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A Theoretical Model for Valuing Preferred Stock

DAVID EMANUEL*

ABSTRACT

This paper develops a model of preferred stock value which includes the possibility of dividends on the preferred stock being omitted. The analytical framework used is based on the option-hedging methodology of Black and Scholes. Precise valuation formulae are obtained for cumulative and noncumulative preferred stock in a variety of contexts. The values obtained are quite different from those for either riskless or risky perpetual bonds, which have previously been proposed as being similar to preferred stock.

SINCE 1970, AN AVERAGE of 3 billion dollars worth of capital has been raised annually by corporations in the United States in the form of preferred stock. In spite of this impressive figure, there has been very little theoretical analysis of preferred stock valuation.

While previous analysis (e.g., Merton [9] and Ingersoll [7]) has treated preferred stock as a corporate consol bond, this view is not taken here, for an essential feature of preferred stock is that omission of the dividend is permissible and does not precipitate bankruptcy, whereas this would happen if a bond coupon payment were missed. Corporations can and do fail to pay preferred dividends from time to time, particularly in times of severe economic depression.

Herein is developed a theory of preferred stock valuation. The return-generating process for the firm is assumed to be a continuous time diffusion and the option hedging arguments of Black and Scholes [2], Cox and Ross [4], and other authors can be used to derive a functional form of preferred stock value in terms of firm value and dividend arrearage. The key to the valuation theory lies in a particular assumption about the conditions under which preferred stock dividends are omitted. This assumption is that all dividends are omitted whenever firm value falls below some critical amount. All the theory is developed subject to this assumption. Section I of this paper discusses whether such an assumption is reasonable. This section carefully examines conditions under which rational management, seeking to maximize common shareholder wealth, would pay dividends as suggested.

In Section II, the basic model is described and solved. Some generalization is obtained in Section III, where allowance is made for the possible coexistence of other claims such as bonds. Illustrative solutions to the model are provided

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subsequently in Section IV. Securities are valued when cumulative preferred, noncumulative preferred, and consol bonds are outstanding claims in addition to common equity claims. The paper concludes with a summary and a discussion of further extensions of the model. Further avenues for the development of theory, application, and empirical research are suggested and discussed.

I. The Dividend Policy of a Firm with Preferred Stock Outstanding

Generally, in the idealized world of the models discussed here, the Modigliani-Miller irrelevance propositions for capital structure and dividend policy hold true (see Modigliani and Miller [11], Miller and Modigliani [10]). If these propositions hold, one is then entitled to wonder what criterion rational management might use to help decide on a dividend policy. Since the management is normally appointed by the common shareholders, one tends to think of the maximization of common share value as the criterion by which management judges its decisions.[1] Management can either maximize common share value or, equivalently (given that the Modigliani and Miller propositions are assumed to hold), minimize the aggregate value of all other claims on the firm. Surprisingly, this simple restatement of the management's objective enables one to draw a number of useful conclusions.

Consider a firm that is financed by common shares and noncumulative preferred shares. The sole rights of the preferred shareholders are assumed to be the right to receive steady fixed dividends when and if declared, and the right to prevent common shareholders from receiving a dividend whenever the preferred dividend is not paid. The maximum possible value of the preferred shares is given by the perpetuity formula c/r where c is the rate at which dividends are paid and r is the riskless rate of interest. By refusing to pay dividends for time T, management can reduce this upper bound on preferred value to $e^{-rT}c/r$. Clearly, by making T very large, the managers can reduce the value of the preferred shares to zero. This is what rational management would do.

If the preferred dividends are cumulative, then the upper bound on preferred share value now becomes $e^{-rT}(cT + c/r)$, where cT represents the amount of dividend arrearage that must be repaid when dividend payments are resumed. Again, indefinite deferral of dividends reduces the preferred share value to zero. Further contractual rights must clearly be specified; otherwise, preferred stock is simply worthless.

Another provision that might protect the value of cumulative or noncumulative preferred stock is convertibility into common stock. This is quite a common feature and, at the very least, guarantees that the preferred share is worth at least as much as the number of common shares into which it is convertible. Equally, the right of participation in common dividends, although a less common

[1] It has been noted by Stiglitz [17], Galai and Masulis [6], and others that the fundamental conflict of interest between shareholders and owners of other claims on the firm can lead management, interested in maximizing share value, into decisions that do not maximize firm value. Also, Stiglitz [18] has cast some doubt on the desirability of the goal of maximizing firm value. Without denying the profound implications of such observations, this issue is completely ignored here.

provision, will provide a similar lower bound on preferred stock value. However, these lower bounds will be strictly binding as it will still be optimal to defer the payment of dividends indefinitely. Therefore, convertible and participating preferred stock will merely be a disguised form of common stock.

A more direct method of safeguarding the interests of preferred shareholders is to have an explicit contractual provision that if retained earnings are positive then the preferred dividend must be paid. In other words, if shareholders' funds exceed initial capitalization, that excess must be applied first to satisfying the dividend obligation to the preferred shareholders. This is a common provision in Great Britain and elsewhere (e.g., Australia). However, it is not common in America.

Given the aforementioned contractual provision, it will then be optimal for management, having met the obligation, to pay any excess out as a common dividend. Failure to do this will merely create an earnings cushion out of which future preferred dividend obligations may be paid. This cushion would only serve to benefit the preferred shareholders at the expense of the common shareholders. One would then observe that, as soon as shareholders' funds momentarily exceeded initial capitalization, the excess would be paid out as a combination of preferred and common dividends. This corresponds very closely to the dividend policy that is assumed here in the initial valuation model. The sole difference is that the contractual provisions one observes in practice are couched in terms of book value whereas here the dividend policy constraint is contingent upon the market value of the firm.

Quite often, particularly where bonds have also been issued by the firm, this provision is also backed up by an explicit provision limiting common dividends. In such a case, management will not be able to reduce shareholders' funds to initial capitalization, but only as low as allowed by the extra provision. Clearly, this is what management will do. Some of the illustrative examples presented in Section IV correspond quite closely to that situation.

Thus, a provision requiring preferred dividends to be paid out of retained earnings does afford the preferred shareholder some protection and makes the assumptions of the models presented here seem reasonable. However, since this type of provision is rare in the United States, it seems likely that there are other provisions that serve the same purpose.

Indeed, we find that in the United States preferred shareholders are often protected by voting rights. Unless specifically excluded, preferred shareholders have all the rights of common shareholders including the right to vote at shareholders' meetings. Quite often these rights are restricted. However, even if preferred shareholders possess the same voting rights as common shareholders, the right to vote is not of much use as, almost invariably, the preferred shareholders will be in the minority by a considerable margin. Common shareholders, being the majority, will still call the shots. However, a frequent provision allows preferred shareholders to vote as a separate class whenever dividends have been in arrears for a specific period of time (normally four quarters) and to elect a specific number of directors. In the vast majority of cases of preferred stock issued by industrial corporations, that number represents a minority of the board and may therefore be assumed to be relatively ineffectual. The reverse is true for public utilities (which, in any event, are the main corporate issuers of preferred stock) where the norm is for preferred shareholders to have the right to elect a majority of directors when the dividend is in arrears (cf. Moody's [12]).

