_n), \quad (1)$$

where

P_j = the price of the j th fixed-income security,

n = the number of cash flows (twice a year for the U.K. and U.S.

⁴ The approach followed in this study is "static" in the sense that it assumes a buy-and-hold strategy. Constantinides and Ingersoll [2] have analyzed the impact of taxation on bond prices in the framework of a continuous-time model. Bondholders who are neither tax-exempt nor government bond dealers are taxed on the basis of realized capital gains. These bondholders have a tax incentive to sell bonds with unrealized losses, and a tax disincentive to sell bonds with unrealized gains. Constantinides and Ingersoll have noted that the corresponding effect may be characterized as an option to realize gains or losses prior to maturity. In an analysis of the pricing of government bonds with the same maturity, the authors [5] give some empirical evidence on this effect.

⁵ The time-length of each discounting would reflect the exact timing of the cash-flows.

and once a year for Germany and Japan) during the remaining life of the bond,

$[T_1, \dots, T_n]$ = the sequence of cash flow dates on the j th bond,

C_j = the interest payment at the end of each period (half of the coupon for the U.K. and U.S., the full coupon for Germany and Japan),

t = an interest payment date in the interval $[T_1, \dots, T_n]$,

$d(t)$ = the price of a pure claim on one unit of non-taxable wealth at time t ,

$d_i(t)$ = the price of a pure claim on one unit of interest income payable at time t , and

$d_g(T_n)$ = the price of a pure claim on one unit of capital gains income payable at time T_n (i.e., at maturity).

Expression (1) would hold for all coupon bonds in all four markets, with the sole exception of bonds selling at a premium in the United States.⁶ Rearranging Expression (1) to solve for P_j yields

$$P_j = [100 d_g(T_n) / \{1 + d_g(T_n) - d(T_n)\}] + [\sum_{t=T_1}^{T_n} d_i(t) / \{1 + d_g(T_n) - d(T_n)\}] C_j. \quad (1a)$$

Equation (1a) demonstrates that in a complete bond market the prices of coupon bonds of a given maturity would be a linear function of their coupons.

II. Implied Tax Rates in a Complete Bond Market

If the bond market were complete, in equilibrium, the k th individual's marginal rates of substitution between present consumption and (a) interest income payable in period t , $m_i^k(t)$, (b) capital gains payable in period t , $m_g^k(t)$, and (c) non-taxable wealth payable in period t , $m^k(t)$, would equal the respective prices of these pure claims, $d_i^k(t)$, $d_g^k(t)$, and $d(t)$.

In a complete market these pure claims could be spanned by existing bonds and all individuals would face the same set of bond prices. Therefore, the marginal rates of substitution between present consumption and a given pure claim would be the same for all individuals, assuming no constraints on short sales. For the k th individual, who is subject to an interest income tax rate of $\tau_i^k(t)$ and a capital gains tax rate of $\tau_g^k(t)$, a dollar claim on interest income yields $[1 - \tau_i^k(t)]$ of after-tax wealth, and a dollar claim on capital gains yields $[1 - \tau_g^k(t)]$ of after-tax wealth. Therefore, $m_i^k(t)/m^k(t) = 1 - \tau_i^k(t)$, and $m_g^k(t)/m^k(t) = 1 - \tau_g^k(t)$. In a complete bond market equilibrium, these respective ratios of marginal rates of substitution would be identical for all individuals. It follows that, in the absence of non-negativity restrictions on pure claims, homogeneity of marginal tax rates across individuals is required for market equilibrium in a complete capital market setting.

⁶ In the United States, the price of a coupon bond selling at a premium may be expressed as:

$$P = [C - (P - 100)/n] \sum_{t=T_1}^{T_n} d_i(t) + [(P - 100)/n] \sum_{t=T_1}^{T_n} d(t) + 100d(T_n).$$

Under progressive taxation and in the absence of non-negativity restrictions, tax arbitrage would take place if all individuals were not in the same marginal tax bracket. Lower-income individuals would short-sell pure claims on capital gains income and non-taxable wealth, and purchase pure claims on interest income with the proceeds. Higher-income individuals would short-sell pure claims on interest income and purchase pure claims on capital gains income and non-taxable wealth with the proceeds. This tax arbitrage activity would cease when all individuals are taxed at the same marginal interest income and capital gains tax rates, and the total tax liability is minimized. The uniform marginal interest income and marginal capital gains income tax rates could be imputed as $[1 - \tau_i(t)] = d_i(t)/d(t)$, and $[1 - \tau_g(t)] = d_g(t)/d(t)$, respectively. Under divergent marginal tax rates, non-negativity restrictions on these holdings of pure claims (which are required to prevent tax arbitrage) would result in the weak inequality $m_i^k(t) \leq d_i(t)$, $\forall k$, with an equality holding for individuals having positive claims on non-taxable wealth at date t . Note that this implies that, for an individual holding positive claims on both non-taxable wealth and interest income for the same date, $[1 - \tau_i^k(t)] = d_i(t)/d(t)$. However, for an individual for whom the non-negativity constraint on interest income at t is binding, $[1 - \tau_i^k(t)] \leq d_i(t)/d(t)$. Consequently, $1 - d_i(t)/d(t)$ may be interpreted as the marginal interest income tax rate of an individual holding pure claims on both interest income and on non-taxable wealth at date t . Following an identical argument, $1 - d_g(t)/d(t)$ could be interpreted as the marginal capital gains tax rate of an individual holding both pure claims on capital gains income and pure claims on non-taxable wealth at date t . The implied tax rate on interest income in period t , $\tau_i(t)$, is defined as:

$$\tau_i(t) = 1 - [d_i(t)/d(t)], \quad (2)$$

and the implied tax rate on capital gains in period t , $\tau_g(t)$, as:

$$\tau_g(t) = 1 - [d_g(t)/d(t)]. \quad (3)$$

Note, however, that the prices of these three classes of pure claims cannot be determined from the prices of *coupon* bonds alone. The number of pure claims is three times the number of dates on which coupons are paid. However, it is shown in Equation (1a) that the prices of coupon bonds of a given maturity are a linear function of their coupons. This indicates that the vectors of payoffs of the three classes of time-dated claims from holding a given coupon bond can be replicated by a portfolio of two other bonds of the same maturity.⁷ Thus, the maximum number of coupon bonds having linear independent payoffs (in terms of the three classes of claims) is twice the number of coupon dates. Although the prices of these three classes of pure dated claims cannot be inferred from coupon bonds alone, the additional observation of the price of a default-free tax-exempt bond, or a default-free zero-coupon taxable bond (or, in the case of the U.S., a coupon

