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Author(s): C. G. C. Pitts and M. J. P. Selby
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The Pricing of Corporate Debt: A Further Note

C. G. C. PITTS and M. J. P. SELBY*

IN A MAJOR ARTICLE in 1974, Merton [3] described a model giving a formula for the risk premium H on risky zero-coupon debt. Recently Lee [2] re-examined Merton's model and gave somewhat different graphs for H as a function of time. While Lee eliminated a source of confusion which arose from one of Merton's graphs, Lee's graph is not entirely correct. The purpose of this note is to prove mathematically two properties of H which will settle the disparities.

We now set forth the issue in detail. Merton described *inter alia* how H varied with d, τ and σ^2 where

$$d = Be^{-r\tau}/V;$$

B is the face value of the debt, V is the current value of the firm, τ is the time to maturity of the debt, and r is the risk-free rate. Merton refers to d as the "quasi" debt-to-firm value ratio (see [3] pp. 449–455). The risk premium is defined as $H = R - r$ where R is the yield to maturity on the risky debt, provided the firm does not default. Merton gave a diagram ([3] Figure 3), to show how, for different values of d, H varies with τ.

Lee correctly pointed out that, because H is an increasing function of d (see (23) of [3]), there could be no intersection between the curves for $d < 1$ and those for $d \geq 1$, and redrew the graph of H (see [2]). Unfortunately the behaviour of H is not correctly depicted when τ is small.

There are two aspects of Lee's graph which are erroneous or at least misleading. Firstly, when $d = 1$, the risk premium H increases without bound as the time to maturity decreases to zero. Secondly, when $0 < d < 1$, so the firm is currently solvent, the slope of the graph of H should become zero as the time to maturity decreases to zero. These two aspects of the behaviour of H were correctly depicted by Merton, but not by Lee.

Since there is no way of deciding from [2], [3] which graph is correct, we prove the following result which settles both points; it shows that these features of H are direct consequences of the model set up by Merton in [3].

PROPOSITION. *If* $H \equiv H(d, \tau, \sigma^2)$ *then*

$$\text{(i)} \quad H(1, \tau, \sigma^2) \to \infty \quad as \quad \tau \to 0;$$

$$\text{(ii)} \quad H_\tau(d, 0, \sigma^2) = 0 \quad whenever \quad 0 < d < 1,$$

where the subscript τ *denotes partial differentiation with respect to* τ.

* London Business School, London, and University of East Anglia, Norwich, U.K.; and London Business School, London, respectively. The authors are very grateful to Professor Richard Brealey, Dr. Stewart Hodges and Dr. Stephen Schaefer for their helpful comments.

Proof of (i). From [4] pp. 454, 455

$$H = -\tau^{-1}\cdot\log P, \tag{1}$$

$$P = \Phi(h_2) + d^{-1}\cdot\Phi(h_1), \tag{2}$$

$$\left.\begin{aligned} h_1 &\equiv h_1(d, T) = -(\tfrac{1}{2}T - \log d)/\sqrt{T}, \\ h_2 &\equiv h_2(d, T) = -(\tfrac{1}{2}T + \log d)/\sqrt{T}. \end{aligned}\right\} \tag{3}$$

where $T = \sigma^2\tau$, and Φ is the normal distribution. Thus when $d = 1$ we have $h_1 = h_2 = -\frac{1}{2}\sqrt{T}$ so $P = 2\Phi(-\frac{1}{2}\sqrt{T})$. Now for small values of x

$$\Phi(x) \simeq \Phi(0) + x\Phi'(0) = \tfrac{1}{2} + x/\sqrt{2\pi};$$

hence for small values of τ we have

$$P \simeq 1 - \sigma\sqrt{\tau/(2\pi)}. \tag{4}$$

Thus

$$H \simeq \sigma/\sqrt{2\pi\tau} \tag{5}$$

for small τ, so $H \to \infty$ as $\tau \to 0$.