If preferred shareholders gain effective control of the company when dividends are in arrears then, ignoring the four-quarter rule, it makes little sense for management to omit a dividend whenever there are funds available for disbursement of dividends, since rational preferred shareholders would, in that case, immediately vote for payment of the dividend arrearage as soon as they gained control of the board of directors. Therefore, management will pay preferred dividends as required and, of course, common dividends as fast as possible as long as there are funds available for paying dividends. Unless there are other constraints, funds are available for distribution as dividends whenever shareholders' funds exceed initial capitalization. Any payments beyond that point would not be a dividend but a capital distribution. Once shareholders' funds dip below initial capitalization, management will be unable to pay dividends and the preferred shareholders will assume control of the firm. However, once in control, the preferred shareholders will not immediately be able to pay off the arrearage as the only funds available will be common shareholders' initial capital and the prior claim of the preferred stock certainly does not extend to preemption of those funds. The preferred shareholders will have to wait until shareholders' funds again exceed initial capitalization and then the excess may be used to pay off dividend arrearage as fast as possible. Once the arrearage is fully paid off, control will revert to the common shareholders. Thus, the provision of voting control whenever dividends are in arrears leads to exactly the same dividend policy as an explicit requirement that preferred dividends be paid whenever retained earnings are available.

If neither of the two provisions discussed above is available, preferred shareholders may still be afforded some protection if there are also bonds outstanding. While a policy of not paying dividends minimizes preferred share value, it also serves to maximize the equity cushion protecting the debt outstanding and, therefore, to maximize bond value. Particularly, when the amount of preferred stock is small relative to the amount of debt capital, rational management, seeking to maximize common shareholder wealth, will be more concerned with minimizing bond value than with minimizing preferred share value. The precise analytic form of the optimal dividend policy cannot be obtained. However, if the amount of preferred stock is very small relative to the amount of debt, then the dividend policy that is optimal involves paying common and preferred dividends as fast as possible. In that case, the values obtained by the models presented here will be good approximations to the value of preferred stock that is protected by a large bond issue. It is interesting to note that, with the exception of public utilities, the amount of preferred stock financing is typically very small relative to the amount of both debt and common stock financing.

In the real world, common shareholders may actually prefer to receive steady dividends to minimize both taxes and transactions costs. Given a preference for dividends, it is no longer clear that management will defer paying dividends indefinitely. However, the resulting form of the optimal dividend policy is far from obvious.

If one is prepared to drop some of the idealized assumptions made here, and summarized below as Conditions 1, 2, and 3, one might argue that dividend policies of the type assumed here are adopted by management because they seem reasonable. Management feels that preferred dividends can be omitted if the firm has fallen on hard times, but in normal circumstances management feels obliged to honor dividend commitments to preferred shareholders. A justification for this behavior would be a belief that, if these commitments were not normally honored, then future attempts by the firm to issue capital would be viewed with suspicion in the marketplace and would therefore fail. Such a line of reasoning, however, clearly does not hold water in anything approaching a frictionless market.

The conclusion to be drawn from this discussion is that the dividend policy assumed by the models presented in this paper will be followed by rational management if the following conditions are met:

1. Market value and book value are equivalent

and either

2. It is explicitly required that preferred dividends be paid if net retained earnings are available

or

3. It is provided that preferred shareholders obtain majority voting rights whenever dividends are in arrears.

In practice, a similar dividend policy will be followed if neither Condition 2 nor 3 is met, providing preferred stock financing is very small relative to debt or common stock financing.

II. The Basic Model

A. Assumptions and Solution of the Basic Model

The assets of the firm have a value of $V(t)$ which is assumed to be the realization of a diffusion process defined, in the absence of payouts, as follows

$$dV = V\mu \, dt + \sigma V \, dz(t) \tag{1}$$

where σ is a constant and

$$E(dz(t)) = 0, \qquad E(dz(t)^2) = dt$$

and

$$E((z(t_3) - z(t_2))(z(t_2) - z(t_1))) = 0 \qquad \forall t_3 \geq t_2 \geq t_1$$

In other words, $V(t)$ follows the familiar "Geometric Brownian Motion" (cf. Black and Scholes [2], Samuelson [14], and other authors).

There exist two perpetual claims on the assets, namely a block of common stock (denoted by S), and a block of nonconvertible, noncallable cumulative preferred stock (P) with rights to a dividend accruing continuously at rate c. The dividend obligation may be in arrears, in which case let a denote the amount of arrearage. At time t, the arrearage is equal to ct minus the total amount of preferred dividends paid between time 0 and time t. It is assumed that no dividend may be paid on the common stock as long as a is strictly positive.

Modification of the approach to include conversion and call provisions is quite straightforward as Ingersoll [7] and others have shown. For clarity of exposition, these possibilities are not considered here.

Other assumptions are

A1. The interest rate r is constant.

A2. There are no taxes.

A3. V, S, and P may each be traded continually in any amount. The continuous sample path dynamics governing V have been specified above in Equation (1).

A4. There are no transaction costs.

A5. The management of the firm will pay off the arrearage on the preferred stock dividend as fast as possible whenever V rises to the level 1 or higher.

A6. σ is unanimously known with certainty and is constant.

A7. Subject to all of the above, all participants are rational and will not allow the coexistence of two security combinations, the rate of return on one of which dominates that on the other.

Assumption A5 is, of course, critical. In the preceding section, possible conditions and provisions that make this assumption reasonable were discussed at considerable length. The assumption is that management will pay off the preferred dividend if it can, but is relieved of the binding obligation if V should fall below the arbitrary level 1.

The rationality Assumption A7 implies that common dividends will be paid as fast as possible, once the arrearage is paid off. This will prevent V rising above 1. The examples developed in Section IV allow a more flexible specification of dividend policy which may permit V to rise above 1.

Subject to these assumptions, we have the following relationships for $S(V, a)$ and $P(V, a)$ (see the Appendix for the derivation):

$$S + P = V \tag{2}$$

$$rV\frac{\partial S}{\partial V} - rS + \frac{\partial^2 S}{\partial V^2}\frac{\sigma^2 V^2}{2} + c\frac{\partial S}{\partial a} = 0 \tag{3}$$

S must satisfy the above partial differential equation as long as the firm makes no payouts. (i.e., as long as $V < 1$). When payouts are being made, neither Equation (1) nor (3) applies, but the effect of payouts is properly accounted for in the boundary conditions. The following boundary conditions must be satisfied (see the Appendix for details):

$$S(0, a) = 0 \tag{4}$$

$$\lim_{a\to\infty} S(V, a) = 0 \tag{5}$$

$$\frac{\partial S}{\partial V}(1, a) + \frac{\partial S}{\partial a}(1, a) = 0 \tag{6}$$

$$\frac{\partial S}{\partial V}(1, 0) = 1 \tag{7}$$

The solution to Equation (3) and Conditions (4) to (7) is (see the Appendix):

$$S = \frac{1}{\lambda} e^{-\lambda a} V^{\lambda} \tag{8}$$ [2]

where

$$\lambda = \frac{\sqrt{((2c - 2r + \sigma^2)^2 + 8r\sigma^2)} + 2c - 2r + \sigma^2}{2\sigma^2} > 1 \tag{9}$$

and, of course

$$P = V - \frac{1}{\lambda} e^{-\lambda a} V^{\lambda} \tag{10}$$

Consequently, $\dfrac{\partial P}{\partial V}$ $(= 1 - e^{-\lambda a} V^{\lambda - 1})$ approaches the value 1 when a becomes large and also when V becomes small. (Both cases imply that the firm will spend a long time paying off the arrearage.) When $V = 1$ and $a = 0$, this derivative is zero. The implications of this are that preferred stock should be viewed as a debt instrument when dividends are not in arrears and as an equity instrument when the arrearage is large (or is likely to become so). This seems reasonable and is a somewhat clearer description than the folklore statement that "preferred stock is a hybrid of debt and equity."