⁷ The following identity holds between the prices and the associated cash-flows of three bonds P_1 , P_2 , and P_3 , selling at a discount, carrying coupons of C_1 , C_2 , and C_3 , and maturing on the same day (which also implies that the coupon payments on the three bonds are simultaneous):

$$P_1(C_2 - C_3) + P_2(C_3 - C_1) + P_3(C_1 - C_2) = 0.$$

bond selling at a premium) maturing at every date, is sufficient to identify the prices of the three classes of pure time-dated claims.⁸

III. Implied Tax Rates in an Incomplete Bond Market

Actual bond markets are incomplete. In an incomplete bond market, even in the absence of non-negativity constraints, the marginal rates of substitution $m_i^k(t)$, $m_g^k(t)$, and $m^k(t)$ are not, in general, equal across individuals. In a market equilibrium, the marginal rates of substitution between present consumption and each existing bond, m_j^k 's, are equal across individuals because they face the same set of prices for existing bond, and

$$P_j = m_j^k = C_j \sum_{t=t_1}^{T_n} [1 - \tau_i^k(t)] m^k(t) + [100 - (100 - P_j) \tau_g^k(T_n)] m^k(T_n), \quad \forall k \text{ and } j. \quad (4)$$

However, the set of $\tau_i(t)$'s, $\tau_g(T_n)$'s, and $m(t)$'s consistent with the prices of existing bonds is not unique. In market equilibrium, the $\tau_g^k(T_n)$'s, $\tau_i^k(t)$'s, and $m^k(t)$'s for any individual are consistent with Equation (4). The choice of any particular set of parameters to characterize equilibrium is arbitrary.

Consider the case where there are at least two coupon bonds maturing on each cash-flow date (a "semi-complete" bond market). Rearranging Equation (4) to solve for the prices of coupon bonds with a maturity of T_n yields Equation (4a):

$$P_j(T_n) = A(T_n) + B(T_n)C_j, \quad \forall j \text{ and } T_n, \quad (4a)$$

where

$$A(T_n) = [100\{1 - \tau_g^k(T_n)\}m^k(T_n)]/[1 - \tau_g^k(T_n)m^k(T_n)], \quad \text{and}$$

$$B(T_n) = [\sum_{t=t_1}^{T_n} \{1 - \tau_i^k(t)\}m^k(t)]/[1 - \tau_g^k(T_n)m^k(T_n)].$$

The intercept term, $A(T_n)$, may be interpreted as the price of a portfolio consisting of bonds with a T_n maturity date (with portfolio weights adding to one) and a weighted coupon yield of zero. Such a portfolio can be constructed from any two bonds with a T_n maturity date. Since all individuals face the same bond prices, $A(T_n)$ is a constant across individuals. The slope $B(T_n)$ may be interpreted as the difference between the price of a portfolio having a 100% coupon yield and the price of a portfolio having a zero-coupon yield, each having a T_n maturity date. Such portfolios can be constructed from any two bonds with a T_n maturity date. Since all individuals face the same coupon bond prices, $B(T_n)$ is a constant across individuals. In a semi-complete bond market, the $A(T_n)$'s and the $B(T_n)$'s

⁸ A zero coupon taxable bond would be taxed either as a Treasury Bill with a maturity above a year, the price appreciation being treated as income at maturity, and its price would be:

$$P = (100 - P)d_i(T_n) + Pd(T_n);$$

or, the price appreciation would be amortized over the life of the bond for tax purposes, and its price would be:

$$P = \sum_{t=T_1}^{T_n} [(100 - P)/n]d_i(t) + Pd(T_n).$$

could be derived analytically for each coupon date from the prices of any two bonds with a T_n maturity date.

The $A(T_n)$'s and $B(T_n)$'s do not imply a unique set of interest income tax rate, capital gains tax rate, and marginal rate of substitution between present and future non-taxable wealth for each date. There exists for each date a locus of interest income and capital gains tax rates consistent with the prices of existing bonds. To understand the relationship between points on this tax rate locus and individuals' marginal rates of substitution, consider the special case where each individual's marginal tax rate is constant over time. Using the definitions of $A(T_n)$ and $B(T_n)$ and given that the k th and p th investors face the same bond prices:

$$\frac{(1 - \tau_i^k)}{(1 - \tau_g^k)} = \left[\frac{\sum_{t=T_1}^{T_n} m^p(t)/m^p(T_n)}{\sum_{t=T_1}^{T_n} m^k(t)/m^k(T_n)} \right] \cdot \frac{(1 - \tau_i^p)}{(1 - \tau_g^p)}$$

The term between square brackets would not, in general, be equal to one for a multi-period bond in a semi-complete capital market. This term is the ratio of the p th and k th individuals' marginal rates of substitution between non-taxable wealth in period T_n and an annuity over the dates $[T_1, \dots, T_n]$. Because tax-free claims are not spanned by existing securities in a semi-complete bond market, rates of substitution are not, in general, equal across individuals.⁹

In a semi-complete bond market, conditioning on either a capital gains tax rate, τ_g^k , or a ratio of capital gains to interest income tax rates applicable to a given class of investors, would permit the imputation of the marginal interest income tax rate, $\tau_i^k(t)$, and the marginal rate of substitution between present and future non-taxable wealth, $m^k(t)$, to be inferred analytically for each date. Thus, in a semi-complete bond market, it is rational for investors subject to different tax rates to have positive holdings of all bonds. It is possible that a commercial bank taxed at a 46% rate on both capital gains and interest income, and an individual taxed at 34% on interest income and 17% on capital gains income would each rationally hold a market portfolio of all bonds.¹⁰

In actual bond markets, there do not exist two coupon bonds for each cash flow date; therefore, it is not possible to infer analytically both an interest income tax rate, $\tau_i^k(t)$, and a marginal rate of substitution between present and future non-taxable wealth, $m^k(t)$, conditional on either a capital gains tax rate, τ_g^k , or a ratio of capital gains to interest income tax rate applicable to a given investor. In actual bond markets, the number of coupon bonds having linearly independent payoffs in terms of the three classes of claims is considerably less than twice the number of cash flow dates. However, there are several ways to reduce the number of parameters to be determined from twice the number of cash flow dates. First,

⁹ Note that Equation (4a) would not hold for bonds with one remaining coupon because the entire return would be taxed as ordinary income.