Proof of (ii). We have to show that

$$\lim_{\tau\to 0}\{H(d, \tau, \sigma^2) - H(d, 0, \sigma^2)\}/\tau = 0. \tag{6}$$

To do this, first we define $H(d, 0, \sigma^2) = 0$ (H is not defined by formula (14) of [4] when $\tau = 0$); we then show that $H(d, \tau, \sigma^2)$ tends to zero faster than τ, so establishing (6). For the latter we need some bounds for the magnitude of the normal distribution functions which arise in (2).

From Feller ([2] Lemma 2, p. 175) we have, as $x \to \infty$

$$1 - \Phi(x) \sim \frac{e^{-x^2/2}}{\sqrt{2\pi}\cdot x}$$

where $\sim$ means that the ratio of the two sides tends to 1. Now for any fixed number k, $e^{-x} < 1/x^k$ for all x sufficiently large (and positive); hence as $x \to \infty$

$$1 - \Phi(x) = O(x^{-2k-1}),$$

where O denotes the Landau order symbol[1]. Thus as $x \to -\infty$

$$\Phi(x) = 1 - \Phi(-x) = O(x^{-2k-1}).$$

Now assume d is fixed such that $0 < d < 1$; when $\tau \to 0$ we have (from (3))

$$h_1 \sim (\log d)/\sigma\sqrt{\tau}, \qquad h_2 \sim -(\log d)/\sigma\sqrt{\tau}$$

so $h_1 \to -\infty$, $h_2 \to +\infty$. Hence

$$P = 1 + O(h_2^{-2k-1}),$$

so

$$P = 1 + O(\tau^{k+1/2}), \tag{7}$$

[1] Thus $f(x) = O(g(x))$ for x in some set S means that there exists a constant M such that $|f(x)| < M|g(x)|$ for all x in S.

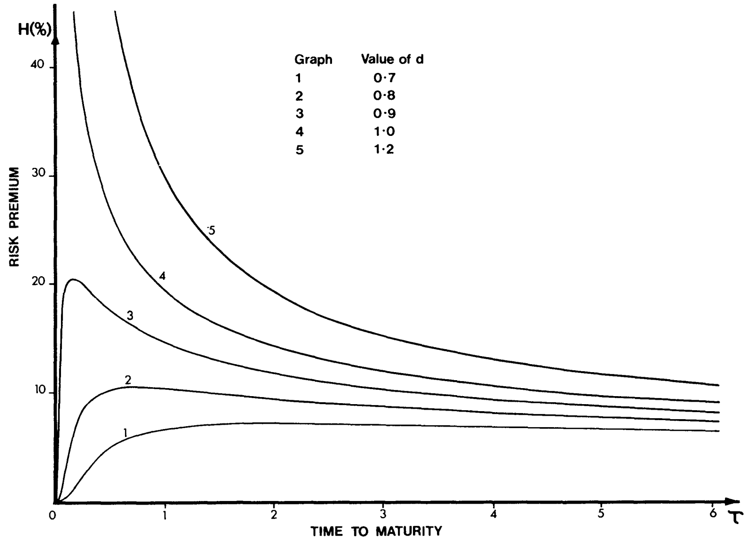

Figure 1. Risk premium as a function of time to maturity.

and thus

$$H = O(\tau^{k-1/2}) \tag{8}$$

for any fixed number k. Taking $k = 2\frac{1}{2}$ we have $H = O(\tau^2)$ so it follows that $H_\tau(d, 0, \sigma^2)$ exists and is zero. This completes the proof.

In fact we have proved a good deal more than we set out to do. We have shown that $H(1, \tau, \sigma^2)$ behaves, as $\tau \rightarrow 0$, like $C/\sqrt{\tau}$ where we give the value of the constant C; for (ii) we have shown that, when $d < 1$, $H(d, \tau, \sigma^2)$ is smaller, again as $\tau \rightarrow 0$, than any positive power of τ.

Figure 1 illustrates the results for $\sigma^2 = 0.2$, and $d = 0.7, 0.8, 0.9, 1, 1.2$.

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