If the value of the firm V is assumed to have a constant level of systematic risk β, then P and S will have systematic risk levels given by

$$\beta \frac{\partial P}{\partial V} \frac{V}{P} = \frac{\lambda V - \lambda e^{-\lambda a} V^{\lambda}}{\lambda V - e^{-\lambda a} V^{\lambda}} \beta = \beta \left(1 - \frac{(\lambda - 1) e^{-\lambda a} V^{\lambda}}{\lambda V - e^{-\lambda a} V^{\lambda}} \right) \tag{11}$$

and

$$\beta \frac{\partial S}{\partial V} \frac{V}{S} = \frac{\lambda e^{-\lambda a} V^{\lambda}}{e^{-\lambda a} V^{\lambda}} \beta = \lambda \beta \tag{12}$$

Thus, the relative volatility of S remains constant while that of P decreases as V increases, and increases as a increases.

Notice that when $V = 1$ and $a = 0$ expression (11), of course, evaluates to zero and when V is small or a is large the expression evaluates to β. Consequently, the ratio of preferred volatility to common stock volatility has asymptotes of zero and $1/\lambda$ as the company's fortunes range from very good to very poor. Qualitatively, this would seem to explain quite well the cross-sectional empirical results of Bildersee [1] summarized in Figure 1 of that paper, particularly in view of his comments about the tendency of the high β preferreds to be of "low quality" and to "lack structural stability." Bildersee found β to be close to zero for most preferred stocks. However, unstable and quite large positive β values were found for preferred stocks whose dividends had been omitted. These unstable β values were apparently related to but lower than the associated common stock β values.

[2] This simple solution can be given some intuitive meaning by repeating the analysis from a probabilistic viewpoint. This is done in Emanuel [5].

III. More General Payout Policy

In the preceding section, the firm was assumed not to make payments when $V < 1$. If debt claims are also outstanding, this assumption must be modified. The assets are now assumed to follow a diffusion defined for $V < 1$ as follows:

$$dV = ((\mu - \alpha)V - D)\,dt + \sigma V\,dz(t) \tag{13}$$

where α and σ are constants. As in the basic model, Assumptions A1 to A7 ensure that a riskless hedging argument can be constructed and that the pricing of claims will be independent of preferences. That is to say, risk neutrality may be assumed.[3] Then, the process above would be consistent with risk neutrality if payments of the form $\alpha V + D$ were being made to someone holding V and if μ had the value r. The expected return to a holder of V would then equal rV, being the sum of payments $\alpha V + D$ and expected capital gain $(r - \alpha)V - D$.

S must now satisfy the following partial differential equation:

$$\frac{\sigma^2}{2} V^2 \frac{\partial^2 S}{\partial V^2} + ((r - \alpha)V - D)\frac{\partial S}{\partial V} - rS + c\frac{\partial S}{\partial a} = 0 \tag{14}$$

subject to Conditions (4) to (7). Equation (14) may be derived by analogous logic to that used in Appendix 1 to obtain Equation (3).
Also,

$$P(V, a) = \phi(V) - S(V, a) \tag{15}$$

where ϕ is a solution of Equation (14) subject to the boundary conditions:

$$\phi(0) = 0 \qquad \phi'(1) = 1 \tag{16}$$

$\phi(= P + S)$ is necessarily a function of V only. The arrearage a serves only to redistribute dividends flowing to P and S and does not affect their aggregate value. The first condition in (16) is obvious and the second may be inferred using similar logic to that used in deriving Equation (7). $V - \phi(V)$ represents the value of other claims against the firm that is receiving the previously mentioned payment stream $\alpha V + D$. The form $S = e^{-\lambda a}g(V)$ should be substituted in Equation (14) to yield

$$\frac{\sigma^2}{2} V^2 g'' + ((r - \alpha)V - D)g' - (r + \lambda c)g = 0 \tag{17}$$

Equation (17) is similar to Equation 44 of Merton [8] and can be solved by substitutions similar to those used in Footnote 66 of that paper. Define

$$\left.\begin{aligned} \gamma &= \frac{2r - 2\alpha - \sigma^2 + \sqrt{((\sigma^2 - 2r + 2\alpha)^2 + 8(r + \lambda c)\sigma^2)}}{2\sigma^2} \\ \delta &= \frac{2D}{\sigma^2} \qquad \beta = \frac{2(r + \lambda c)}{\sigma^2} \end{aligned}\right\} \tag{18}$$

[3] Further insight into constructing partial differential equations using the risk-neutral world argument may be obtained by reference to Cox and Ross [4].

and

$$\eta = \frac{2r - 2\alpha}{\sigma^2}$$

Then, the Equation (17) becomes:

$$\beta g - V^2 g'' - \frac{((r - \alpha)V - D)2}{\sigma^2} g' = 0 \tag{19}$$

and, redefining

$$x = \frac{\delta}{V}$$

and

$$h(x) = e^x x^{-\gamma} g$$

tedious substitution and elimination (see the Appendix) in Equation (19) leads to

$$xh'' + (2\gamma - \eta + 2 - x)h' + (\eta - \gamma - 2)h = 0 \tag{20}$$

Equation (20) is a form of Kummer's Equation (cf. Slater [15]) and its solution is the confluent hypergeometric function M.

$$M(a, b, z) = 1 + \frac{az}{b} \cdots + \left(\prod_{i=0}^{n-1} \frac{a + i}{b + i}\right) \frac{z^n}{n!} \cdots \tag{21}$$

Substituting S back into the general solution of Equation (20), we obtain

$$S = e^{-\lambda a}\left(c_1\left(\frac{\delta}{V}\right)^{\gamma} M\left(\gamma, 2\gamma - \eta + 2, \frac{-\delta}{V}\right)\right.$$
$$\left. + c_2\left(\frac{\delta}{V}\right)^{\eta-\gamma-1} M\left(\eta - \gamma - 1, \eta - 2\gamma, \frac{-\delta}{V}\right)\right) \tag{22}$$

The unknowns in (22), namely c_1, c_2, and λ, can be determined by applying the boundary Conditions (4) to (7). A unique positive real root λ is guaranteed (see the Appendix) and the application of the Conditions (4), (6), and (7) is made easier once it is realized that

$$\lim_{z \to -\infty} (-z)^a M(a, b, z) = \frac{\Gamma(b)}{\Gamma(b - a)} \tag{23}$$

and

$$\frac{\partial}{\partial z} M(a, b, z) = \frac{a}{b} M(a + 1, b + 1, z) \tag{24}$$

Finally, $\phi(V)$ has the same form as Equation (22) with $\lambda = 0$ and different constants to satisfy the boundary conditions (16).