¹⁰ Note that Schaefer's example in [13] showing that, unless two bonds are identical, their returns cannot be equal unless

$$(1 - \tau_i^k)/(1 - \tau_g^k) = (1 - \tau_i^p)/(1 - \tau_g^p),$$

does not extend to the multiperiod case when marginal rates of substitution differ between individuals.

the marginal tax rate on interest income faced by a given individual may be assumed to be an intertemporal constant. If there are n cash-flow dates, this *a priori* restriction reduces the number of parameters to be estimated by $(n - 1)$. Second, the functional relationship between the individuals' marginal rate of substitution and time can be restricted.

Some restrictions on individuals' marginal rates of substitution can be imposed based on *a priori* economic considerations. The existence of money assures that each individual's marginal rate of substitution between present and future non-taxable wealth is a monotone decreasing function of time. A continuous first derivative is necessary for the existence of marginal rates of substitution between non-taxable wealth in adjacent (continuous) periods. The continuity of the second derivative is necessary for the marginal rate of substitution between non-taxable wealth in adjacent (continuous) periods to be a smooth function of time.

Even when the discount function for a given individual is constrained to have continuous first and second derivatives, there is still considerable discretion in the choice of a functional form. Equation (5) is obtained by constraining the tax rates applicable to a market participant to be intertemporal constants, τ_i and τ_g :

$$P_j = C_j(1 - \tau_i) \sum_{t=T_1}^{T_n} m(t) + [100 - (100 - P_j)\tau_g]m(T_n), \quad \forall j. \quad (5)$$

where the $m(t)$'s for a given market participant are restricted to a smooth function of maturity.

The choice of a specific functional form for the discount function is arbitrary to a large extent. For instance, Vasicek and Fong [19] prefer an "exponential spline" procedure on the grounds that the discount function is of an exponential type rather than of a polynomial one. Hence, they transform the discount function into a power function which is in turn estimated by a cubic spline. Poirier [9] notes that draftsmen have long made use of the smoothness property of splines [9, p. 3]. He demonstrates [9, pp. 49–50] that fitting a cubic spline minimizes the integral of the square of the second derivative, which is a good approximation of the curvature when the absolute value of the first derivative is small compared to 1 ("minimum curvature" property). Consequently, a cubic spline ensures a high fitting quality and a "smooth" shape.

However, each of the imposed restrictions is a potential source of misspecification. For example, if these restrictions were imposed in a semi-complete capital market, the pricing relationship given in Relation (4a) would not hold, in general, without error. The resulting error term would be attributable to the incorrect specification of a stationary marginal tax rate/or to the functional form between individuals' marginal rates of substitution and time.

In an incomplete bond market, it is possible to reduce the number of parameters to equal exactly the number of bonds having linearly independent payoffs in terms of the three classes of claims. However, such an approach would not necessarily result in the most accurate estimates of the individuals' marginal rates of substitution. For example, any misspecification resulting from constraining the marginal tax rates on interest income to be an intertemporal constant would affect the estimation of the parameters of the marginal rates of substitution function. Furthermore, the price data used in the study are based on either averages of dealer bid-ask quotes (for the U.S. and the U.K.) or non-synchronous

closing prices (for Germany and Japan). Under such a procedure, the measurement errors associated with the data would also be reflected in parameter estimates.

An alternative approach is to choose a functional form of the discount function where the number of parameters to be estimated is substantially less than the number of bonds having linearly independent payoffs. The marginal tax rate on interest income and the marginal rates of substitution for a "representative investor" who holds all bonds in positive amounts could then be estimated econometrically, conditional on capital gains tax rates applicable to major holders of coupon bonds.

Since the number of parameters of the cubic spline is substantially less than the number of existing bonds, the system of equations is overidentified. An error term is added to Equation (5), and a goodness-of-fit criterion is employed to estimate econometrically the tax rates, τ_i and τ_g , and the discount function $m(t)$ that best explain the structure of coupon bond prices.

IV. Estimation Procedure

As previously discussed, the estimation procedure used in this paper is based on the representation of the discount function $m(\cdot)$ by a cubic spline. Over each of $k + 1$ consecutive intervals defined by $k + 2$ knots placed at time 0, $t_1, \dots, t_{k+1}$, the discount function is a cubic polynomial. At each knot point, the discount function and its first and second derivatives are continuous to ensure that two adjacent cubic functions "connect" smoothly.

The general functional form of a succession of cubic polynomials can be written as:

$$m(t) = a_1 + b_1 t + c_1 t^2 + d_1 t^3 + \sum_{i=1}^k [A_{i+1} + B_{i+1}(t - t_i) + C_{i+1}(t - t_i)^2 + F_{i+1}(t - t_i)^3] D_i(t), \quad (6)$$

where $D_i(\cdot)$'s $i = 1$ to k , are step functions at the knot points:

$$D_i(t) = 0, \quad t < t_i, \quad \text{and} \quad D_i(t) = 1, \quad t \geq t_i, \quad i = 1 \text{ to } k. \quad (7)$$

This formulation introduces 4 additional parameters for each interval. To ensure the continuity of the discount function at the knot points, all A_i 's are set equal to zero, as can readily be seen by equating the limits of $m(t)$ when t tends toward t_i from the left side and from the right side. Likewise, to ensure the continuity of the first and second derivatives of the discount function at all knot points, B_i 's and C_i 's need to be taken equal to zero. In addition, since the marginal rate of substitution between non-taxable wealth and money is unity, $m(0) = 1$, which requires that a_1 be equal to one.

Using these restrictions on Equation (6):

$$m(t) = 1 + b_1 t + c_1 t^2 + d_1 t^3 + \sum_{i=1}^k F_{i+1}(t - t_i)^3 D_i(t). \quad (8)$$

The discount function is uniquely defined by the $k + 3$ parameters $b_1, c_1, d_1, F_2, \dots, F_{k+1}$. Prices for non-callable bonds can be expressed as linear combinations

of the discount function valued at the associated cash flow dates:

$$P = C(1 - \tau_i) \sum_{t=T_1}^{T_n} m(t) + [100 - \tau_g(100 - P)]m(T_n), \quad (9)$$

for all bonds with the exception of bonds selling at a premium in the U.S. market.¹¹

After substituting the right-hand side of Relation (8) into Equation (9), the estimation problem is reduced to a multiple regression on a set of $k + 3$ parameters, with the dependent variable defined as $(P - nA - B)$, and the constraint that the constant term be suppressed:

$$\begin{aligned} P - nA - B = & b_1[A \sum_{t=T_1}^{T_n} t + BT_n] \\ & + c_1[A \sum_{t=T_1}^{T_n} t^2 + BT_n^2] + d_1[A \sum_{t=T_1}^{T_n} t^3 + BT_n^3] \\ & + \sum_{i=1}^k F_{i+1}[A \sum_{t=T_1}^{T_n} (t - t_i)^3 D_i(t) + B(T_n - t_i)^3 D_i(T_n)] + u, \end{aligned} \quad (10)$$

where $A = C(1 - \tau_i)$, $B = 100 - \tau_g(100 - P)$, and u is an error term. For non-callable bonds selling at a premium in the United States, A and B are redefined as $C(1 - \tau_i) + \tau_i(P - 100)/n$, and 100, respectively.

Note that the price component P which contains some measurement error appears both in the dependent variable and in the independent variable which are the terms in square brackets. An instrumental variable approach is used to obtain consistent estimates of the coefficients. The two desirable properties of an instrumental variable are a high correlation with the true independent variable and no correlation with the measurement error. For each independent variable, an instrument is constructed by replacing P by 100. The resulting instrumental variables are uncorrelated with the measurement error in the observed price. Furthermore, they are highly correlated with the true independent variable because only the observed prices are replaced, and the "true" prices are highly correlated with coupon yield (which is included in the instrument).¹²

The approach followed by McCulloch is formally different in that he chooses a different basis for the cubic spline; however, the two approaches are mathematically equivalent (see Shea [15]). A modified version of McCulloch's NBER computer program is used in this study. Differences between the various markets are incorporated in the estimation procedures. Coupons are paid annually in the Federal Republic of Germany and Japan, and semi-annually in the U.K. and U.S. Amortization of premium is specific to the U.S. market. In markets outside the U.S., the same formulation holds whether the bond is at a premium or at a discount. The constraint that the capital gains tax rate be half the income tax

¹¹ In the case of bond selling at a premium in the U.S.,

$$P = [C(1 - \tau_i) + \tau_i(P - 100)/n] \sum_{t=T_1}^{T_n} m(t) + 100m(T_n).$$

¹² McCulloch [8] and Jordan [3] suggest that this approach, which was first used by McCulloch, results in estimates similar to those obtained through OLS estimation.

rate applies only to the "individual" segment of the U.S. market participants (after October 1978, the ratio changes to 40%).

For a given income tax rate and a given capital gains tax rate, the estimation equations are linear in the parameters of the cubic splines. However, the estimation equations are not simultaneously linear in both the parameters of the cubic spline and the tax rates, as can be seen from (4) and (5). An iterative procedure based on an instrumental variable regression technique is used to calculate the interest income and capital gains tax rates which provide the best fit.

While it is technically possible to determine simultaneously two tax rates and the parameters of the after-tax term structure, a locus of capital gains and interest income tax rates which produce approximately the same explanatory power would be expected given our previous discussion of the identification problem. This is confirmed by our empirical results which are discussed in the next section. Slight differences in explanatory power between different points on this locus would be a reflection of a slight misspecification of the after-tax term structure functional relationship rather than a reflection of a "better" combination of tax rates.

The criterion measuring the quality of the fitting is defined as the root of the mean squared relative error, i.e., $\sqrt{[(p_j^o - p_j^e)/p_j^o]^2/T}$ where p_j^o and p_j^e are the observed and estimated prices, respectively, and T is the number of bonds included in the estimation. This procedure gives the same relative importance to each security. McCulloch [8] divides the price error by one half the bid-ask spread, which is also recommended by Jordan [3] as a means to reduce potential heteroscedasticity. There are several reasons why such an approach is not appropriate in the context of the present paper. Bid-ask spreads are not available for the Federal Republic of Germany and Japan. In these markets, quotations usually refer to the last trade. Although bid-ask spreads are available for the U.S. and the U.K. bond market, using them to deflate price errors is not justified. Typically, bid-ask spreads increase with maturity, and are larger for the least frequently traded ("off-the-run") than for the most frequently traded ("on-the-run") issues. Giving less weight to "on-the-run" longer issues than to "on-the-run" shorter issues is not very intuitive. Bid-ask spreads can be viewed as a dealer insurance (or, in a stable market, as a profit-making mechanism for financial intermediaries) around a mid-market price. There does not appear to be any compelling economic rationale for assuming that the standard deviation of the error term is proportional to the bid-ask spread.

The present analysis also differs from McCulloch's in that the knots corresponding to the boundaries of the successive estimation intervals for the three cubic functions have been set exogenously at maturities corresponding to "natural" market boundaries; i.e., 1, 5, 10 years and longest maturity, if applicable. Although the notion of "natural market boundary" is to some extent subjective, it seems preferable to McCulloch's procedure of spacing so that an equal number of issues fall between adjacent knots. The latter is likely to result in an inefficient fitting for longer maturities due to the larger number of securities with short maturities.

V. An Example of the Estimation of Implied Ordinary and Capital Gains Tax Rates, and of the Surface of Standard Errors of Estimates

To generate the surface of standard errors of estimate associated with various combinations of ordinary income and capital gains tax rates, the discount function parameters are estimated for income and capital gains tax rates between -5% and 80% with increments of 5% . McCulloch [8] assumes that the capital gains tax rate is half the income tax rate. Introducing a constraint on the ratio of the capital gains and ordinary income tax rates is tantamount to considering the intersection of the standard errors of estimate surface by the vertical plane representing the constraint.

Consistent with the previous discussion, the surface does not present a global minimum, but rather an approximately linear, usually flat, segment. The choice of a point on this segment has little effect on the quality of the fitting as measured by the standard error of estimate. As an example, the case of the U.S. Government securities market on 1/31/80 is presented for capital gains and ordinary income tax rates varying independently between -5% and 80% (see Table I). Treasury Bills, flower bonds, and callable bonds are excluded.

The standard error of estimate corresponding to a model without taxes (point of coordinates $[0\%, 0\%]$) is 0.00794 . The minimum standard error locus, which is almost linear, connects the points of coordinates $[0\%, 10-15\%]$ and $[80\%, 80\%]$. On this locus, the standard error of estimate varies between 0.00632 (point $[60\%, 60\%]$) and 0.00702 (point $[0\%, 10-15\%]$). The improvement over the feasible segment varies between 11.6% and 20.4% compared to the no-tax situation. The point that McCulloch's procedure would identify, i.e., $CG = Inc/2$, corresponds to an ordinary income tax rate close to 20% (i.e., a capital gains tax rate of 10%) with a standard error of estimate near 0.00683 . The quality of the fitting is quite stable along the minimum standard error locus.

The existence of a locus of points rather than of a unique global minimum for the standard error of estimate surface demonstrates the empirical relevance of our previous discussion of the identification problem. Indeed, the full range of variability of the standard error of estimate surface is exhibited in the intersection of the surface by a "vertical" plane perpendicular to the minimum standard error locus (see [4] for a general discussion of this class of problems). As seen above, the economic interpretation is that, for a given functional form of the time value of after-tax wealth, the model cannot determine a unique set of an income tax rate, a capital gains tax rate, and a discount function.