As an example to illustrate finding the unknowns in Equation (22), let us consider the limiting case when $D = 0$. Although $\delta = 0$ in this case, Equation (22) still has meaning in that the leading coefficient in the expansion as a power series in δ can be taken for each term, i.e.,

$$S = e^{-\lambda a}(c_1 V^{-\gamma} + c_2 V^{1+\gamma-\eta}) \tag{25}$$

Then, from Equation (4), $c_1 = 0$ and from Equation (6)

$$\lambda = 1 + \gamma - \eta = \frac{\sigma^2 - 2r + 2\alpha + \sqrt{((\sigma^2 - 2r + 2\alpha)^2 + 8r\sigma^2)}}{2\sigma^2} \tag{26}$$

Rearranging, squaring, and solving

$$\lambda = \frac{2c - 2r + 2\alpha + \sigma^2 + \sqrt{((2c - 2r + 2\alpha + \sigma^2)^2 + 8r\sigma^2)}}{2\sigma^2} \tag{27}$$

Thus

$$S(V, a) = \frac{1}{\lambda} e^{-\lambda a} V^{\lambda} \tag{28}$$

and

$$\phi(V) = \frac{1}{\xi} V^{\xi} \tag{29}$$

where

$$\xi = \frac{\sigma^2 - 2r + 2\alpha + \sqrt{((\sigma^2 - 2r + 2\alpha)^2 + 8r\sigma^2)}}{2\sigma^2} \tag{30}$$

and

$$P(V, a) = \phi(V) - S(V, a) \tag{31}$$

The above solution is identical to that of the basic model when $\alpha = 0$.

IV. Illustration of the Models

The models described in Sections II and III have relatively straightforward solutions. However, neither model permits V to rise above 1 because dividends on common would be paid at an infinite rate in that region. This behavior is dictated by the assumptions.

It might be more realistic to restrict the rate of common dividends to be a linear function of V whenever $V \geq 1$. This constraint would then permit V to rise above 1. The solution becomes more complicated because there are now two regions of interest ($V < 1$ and $V \geq 1$). However, the basic form of the solution is similar to that implied by Equations (15) and (22).

Let the asset diffusion process be defined as follows:

$$\forall V < 1 \quad dV = ((\mu - \alpha)V - D)\, dt + \sigma V\, d\mathcal{Z}(t)$$

$$\forall V > 1 \quad dV = ((\mu - \beta)V - E)\, dt + \sigma V\, d\mathcal{Z}(t) \tag{32}$$

Comparison with Equation (13) will show that (32) is consistent with a payout rate of $\alpha V + D$ when $V < 1$ and $\beta V + E$ when $V > 1$.

In the examples that follow, perpetual bonds, common stock, and preferred stock may exist as claims against the firm. The payout functions $\alpha V + D$ and $\beta V + E$ must equal the total payout rates on these claims. In these examples:

$$\alpha = 0 \text{ and } D = \text{Bond Coupon Payment Rate}$$

and

$$\begin{aligned} \beta V + E = {} & \text{linear payout function on common stock} \\ & + \text{Bond Coupon Payment Rate } (D) \\ & + \text{Preferred Dividend Rate } (c) \end{aligned}$$

Thus, when $V < 1$, only bond coupon payments (if a bond claim is outstanding) are made by the firm. When $V > 1$, bond coupons continue to be paid at rate D, preferred dividends are paid at rate c, and the common stock receives dividends at a rate that varies linearly with V (rate $= \beta V + E - D - c$). On cumulative preferred stock, dividend arrearage is paid off whenever $V = 1$. As before, an infinitesimal fraction of time is spent with $V = 1$. The dynamics of arrearage payment are handled by the boundary conditions.

We can now specify partial differential equations and boundary conditions for the claim values. For the common stock

$$\left.\begin{aligned} \forall V > 1 \quad & \frac{\sigma^2}{2} V^2 \frac{\partial^2 S}{\partial V^2} + ((r - \alpha)V - D)\frac{\partial S}{\partial V} - rS + c\frac{\partial S}{\partial a} = 0 \\ \forall V > 1 \quad & \frac{\sigma^2}{2} V^2 \frac{\partial^2 S}{\partial V^2} + ((r - \beta)V - E)\frac{\partial S}{\partial V} - rS + \beta V + E - D - c = 0 \end{aligned}\right\} \tag{33}$$

Equation (33) is similar to (14), except that allowance has been made for the common dividend rate of $\beta V + E - D - c$ when $V > 1$. Also, arrearage a is not a relevant factor when $V > 1$. The adjustment of pricing differential equations for payouts is discussed in Cox and Ross [4]. When the preferred stock is noncumulative, Equation (33) should be modified by removing any term that involves a $\left(\text{i.e., } c\frac{\partial S}{\partial a}\right)$. Similar changes must be made in the equations that follow and elsewhere as specifically noted.

The preferred stock value will be governed by

$$\left.\begin{aligned} \forall V < 1 \quad & \frac{\sigma^2}{2} V^2 \frac{\partial^2 P}{\partial V^2} + ((r-\alpha)V - D)\frac{\partial P}{\partial V} - rP + c\frac{\partial P}{\partial a} = 0 \\ \forall V > 1 \quad & \frac{\sigma^2}{2} V^2 \frac{\partial^2 P}{\partial V^2} + ((r-\beta)V - E)\frac{\partial P}{\partial V} - rP + c = 0 \end{aligned}\right\} \tag{34}$$

The bond value B can be represented as a solution to a similar equation. However, it is simpler to use the relationship

$$B = V - S - P \tag{35}$$

Boundary conditions can also be specified. These are analogous to Conditions (4) to (7), but are more complicated because the state space has more regions. They are provided here in a sequence that matches (4) to (7).

$$\lim_{V\to 0} S(V, a) = 0 \tag{36}$$

$$\lim_{V\to 0} P(V, a) = 0 \tag{37}$$

$$\lim_{a\to\infty} S(V, a) = 0 \tag{38}$$

$$\lim_{a\to\infty} \frac{\partial P}{\partial a} = 0 \tag{39}$$

$$\frac{\partial S}{\partial V}(1, a) + \frac{\partial S}{\partial a}(1, a) = 0 \tag{40}$$

$$\frac{\partial P}{\partial V}(1, a) + \frac{\partial P}{\partial a}(1, a) = 1 \tag{41}$$

$$\lim_{V\to\infty} S = V - \frac{D}{r} - \frac{c}{r} \tag{42}$$

$$\lim_{V\to\infty} P = \frac{c}{r} \tag{43}$$

(42) and (43) use the fact that bond and preferred stock values asymptotically approach the values of riskless perpetuities. Needless to say, (38) to (41) do not apply to noncumulative preferred stock.