VI. Presentation of Empirical Results

The portfolio distributions of publicly held government debt in the Federal Republic of Germany, Japan, the United Kingdom, and the United States are presented in Table II. Table III displays the annual averages of the monthly estimation results (from January 1973 to March 1980) for these four countries for prespecified values of the capital gains tax rates as applicable to major market participants. The standard errors of estimates relative to those corresponding to

Table I
U.S. Government Securities Tax Grid for 1/31/80; Standard Errors of Estimate for Selected Values of Capital Gains and Income Tax Rates*

Capital Gains Tax	Income Tax														
	-0.05	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.60	0.70	0.80
-0.050	0.00815	0.00754	0.00715	0.00705	0.00730	0.00790	0.00885	0.01009	0.01161	0.01337	0.01540	0.01772	0.02351	0.03186	0.04630
0.000	0.00870	0.00794	0.00735	<u>0.00702</u>	0.00702	0.00741	0.00818	0.00929	0.01072	0.01242	0.01440	0.01667	0.02233	0.03048	0.0448
0.050	0.00940	0.00850	0.00773	<u>0.00717</u>	<u>0.00691</u>	0.00703	0.00758	0.00853	0.00983	0.01145	0.01336	0.01558	0.02113	0.02908	0.04263
0.100	0.01024	0.00922	0.00830	0.00753	<u>0.00700</u>	0.00683	0.00710	0.00782	0.00896	0.01047	0.01230	0.01446	0.01989	0.02763	0.04075
0.150	0.01122	0.01011	0.00906	0.00811	0.00734	<u>0.00686</u>	0.00679	0.00722	0.00814	0.00949	0.01122	0.01330	0.01860	0.02615	0.03883
0.200	0.01235	0.01116	0.01000	0.00890	0.00792	0.00716	<u>0.00674</u>	0.00680	0.00741	0.00855	0.01013	0.01212	0.01727	0.02463	0.03687
0.250	0.01364	0.01238	0.01114	0.00992	0.00876	0.00775	<u>0.00699</u>	<u>0.00664</u>	0.00685	0.00768	0.00906	0.01090	0.01589	0.02305	0.03486
0.300	0.01508	0.01378	0.01246	0.01114	0.00985	0.00864	0.00759	<u>0.00684</u>	0.00658	0.00697	0.00803	0.00968	0.01445	0.02142	0.03280
0.400	0.01854	0.01715	0.01573	0.01427	0.01279	0.01130	0.00983	0.00846	<u>0.00731</u>	<u>0.00658</u>	<u>0.00657</u>	0.00742	0.01140	0.01792	0.02850
0.500	0.02293	0.02148	0.01997	0.01841	0.01680	0.01515	0.01345	0.011873	0.01002	<u>0.00842</u>	<u>0.00710</u>	0.00640	0.00828	0.01406	0.02384
0.600	0.02865	0.02714	0.02557	0.02393	0.02223	0.02045	0.01861	0.01668	0.01469	0.01264	0.01057	<u>0.00860</u>	<u>0.00632</u>	0.00978	0.01864
0.700	0.03644	0.03489	0.03327	0.03157	0.02979	0.02792	0.02596	0.02389	0.02171	0.01942	0.01700	0.01447	0.00925	<u>0.00635</u>	0.01258
0.800	0.04787	0.04631	0.04467	0.04295	0.04113	0.03921	0.03717	0.03500	0.03270	0.03023	0.02759	0.02475	0.01838	0.01108	0.00649

Note: Points corresponding to the constraint $CG = Inc/2$ are underlined once. Interior points underlined twice correspond to the minimum standard error of estimate for a given value of capital gain tax rate.

* Excluding Treasury Bills, callable and flower bonds.

the no-tax case and the implied income tax rates associated with the capital gains tax status of the major market participants are also given in Table III.¹³

For all markets, specifying a zero capital gains tax rate is a case of major interest because it typically quantifies the improvement in the econometric fitting once taxes are introduced.¹⁴ A zero capital gains tax rate is also of economic interest because individuals in the Federal Republic of Germany and Japan, and individuals, life insurance companies, and non-financial corporations in the United Kingdom are taxed on interest but not on capital gains. For every calendar year, and over the whole data base, "averages" are provided which correspond to the minimization of the sum of the standard errors of estimate during the period under consideration. Overall estimates, mean monthly estimates, and standard errors of monthly estimates of implied income tax rates are calculated. Standard errors of the means are estimated by assuming that the monthly estimates follow a first-order autoregressive process.¹⁵

During the search, the standard errors of estimate are calculated with a precision of 0.00001. When the same standard error of estimate is obtained for several income tax rates, the implied income tax rate column reports the segment over which the standard error of estimate is constant at the 0.00001 level. For instance, in Table III, on 1/31/73, the implied income tax rate for a prespecified zero capital gains tax rate is reported as 20–22%, which means that the income tax rates of 20%, 21%, and 22%, associated with a zero capital gains tax rate, provided the same fitting quality at the 0.00001 level. (Note that increments in income tax rates are in percentage points.) All callable bonds are excluded from

¹³ That is, the set of income taxes providing the lowest standard error of estimates for prespecified capital gains tax rates.

¹⁴ Recall that the minimum absolute standard error of estimate does not vary much for various prespecified capital gains tax rates.

¹⁵ Assuming a first-order relationship in the implied tax series of the form:

$$\tau_i(t) = a + b \tau_i(t-1) + e_t,$$

estimated over T observations, the mean, τ_i , and the standard error of the mean (adjusted for serial correlation), $\sigma(\tau_i)$, of the implied tax series are computed as

$$\tau_i = [\sum_{t=1}^T \tau_i(t)]/T,$$

and

$$\sigma(\tau_i) = \sigma(e_t) [1 + ((\sum_{t=1}^T \sum_{s=1, s \neq T}^T b^{(t-s)})/T) / \{T(1 - b^2)\}]^{1/2}.$$

The standard error, $\sigma(\tau_i)$, is derived based on the relationship

$$\sigma^2(\tau_i) = \sigma^2[\sum_{t=1}^T \tau_i(t)/T] = \frac{1}{T^2} [T\sigma^2(\tau_i) + \sigma^2(\tau_i) \sum_{t=1}^T \sum_{s=1, s \neq T}^T \text{cov}\{\tau_i(t), \tau_i(s)\}],$$

assuming homoscedasticity of the implied tax series and a first-order autoregressive structure. This approach has been used by Litzenberger and Sosin [6]. The standard error of estimates of the first-order relationships are as follows in the case of the Federal Republic of Germany:

$$\begin{aligned} \tau_g = 0: & \quad \tau_i(t) = 1.1468 + 0.8988 \tau_i(t-1) + e_t, \quad \bar{R}^2 = 0.79, \quad \text{and} \quad \sigma(\tau_i) = 3.95; \\ \tau_g = 36\%: & \quad \tau_i(t) = 4.484 + 0.8882 \tau_i(t-1) + e_t, \quad \bar{R}^2 = 0.76, \quad \text{and} \quad \sigma(\tau_i) = 2.47; \\ \tau_g = 56\%: & \quad \tau_i(t) = 7.53 + 0.8693 \tau_i(t-1) + e_t, \quad \bar{R}^2 = 0.72, \quad \text{and} \quad \sigma(\tau_i) = 1.67. \end{aligned}$$

Table II
Portfolio Distributions of Publicly Held Government Debt

	Federal Republic of Germany				Japan		United Kingdom			United States		
	Tax Rate on Capital Gains %	Tax Rate on Interest %	Average Hold- ings 1973-79 %	Tax Rate on Capital Gains %	Tax Rate on Interest %	Average Holdings 1973-79 ^b %	Tax Rate on Capital Gains %	Tax Rate on Interest %	Average Net Purchase 1973/4-1978/9 %	Tax Rate on Capital Gains %	Tax Rate on Interest %	Average Hold- ings 1973-79 %
Domestic banks	36 ^a	61	48.5	40	40	79.5 ^c	52	52	11.9	48 ⁱ	48 ⁱ	31.5
Insurance companies	36	61	11.9	40	40	5.2	52 ^d	52	35.7	30 ⁱ	48 ⁱ	4.0
							0 ^{e,*}	37.5				
							0 ^f	0				
Individuals	0	≤56	19.9	0	10-75	12.9	0 ^g	34	50% of inc. tax ^k	50% of inc.	≤70	9.4
Non-financial corporations	36	61	4.7	n.a.	n.a.	n.a.	0	52		30 ⁱ	48 ⁱ	6.9
Foreigners	Misc.	Misc.	5.6	n.a.	n.a.	n.a.	Misc. ^h	Misc. ^h		30 ⁱ	48 ⁱ	30.7
Others	Misc.	Misc.	9.4	Misc.	Misc.	2.4	Misc. ^h	Misc. ^h		Misc.	Misc.	17.5 ^l

Sources: *Federal Republic of Germany*; Monthly Report of the Deutsche Bundesbank, May issues of 1977, 1978, 1980, "Security Ownership in 19.." It includes Federal, Railways, Post Office, States and Municipalities issues. The share of private offerings is negligible. *Japan*; Bond Review, Monthly Publication by the Bond Underwriters Association of Japan. *United Kingdom*; Adapted from Consecutive Tables 8, Bank of England, Quarterly Bulletin. *United States*; Table I.41, A32, Gross Public Debt of U.S. Treasury, Federal Reserve Bulletin, June 1980.

^a 36% if profits are disbursed, and 56% if profits go to reserve. Also, includes investment funds (4% of average holdings).

^b Excludes medium-term Government Bonds (2-4 years), for which price quotes were unavailable.

^c Also includes City Banks and Long-Term Credit Banks; Central Cooperative Banks for Agriculture and Forestry, and Agricultural Cooperatives (CCBAF), Central Bank for Commercial and Industrial Cooperatives, and Credit Associations (CBCIC), Regional Banks, Trust Banks, Mutual Loan and Savings Banks, and investment funds.

^d Casualty.

^e Life.

^f Pension Funds.

^g Capital gains tax rate of 30% for holding periods under a year.

^h Including pension funds, which are tax-exempt and account for 20.2% of the average net purchases, and Building Societies, which are subject to progressive tax rates and account for 8.5% of average net purchases and investment unit trusts.

ⁱ 46% after December 31, 1978.

^j 28% after December 31, 1978.

^k 40% after December 31, 1978.

^l Includes pension funds and investment funds.

Table III
Annual Averages of the Monthly Estimation Results for Prespecified Values of the Capital Gains Tax Rates

Federal Republic of Germany										United Kingdom			
Year	Minimum Standard Errors of Estimates Relative to the No-Tax Case for Prespecified Capital Gains Tax Rates of:					Implied Income Tax Rates for Prespecified Capital Gains Tax Rates of:					Minimum Standard Errors of Estimates Relative to the No-Tax Case for Prespecified Capital Gains Tax Rates of:		
	0%	36%	56%	0%	56%	0%	36%	56%	0%	56%	0%	52%	52%
1973	0.899	0.893	0.891	21-22	46	61-62	1973	0.673	0.725	0.725	28	61	61
1974	0.901	0.904	0.906	18-19	44-45	60	1974	0.530	0.674	0.674	32	60	60
1975	0.740	0.726	0.720	22	47	62	1975	0.352	0.574	0.574	28	57	57
1976	0.732	0.707	0.695	18	45	61	1976	0.416	0.649	0.649	33	59	59
1977	0.924	0.895	0.881	7-8	38-39	57	1977	0.277	0.398	0.398	39	64	64
1978	0.970	0.950	0.943	5-6	36-37	55-56	1978	0.342	0.430	0.430	35	63	63
1979	0.975	0.990	1.006	4	34	53	1979	0.378	0.453	0.453	34	62	62
1980-I	1.000	1.021	1.044	0-1	31-32	51	1980-I	0.463	0.490	0.490	32	61	61
Overall	0.907	0.893	0.953	13-14	41-42	58-59	Overall	0.433	0.560	0.560	33-34	61	61
Arithmetic mean of monthly estimates	13.48					58.44	Arithmetic mean of monthly estimates					32.98	60.78
Standard deviation of mean	3.95					1.61	Standard Deviation of mean					1.26	0.81

* These estimates minimize the sum of the squared standard error terms of the twelve monthly observations.

the sample.¹⁶ In Japan there are no callable bonds, and in the Federal Republic of Germany, the United Kingdom, and the United States, there are relatively few callable bonds. In addition, flower bonds and Treasury Bills in the United States are excluded. Sources of data are given.¹⁷

VII. Interpretation of Results

In Germany and Japan there has been considerably less variation in interest rates than in the United States and the United Kingdom. Therefore, the introduction of taxes into a model of the term structure of interest rates improves the model's explanatory power much more substantially in the United States and in the United Kingdom than in Germany or in Japan.