The added complication in boundary conditions comes from specifying continuity and differentiability requirements at $V = 1$, namely

$$P(1^-, 0) = P(1^+, 0) \tag{44}$$

$$S(1^-, 0) = S(1^+, 0) \tag{45}$$

$$\frac{\partial P}{\partial V}(1^-, 0) = \frac{\partial P}{\partial V}(1^+, 0) \tag{46}$$

$$\frac{\partial S}{\partial V}(1^-, 0) = \frac{\partial S}{\partial V}(1^+, 0) \tag{47}$$

Solving (33) and (34) subject to their respective boundary Conditions (36) to (47) is quite lengthy. A general framework for analyzing this and more complicated problems is provided in Emanuel [5]. It can be shown that both S and P have the following form, which is similar to that developed in Section III

$\forall\ V \leq 1$

$$P(V, a) = e^{-\lambda a}\left(C_{1A} V^{-\gamma_A} M\left(\gamma_A,\ 2\gamma_A - \eta_A + 2,\ \frac{-\delta_A}{V}\right)\right.$$
$$\left. + C_{2A} V^{1+\gamma_A-\eta_A} M\left(\eta_A - \gamma_A - 1,\ \eta_A - 2\gamma_A,\ \frac{-\delta_A}{V}\right)\right)$$
$$+ F_{1A} + F_{2A} V + C_{1C} V^{-\gamma_C} M\left(\gamma_C,\ 2\gamma_C - \eta_C + 2,\ \frac{-\delta_C}{V}\right)$$
$$+ C_{2C} V^{1+\gamma_C-\eta_C} M\left(\eta_C - \gamma_C - 1,\ \eta_C - 2\gamma_C,\ \frac{-\delta_C}{V}\right) \tag{48}$$

$\forall\ V > 1,\ a = 0$

$$P(V, 0) = C_{1B} V^{-\gamma_B} M\left(\gamma_B,\ 2\gamma_B - \eta_B + 2,\ \frac{-\delta_B}{V}\right)$$
$$+ C_{2B} V^{1+\gamma_B-\eta_B} M\left(\eta_B - \gamma_B - 1,\ \eta_B - 2\gamma_B,\ \frac{-\delta_B}{V}\right)$$
$$+ F_{1B} + F_{2B} V \tag{49}$$

where

$$\eta_A = \eta_c = \frac{2r - 2\alpha}{\sigma^2} \qquad \eta_B = \frac{2r - 2\beta}{\sigma^2}$$

$$\delta_A = \delta_c = \frac{2D}{\sigma^2} \qquad \delta_B = \frac{2E}{\sigma^2}$$

$$\gamma_A = \frac{2r - 2\alpha - \sigma^2 + \sqrt{((\sigma^2 - 2r + 2\alpha)^2 + 8(r + \lambda c)\sigma^2)}}{2\sigma^2}$$

$$\gamma_B = \frac{2r - 2\beta - \sigma^2 + \sqrt{((\sigma^2 - 2r + 2\beta)^2 + 8r\sigma^2)}}{2\sigma^2}$$

and

$$\gamma_c = \frac{2r - 2\alpha - \sigma^2 + \sqrt{((\sigma^2 - 2r + 2\alpha)^2 + 8r\sigma^2)}}{2\sigma^2}$$

These definitions clearly resemble those given in (18). The constants C_1, C_2, γ, F_1, and F_2 in the solution for S or P are fixed by applying the boundary Conditions (36) to (47). Setting the constants can be greatly facilitated by using a computer program. This does not involve approximation but merely uses automatic calculation of coefficients and verification of conditions in the exact analytical solution.

Table I and Figures 1 to 4 show computer generated solutions for a number of examples. In Table I, pertinent parameters and solution coefficients are tabulated. In any of the examples where no cumulative preferred claim is present, λ and a are identically zero and there may be some redundancy in the above formulae.

In each graph, the value of each security is plotted as a function of the value of the firm. (The arrearage is taken to be zero in each graph.) Graphs are plotted for varying values of the variance rate (σ^2) from 0.05 to 0.15.

In Figure 1, graphs are plotted for a firm financed by common stock (labeled C) and cumulative preferred stock (P). An interest rate r of 0.05 is assumed. The preferred stock accumulates arrearage (when $V < 1$) or is paid dividends (when $V > 1$) at a rate of 0.05. The common stock receives dividends only when $V > 1$ and is paid at a rate of 0.03 $(V - 1)$—a rate which increases as the firm grows.

Naturally, the graph of P has an asymptotic value of a riskless consol bond, i.e., 1 (1 = 0.05/0.05). Even when V is as large as 4 (i.e., four times as great as the point at which dividends are suspended), the value of the preferred is considerably less than the asymptote. This suggests that a procedure of valuing preferred stock as a riskless consol bond, as the textbooks suggest, is seriously in error. Table I shows the exact coefficients for the valuation formulae when the variance rate is 0.05.

The tardiness in approaching the asymptotic value is even more apparent in Figure 2. This figure shows graphs for noncumulative preferred and common

Table I

Parameters and Coefficients Used in Generating Figures 1 to 4[a]

Formula for Common Value: Insert these constants in Equations (32) and (33)

Figure	γ_A	δ_A	λ	C_{1A}	C_{2A}	δ_B	C_{1B}	F_{1B}
1	3.0		2.0		0.326	0.8	0.422	−1.0
2	2.0				0.483	0.8	0.626	−1.0
3	3.663	2.0	3.877	0.00105	0.228	2.8	2.391	−2.0
4	2.0	2.0		0.265	0.397	2.8	2.463	−2.0

$\eta_A = 2.0$, $\gamma_B = 1.318$, $\eta_B = 0.8$, and $F_{2B} = 1.0$ in all cases

Formula for Preferred Value: Insert in Equations (32) and (33)

Figure	γ_A	δ_A	λ	C_{1A}	C_{2A}	δ_B	C_{1B}	γ_C	η_C	δ_C	C_{1C}	C_{2C}
1	3.0		2.0		−0.326	0.8	−0.422	2.0	2.0			1.0
2	2.0				0.517	0.8	−0.626					
3	3.663	2.0	3.877	−0.00187	−0.404	2.8	−1.630	2.0	2.0	2.0	0.985	1.478
4	2.0	2.0		0.514	0.772	2.8	−1.760					

$\eta_A = 2.0$, $\gamma_B = 1.318$, $\eta_B = 0.8$, and $F_{1B} = 1.0$ in all cases

[a] *Parameters*: interest rate, 0.05; preferred dividend rate, 0.05; variance rate, 0.05; common dividend rate, 0.03 V − 0.03 (if $V \geq 1$); bond coupon rate = 0.05 (in Figures 3 and 4 only); preferred stock is cumulative (in Figures 1 and 3 only).

All omitted values or variables should be assumed to be zero. The value for the bond in Figures 3 and 4 may be inferred by deducting preferred value and common value from total firm value.

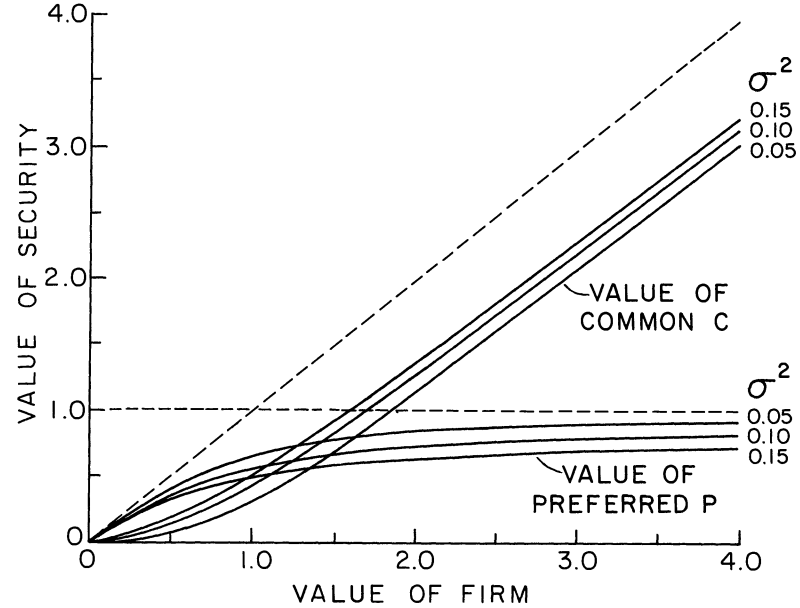

Figure 1. Cumulative Preferred Stock

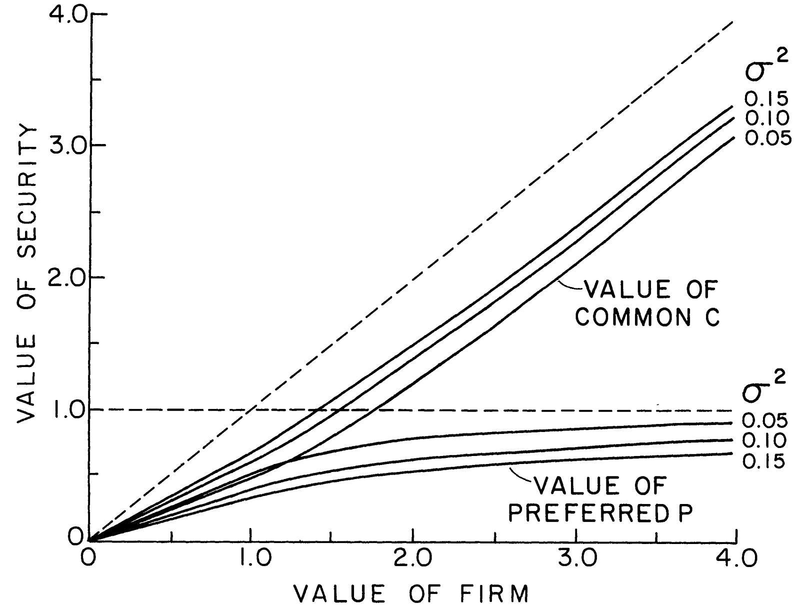

Figure 2. Noncumulative Preferred Stock

stock. Parameters assumed in Figure 2 are identical to those for Figure 1 except that no arrearage is accumulated. Again, Table I shows the coefficients based on a variance rate of 0.05.

It is interesting to contrast the graphs of P in the two figures for $V < 1$. The cumulative preferred value starts with a gradient of 1 when $V = 0$. This reflects the expectation that arrearage earned before V returns to 1 will be extremely large. The noncumulative preferred value graph is exactly linear when $V < 1$. In this region, neither the common nor the preferred receive dividends, and their fortunes are tied together and contingent upon V reaching 1. Ignoring differences in liquidation, noncumulative preferred and common stock will have identical levels of systematic risk whenever $V < 1$, regardless of what other liabilities the firm may have issued.

Figures 3 and 4 represent situations analogous to Figures 1 and 2 respectively, except that now the firm also has a risky consol bond outstanding. This bond is paid coupons at a rate of 0.05 as long as the firm does not go bankrupt. The preferred stock value functions appear to be quite different from either common stock or risky corporate debt. While a corporate consol bond is more like preferred stock than is a riskless perpetuity, the similarity is not particularly striking. Table I provides the valuation coefficients for a variance rate of 0.05. Coefficients for the bond are not provided, but may be readily calculated by subtracting the common value and the preferred value from V.

VI. Summary and Conclusion

A rigorous theory has been developed for the valuation of preferred stock. This is based on the assumption that when the value of the firm falls below a certain arbitrary level, all dividends will be omitted. Such an assumption is consistent with various types of legal or contractual restrictions imposed on the firm. Precise

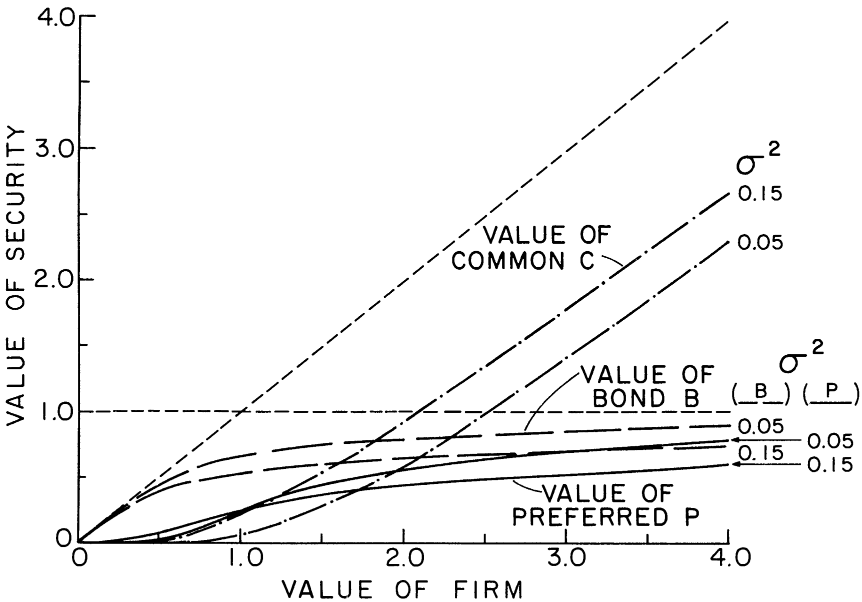

Figure 3. Cumulative Preferred Stock with a Bond also Present

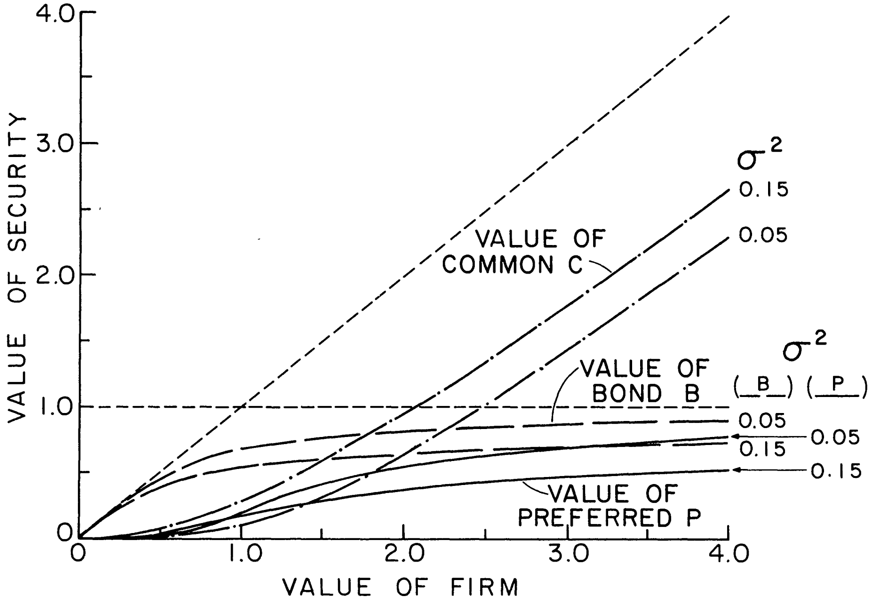

Figure 4. Noncumulative Preferred Stock with a Bond also Present

analytic solutions for preferred stock value have been derived in a number of contexts.