The introduction of taxes into the German term structure model reduces the average standard error of estimate by 9.3%, 10.7%, and 4.7% for prespecified capital gains tax rates of 56%, 36%, and zero, respectively. In the Japanese model, the reduction in the standard error of the estimate is 5.1% and 7.3% for capital gains of 40% and zero, respectively. The standard error of the estimate is reduced by 44% and 56.7%, respectively, for capital gains tax rates of 52% and zero in the United Kingdom model. Reductions ranging between 19.8% and 24.2% for capital gains tax rates ranging from zero to 48% are achieved in the U.S. model.¹⁸

The greatest contrast in the taxation of capital gains on government debt exists between Germany and Japan. In Germany, holders of over 85% of the privately held government debt are granted preferential tax treatment on capital gains, whereas in Japan holders of less than 85% of the privately held government debt are granted preferential tax treatment on capital gains.

In Germany, financial and non-financial corporations hold 65% of the government debt and are taxed at 61% on interest, 56% on undistributed capital gains, and 36% on distributed capital gains. The average of the monthly estimations of

¹⁶ In [7] McCulloch includes callable bonds; following the tradition, the maturity date is assumed to be advanced to the earlier redemption date when bonds sell above par. After observing the distortions created by this simplification, callable bonds have been excluded from our sample.

¹⁷ The data on German federal government securities (Anleihen von Bund-(Federal), Bahn-(Railways), and Post-(Post)) have been gathered from the *Handelsblatt* using the quotations of the Frankfurt market. The number of securities is close to 100 per month (88 in January 1973, and 108 in March 1980). The longest maturity is 10 years. For Japan, this study uses the government bond quotations published in the monthly *Bond Review* of the Bond Underwriters Association of Japan. The maturities of Japanese government bonds are less than 10 years. The data base includes 26 bonds in January 1973 and 35 in March 1980. Data on the U.K. gilt market come from the published quote sheet of Greenwell, a major London bond dealer. The number of securities included in the data base varies between 39 in January 1973 and 49 in March 1980. Finally, the analysis uses the New York Federal Reserve Bank's estimates of the closing bid and ask prices on U.S. government securities (Treasury Notes and Bonds). There are 47 non-callable Treasury bonds and notes included in January 1973 and 95 in March 1980.

¹⁸ Starting in January 1979, the capital gains tax rate is constrained at 28% instead of 30%, and starting in November 1978, the ratio of capital gains tax rate over income tax rate is taken as 40% instead of 50% due to changes in tax laws. The improvement in the estimation of the term structure becomes 29.8% and 30.9%, respectively.

the tax rate on interest income conditional on a 56% capital gains tax rate is 58.4%, which is not statistically different from the 61% statutory rate. However, the average of the monthly estimates of the tax rate on interest income conditional on a 36% capital gains rate is 41.5% which is statistically different from the 61% statutory rate.

In Japan, financial institutions hold 85% of the government debt and are taxed at 40% on both interest and capital gains incomes. The average monthly tax rate conditional on a 40% capital gains rate is 40.6%, which is not statistically different from the statutory rate.¹⁹ Thus, in both Germany and Japan, the relative pricing of coupon bonds is consistent with the taxation of financial and non-financial corporations who are the major holders of government debt in these markets and hold positive amounts of most bonds.

In Germany, individuals hold 20% of the government debt, are taxed at rates up to 56% on interest income, but are not taxed on capital gains. The average monthly estimate of the tax rate on interest income conditional on a zero capital gains tax rate is 13.5%, which is significantly greater than zero. In Japan, individuals hold 13% of the government debt, and are taxed at rates between 10% and 75% on interest income, but are not taxed on capital gains. The average monthly estimate of the tax rate on interest income conditional on a zero capital gains tax rate is 12.6%, which is also statistically greater than zero. Tax brackets of 22% in Germany and 18% in Japan are over two standard errors above the average estimate. Thus, in both Germany and Japan, one would expect individuals in high tax brackets who hold bonds to tilt their bond portfolios toward issues selling at a discount.

Unfortunately, data available in the United Kingdom do not include holdings but only net purchases of government debt, which is not sufficient to provide a complete picture of bondholding by tax status. Banks and casualty insurance companies are taxed at 52% on both interest and capital gains. The average monthly estimate of the tax rate on interest income conditional on a 52% capital gains tax rate is 60.8%, which is statistically greater than the 52% statutory tax rate on interest applicable to these institutions.²⁰ Thus, one would expect banks and casualty companies in the U.K. to tilt their bond holdings away from bonds selling at deep discounts toward bonds selling at or near face value.

In the United Kingdom, bondholders other than banks, casualty companies, and building societies are not taxed on capital gains (for holding periods over 1 year). Individuals are taxed at a minimum of 34% on interest income, non-

¹⁹ The standard error of estimates of the mean income tax rate is derived using the estimated first-order relationships:

$$\begin{aligned}\tau_g = 0: \quad & \tau_i(t) = 4.0532 + 0.6683 \tau_i(t-1) + e_t, \quad \bar{R}^2 = 0.43, \quad \text{and} \quad \sigma(\tau_i) = 2.51; \\ \tau_g = 40\%: \quad & \tau_i(t) = 7.3948 + 0.8083 \tau_i(t-1) + e_t, \quad \bar{R}^2 = 0.59, \quad \text{and} \quad \sigma(\tau_i) = 4.24.\end{aligned}$$

²⁰ The standard error of estimates of the mean income tax rate is derived using the estimated first-order relationships:

$$\begin{aligned}\tau_g = 0: \quad & \tau_i(t) = 8.0007 + 0.7553 \tau_i(t-1) + e_t, \quad \bar{R}^2 = 0.57, \quad \text{and} \quad \sigma(\tau_i) = 1.26; \\ \tau_g = 52\%: \quad & \tau_i(t) = 14.5578 + 0.7594 \tau_i(t-1) + e_t, \quad \bar{R}^2 = 0.59, \quad \text{and} \quad \sigma(\tau_i) = 0.81.\end{aligned}$$

financial corporations are taxed at 52% on interest income, and pension funds are tax exempt. The average monthly estimate of the tax rate on interest is 33%, which is not statistically different from the 34% minimum tax rate that individuals pay on interest; it is significantly less than the 52% tax rate that non-financial corporations pay on interest, and is statistically greater than the zero tax rate than pension funds pay on interest.²¹ A tax rate of 36% is over two standard errors above the estimated tax rate. Thus, high tax-bracket individuals and non-financial corporations would be expected to tilt their bond holdings toward deeply discounted bonds, and pension funds would be expected to tilt their bond holdings away from deeply discounted bonds. For individuals subject to the minimum tax rate on interest income, the rational holding of all bonds in positive amounts would be consistent with these results. The overall results are consistent with bond market clearing although at least some classes of bondholders would not be expected to hold market portfolios of bonds.