It is believed that the primary feature of preferred stock, namely potential omission of dividends, has been adequately modeled in this paper. The models ignore the existence of corporate taxes and cannot therefore distinguish between preferred stock and income bonds. If the tax difference is thought to be important, then this model cannot be used to value income bonds.

It would be quite easy to modify the boundary conditions specified in the models to allow for call, conversion, and liquidation privileges. More complicated would be extensions to allow interest rates to be stochastic, to allow for finite-lived securities, to allow dividend arrearage to earn interest, and to allow dividends to be paid contingent on book value rather than market value. In principle, all such extensions can be handled, but analytic solutions cannot be obtained. Solutions can only be generated by lengthy numerical analysis and the arrearage no longer enters simply as a factor of type $e^{-\lambda a}$. Work on the majority of such extensions is left open for further research.

Appendix

Derivation of Equations (2) and (3)

Assumptions A3 and A4 imply that, if $S + P$ is not equal to V, arbitrage will be possible. However, arbitrage possibilities are ruled out by Assumption A7.

Equation (3) is a little more complicated to derive. However, intuitively, variable a plays a role similar to that played by time in the derivation of the standard Black and Scholes partial differential equation. This explains the familiar appearance of Equation (3).

Applying Ito's Lemma

$$\Delta S = S(V + \Delta V, a + c\Delta t) - S(V, a) = \Delta V \frac{\partial S}{\partial V} + \frac{\sigma^2 V^2}{2} \frac{\partial^2 S}{\partial V^2} \Delta t + \frac{\partial S}{\partial a} c\Delta t$$

By holding S long and $V \dfrac{\partial S}{\partial V}$ short, the familiar riskless hedging argument of Black and Scholes is apparent

$$\Delta\left(S - V \frac{\partial S}{\partial V}\right) = \Delta S - \Delta V \frac{\partial S}{\partial V} = \frac{\sigma^2 V^2}{2} \frac{\partial^2 S}{\partial V^2} \Delta t + \frac{\partial S}{\partial a} c\Delta t = \left(S - V \frac{\partial S}{\partial V}\right) r\Delta t$$

Therefore

$$rV \frac{\partial S}{\partial V} - rS + \frac{\sigma^2 V^2}{2} \frac{\partial^2 S}{\partial V^2} + c \frac{\partial S}{\partial a} = 0$$

Derivation of Equations (4) to (7)

Since the equity in a worthless enterprise is itself worthless, $S(0, a) = 0$. Likewise, when a is very large, it will take forever to pay off the arrearage on the preferred stock. Consequently, $\lim_{a\to\infty} S(V, a) = 0$.

Now, consider what happens between time t (when $V = 1$) and time $t + \Delta t$. At time t, we have $S(1, a)$ as the stock value. During the time interval Δt, the firm realizes a return which is normally distributed in the limit as Δt tends to zero. As equation (1) indicates, this return may be written as

$$\mu\Delta t + \sigma\sqrt{\Delta t}\tilde{e}$$

where $\tilde{e}$ is a drawing from the $N(0, 1)$ distribution. If this return is positive, it will be distributed as a dividend paid first to the preferred shareholders, with any surplus beyond the arrearage paid to the common shareholders. This will leave V unchanged at 1 but will reduce the arrearage. On the other hand, if the return is negative, V will necessarily decline and dividends will not be paid.

The possible states at time $t + \Delta t$ may be summarized as:

If $\tilde{e}$ is:	Then $V =$	And arrearage $=$
$e \leq \dfrac{-\mu\sqrt{\Delta t}}{\sigma}$	$1 + \mu\Delta t + \sigma\sqrt{\Delta t}e$	$a + c\Delta t$
$-\dfrac{\mu\sqrt{\Delta t}}{\sigma} < e \leq \dfrac{a - \mu\Delta t}{\sigma\sqrt{\Delta t}}$	1	$a + c\Delta t - \mu\Delta t - \sigma\sqrt{\Delta t}e$
$e > \dfrac{a - \mu\Delta t}{\sigma\sqrt{\Delta t}}$	1	0

The expected rate of return on the common stock is therefore

$$E\left(\frac{\Delta\tilde{S}}{S(1, a)}\ \Delta t\right)$$

$$= \left[\int_{-\infty}^{(-\mu\sqrt{\Delta t}/\sigma)} S(1 + \mu\Delta t + \sigma\sqrt{\Delta t}e,\ a + c\Delta t)\ \frac{1}{\sqrt{2\pi}}\exp(-e^2/2)\ de\right.$$

$$+ \int_{(-\mu\sqrt{\Delta t}/\sigma)}^{(a-\mu\Delta t/\sigma\sqrt{\Delta t})} S(1,\ a + c\Delta t - \mu\Delta t - \sigma\sqrt{\Delta t}e)\ \frac{1}{\sqrt{2\pi}}\exp(-e^2/2)\ de$$

$$+ \int_{(a-\mu\Delta t/\sigma\sqrt{\Delta t})}^{\infty} (S(1, 0) + \mu\Delta t + \sigma\sqrt{\Delta t}e - a - c\Delta t)\ \frac{1}{\sqrt{2\pi}}\exp(-e^2/2)\ de$$

$$\left. - S(1, a)\right] \Big/ (S(1, a)\Delta t)$$

The extra terms $\mu\Delta t + \sigma\sqrt{\Delta t}e - a - c\Delta t$ in the third integral represent the common dividends that are paid once the arrearage falls to zero.

As Δt tends to zero, $\tilde{e}$ will be less than $\frac{-\mu\sqrt{\Delta t}}{\sigma}$ with probability ½ and will be greater than $\frac{a - \mu\Delta t}{\sigma\sqrt{\Delta t}}$ with probability zero (½) if a is greater than (equal to) zero. The expected rate of return may be evaluated by Taylor series expansion and integration of the resulting terms. When a is greater than zero, the expected rate of return is

$$\frac{-\sigma\sqrt{\Delta t}\,\dfrac{\partial S}{\partial V} - \sigma\sqrt{\Delta t}\,\dfrac{\partial S}{\partial a} + O(\Delta t)}{\sqrt{2\pi}S\Delta t}$$

which can only be finite if $\frac{\partial S(1, a)}{\partial V} + \frac{\partial S(1, a)}{\partial a} = 0$. When a is zero, the expected rate of return is

$$\frac{-\sigma\sqrt{\Delta t}\,\dfrac{\partial S}{\partial V} + \sigma\sqrt{\Delta t} + O(\Delta t)}{\sqrt{2\pi}S\Delta t}$$

which can only be finite if $\frac{\partial S(1, 0)}{\partial V} = 1$. Naturally, the expected rate of return on the common stock must be finite; otherwise Assumption A7 would be violated. Thus, the conditions described in Equations (6) and (7) are "established."