In the U.S. market, the major private domestic holders of government debt are individuals and banks. For individuals, the capital gains tax rate is only a fraction of the income tax rate (50% until October 31, 1978 and 40% afterwards).²² The average of the monthly estimates of the interest income tax rate over the period January 1973–October 1978 is 33.5% based on a prespecified capital gains tax rate of half the ordinary “average” of 23–25% in 1973 to a maximum annual “average” of 45% in 1975–76. This result is consistent with McCulloch’s results [8].²³ Over a more recent but shorter period, November 1978–March 1980, and for a prespecified ratio of 40%, the estimated ordinary income tax rate drops to 12.3%. The average monthly estimate of the ordinary income tax rate conditional on a zero capital gains tax rate is 20.69%, and statistically different from zero. (A tax bracket of 31.2% is two standard errors above the mean estimates.²⁴)

²¹ The quality of the fitting, assuming a zero capital gains tax rate and 37.5% income tax rate, is a well-known fact by gilt brokers who, in their quotation sheets, publish the corresponding net yields in addition to the traditional yield to maturity.

²² Corporations are also taxed at differential tax rates, 48% until December 1978, and 46% afterwards, on capital gains. Although commercial banks are large holders of bonds, and are generally taxed as corporations, they do not benefit from a preferential tax treatment on capital gains. Life insurance companies are required to allocate investment income between policy-holders and owners. The former portion is tax-exempt, the latter is taxed at ordinary corporate income tax rates. (As a result, their average effective tax rate on investment income is well below the ordinary corporate income tax rate.) The average monthly estimate of the ordinary income tax rate conditional on the 30% tax rate paid by corporations until December 31, 1978, 42.2%, is statistically different from the 48% statutory rate ($t = -8$). The same result holds after December 31, 1978 ($t = -18.7$). This suggests that corporations would prefer to hold discounted rather than non-discounted bonds.

²³ Although McCulloch’s estimates are well below Skelton’s, this analysis shows that the two results are consistent because they focus on different categories of bondholders.

²⁴ The standard error of estimates of the mean income tax rate is derived using the estimated first-order relationships:

$$\begin{aligned} \tau_g = 0: & \quad \tau_i(t) = 3.5008 + 0.8375 \tau_i(t-1) + e_i, \quad \bar{R}^2 = 0.71, \text{ and } \sigma(\tau_i) = 5.26; \\ \tau_g = 30\% (1/73-12/78): & \quad \tau_i(t) = 6.2780 + 0.8523 \tau_i(t-1) + e_i, \quad \bar{R}^2 = 0.72, \text{ and } \sigma(\tau_i) = 0.49; \\ \tau_g = 28\% (1/79-3/80): & \quad \tau_i(t) = 21.2478 + 0.3049 \tau_i(t-1) + e_i, \quad \bar{R}^2 = 0.05, \text{ and } \sigma(\tau_i) = 0.96; \\ \tau_g = 48\% (1/73-12/78): & \quad \tau_i(t) = 6.8836 + 0.8757 \tau_i(t-1) + e_i, \quad \bar{R}^2 = 0.77, \text{ and } \sigma(\tau_i) = 3.99; \end{aligned}$$

Commercial banks, which hold 31.5% of the privately held government debt, are taxed at a rate of 46% on both interest income and capital gains (48% before December 31, 1978). The average of the monthly estimates of the interest income tax rate over the period January 1973–December 1978 is 55.1% based on the 48% interest income tax rate paid by commercial banks until December 31, 1978. This estimate is not statistically different from the 48% statutory rate ($t = 1.8$), which is consistent with commercial banks holding positive amounts of all bonds. Commercial banks are the major holders of both government and municipal debt. They are able to issue debt, deduct the interest payments from their profits, and hold municipal bonds. Skelton [16] has argued that high grade corporate tax debt and municipal bonds, aside from tax considerations, are close substitutes, and that tax arbitrage by commercial banks would assure that the tax rate implicit in the high grade corporate and municipal debt would be equal to the corporate statutory rate. The estimates by Skelton [16] and Trzcinka [17] of the tax rates implicit in the relative pricing of corporate and municipal bonds are consistent with the above discussed estimates of the tax rates implicit in the relative pricing of government bonds with different coupons. An equilibrium is possible where banks hold municipal bonds as well as discounted and non-discounted government bonds. High tax-bracket individuals would hold municipal bonds while low tax-bracket investors would hold taxable bonds.

VIII. Conclusion

This paper examines the effect of taxes on the relative pricing of fixed-income securities in the Federal Republic of Germany, Japan, the United Kingdom, and the United States. In the United States and the United Kingdom, the introduction of taxes into an econometric model of the term structure substantially decreases the unexplained variation. In Japan and Germany, the introduction of taxes results in a smaller decrease of the unexplained variation.

It is shown that coupon bonds alone are not sufficient to span time-dated claims on ordinary income, capital gains, and non-taxable wealth. In an incomplete bond market where the pure dated claims are not spanned by existing bonds, marginal rates of substitution between present consumption and pure dated claims on ordinary income, capital gains income, and non-taxable wealth, respectively, can differ across bondholders. Nevertheless, the relative pricing of coupon bonds in each of these countries is shown to be consistent with the tax status of the major (non-tax-exempt) holders of government debt. This result is consistent with the findings of Skelton [16] and Trzcinka [17] for the U.S. that, on a risk-adjusted basis, the pricing of tax-exempt bonds is in accordance with the hypothesis of a tax-exempt interest rate equal to one minus the corporate tax rate times the equivalent taxable interest rate in a market where the predominant category of bond buyers is taxed at the corporate tax rate.

$$\tau_g = 46\% (1/79-3/80): \quad \tau_i(t) = 27.0968 + 0.4165 \tau_i(t-1) + e_i, \quad \bar{R}^2 = 0.15, \text{ and } \sigma(\tau_i) = 0.86;$$

$$\tau_g = \text{Inc}/2 (1/73-10/78): \quad \tau_i(t) = 4.8023 + 0.8627 \tau_i(t-1) + e_i, \quad \bar{R}^2 = 0.76, \text{ and } \sigma(\tau_i) = 7.52;$$

$$\tau_g = 0.4\% (11/78-3/80): \quad \tau_i(t) = 51.1832 + 0.6556 \tau_i(t-1) + e_i, \quad \bar{R}^2 = 0.42, \text{ and } \sigma(\tau_i) = 3.61.$$

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