The above two conditions are essentially smoothness conditions. As long as σ and μ in Equation (1) are continuous, then the same argument applied for $V < 1$

leads to the requirement that S be twice differentiable in V. This comment would appear to settle any questions raised in Footnote 21 of Smith [16]. He repeats the argument that the option pricing formula has been assumed to be twice differentiable everywhere. He argues that dropping that assumption would lead to a nonunique solution to the pricing differential equation. It is true that Itô's Lemma requires that functions be twice differentiable. However, it should also be clear from the foregoing discussion that functions such as S and the Black-Scholes formula must be twice differentiable in their interiors. If not, infinite expected returns may be locally available. The double differentiability condition may only be invalid at boundaries and regions of discontinuity of the parameters generating the stochastic process.

Derivation of Equations (8) to (10)

Let us start with the trial solution:

$$S = ke^{-\lambda a}g(V)$$

Then, substituting in Equation (3), we obtain

$$rVg' - rg + \frac{\sigma^2 V^2}{2} g'' - c\lambda g = 0 \quad \left(\text{where } g' = \frac{dg}{dV}, \text{ etc.}\right)$$

This has solutions of the form: $g = V^\theta$, where θ may be deduced by substitution in Equation (6)

$$\frac{\partial}{\partial V}(ke^{-\lambda a}V^\theta) + \frac{\partial}{\partial a}(ke^{-\lambda a}V^\theta) = 0$$

at the point $(1, a)$, i.e.,

$$\theta = \lambda$$

Consequently, substituting $g = V^\lambda$ in the differential equation above

$$rV\lambda V^{\lambda-1} - rV^\lambda + \frac{\sigma^2 V^2}{2}\lambda(\lambda - 1)V^{\lambda-2} - c\lambda V^\lambda = 0$$

Therefore

$$\lambda = \frac{-(2r - 2c - \sigma^2) \pm \sqrt{((2r - 2c - \sigma^2)^2 + 8r\sigma^2)}}{2\sigma^2}$$

From Conditions (4) and (5) we require that $\lambda > 0$, i.e., that

$$\lambda = \frac{-(2r - 2c - \sigma^2) + \sqrt{((2r - 2c - \sigma^2)^2 + 8r\sigma^2)}}{2\sigma^2}$$

Finally, applying Condition (7)

$$\lambda k = 1$$

Therefore

$$S(V, a) = \frac{1}{\lambda} e^{-\lambda a} V^\lambda$$

Derivation of Equation (20) from Equation (19)

$$g = x^{\gamma} e^{-x} h(x)$$

$$g' = \frac{dg}{dx}\frac{dx}{dV} = \frac{-\delta}{V^2}\left(x^{\gamma} e^{-x} h' - g + \frac{\gamma g}{x}\right)$$

where g', g'', etc., denote derivatives with respect to V and h', h'' denote derivatives with respect to x

$$g'' = \frac{dg'}{dx}\frac{dx}{dV} = \frac{-\delta}{V^2}\frac{d}{dx}\left(\frac{-\delta}{V^2}\left(x^{\gamma} e^{-x} h' - g + \frac{\gamma g}{x}\right)\right)$$

$$= \frac{x^2}{\delta}\frac{d}{dx}\left(\frac{x^2}{\delta}\left(x^{\gamma} e^{-x} h' - g + \frac{\gamma g}{x}\right)\right)$$

$$= \frac{e^{-x}x^2}{\delta^2}\,(h'((2\gamma + 2)x^{\gamma+1} - 2x^{\gamma+2}) + h''x^{\gamma+2}$$

$$+ h(x^{\gamma+2} - (2\gamma + 2)x^{\gamma+1} + \gamma(\gamma + 1)x\gamma))$$

Substituting in Equation (44) and collecting terms, one obtains:

$$\text{Coefficient of } h(x) = e^{-x}\left(\beta x^{\gamma} - x^{\gamma+2} + (2\gamma + 2)x^{\gamma+1} - \gamma(\gamma + 1)x^{\gamma}\right.$$
$$\left. + \frac{2((r - \alpha)V - D)}{\sigma^2}\frac{\delta}{V^2}(-x^{\gamma} + \gamma x^{\gamma-1})\right)$$

$$\text{Coefficient of } h'(x) = e^{-x}\left(-(2\gamma + 2)x^{\gamma+1} + 2x^{\gamma+2} + \frac{2((r - \alpha)V - D)}{\sigma^2}\frac{\delta}{V^2}x^{\gamma}\right)$$

$$\text{Coefficient of } h''(x) = -e^{-x}x^{\gamma+2}$$

The value of γ has been chosen so that:

$$\beta - \gamma(\gamma + 1) + \frac{2r - 2\alpha}{\sigma^2}\gamma = 0$$

Therefore, the coefficient of $h(x)$ may be written as

$$\text{Coefficient of } h(x) = e^{-x}\left(\beta x^{\gamma} - x^{\gamma+2} + (2\gamma + 2)x^{\gamma+1} - \gamma(\gamma + 1)x^{\gamma}\right.$$
$$\left. + \left(\frac{2(r - \alpha)}{\sigma^2}\frac{\delta}{V} - \frac{2D}{\sigma^2}\frac{\delta}{V^2}\right)(-x^{\gamma} + \gamma x^{\gamma-1})\right)$$

$$= e^{-x}\left(\beta x^{\gamma} - x^{\gamma+2} + (2\gamma + 2)x^{\gamma+1} - \gamma(\gamma + 1)x^{\gamma}\right.$$
$$\left. + \left(\frac{2(r - \alpha)}{\sigma^2}x - x^2\right)(-x^{\gamma} + \gamma x^{\gamma-1})\right)$$

i.e.,

$$\text{Coefficient of } h(x) = e^{-x}\left(x^{\gamma}\left(\beta - \gamma(\gamma + 1) + \frac{2(r - \alpha)}{\sigma^2}\,\gamma\right) + x^{\gamma+1}\left(\gamma + 2 - \frac{2(r - \alpha)}{\sigma^2}\right) + x^{\gamma+2}(-1 + 1)\right)$$

$$= e^{-x}x^{\gamma+1}(\gamma + 2 - \eta)$$

Likewise

$$\text{Coefficient of } h'(x) = e^{-x}(-(2\gamma + 2)x^{\gamma+1} + 2x^{\gamma+2} + (\eta x - x^2)x^{\gamma})$$

$$= e^{-x}x^{\gamma+1}(\eta - 2\gamma - 2 + x)$$

and

$$\text{Coefficient of } h''(x) = -e^{-x}x\gamma{+}2$$

Therefore, the equation reduces to

$$-e^{-x}x^{\gamma+1}((\eta - \gamma - 2)h + (2\gamma - \eta + 2 - x)h' + xh'') = 0$$

which immediately reduces to Equation (20).

Demonstration of the Uniqueness of λ *in Section III*

Since $S(V, a) = ke^{-\lambda a}g(V)$ (where g may be viewed as a Laplace transform of the first passage time density for V changing to 1, evaluated at $r + \lambda c$), Equation (6) is equivalent to solving the equation:

$$\frac{\lambda g(1)}{r + \lambda c} = \lambda = \frac{\partial}{\partial V}\,\frac{g(1)}{r + \lambda c}$$

The right-hand side of this equation is strictly positive, and analysis of the properties of Laplace transforms of first passage time density functions shows that

$$\frac{\partial}{\partial \lambda}\,\frac{\partial}{\partial V}\,\frac{g(1)}{r + \lambda c} < 0.$$

Consequently, there can be at most one real root in λ.[4]

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